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ON PERFECT POWERS IN  $k$ -GENERALIZED PELL SEQUENCE

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*Abstract.* Let  $k \geq 2$  and let  $(P_n^{(k)})_{n \geq 2-k}$  be the  $k$ -generalized Pell sequence defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \dots + P_{n-k}^{(k)}$$

for  $n \geq 2$  with initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \dots = P_{-1}^{(k)} = P_0^{(k)} = 0, P_1^{(k)} = 1.$$

In this study, we handle the equation  $P_n^{(k)} = y^m$  in positive integers  $n, m, y, k$  such that  $k, y \geq 2$ , and give an upper bound on  $n$ . Also, we will show that the equation  $P_n^{(k)} = y^m$  with  $2 \leq y \leq 1000$  has only one solution given by  $P_7^{(2)} = 13^2$ .

*Keywords:* Fibonacci and Lucas numbers; exponential Diophantine equation; linear forms in logarithms; Baker's method

*MSC 2020:* 11B39, 11D61, 11J86

## 1. INTRODUCTION

Let  $k, r$  be integers with  $k \geq 2$  and  $r \neq 0$ . Let  $(G_n^{(k)})_{n \geq 2-k}$  be the linear recurrence sequence of order  $k$  defined by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \dots + G_{n-k}^{(k)}$$

for  $n \geq 2$  with the initial conditions  $G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \dots = G_{-1}^{(k)} = G_0^{(k)} = 0$  and  $G_1^{(k)} = 1$ . For  $r = 1$ , the sequence  $(G_n^{(k)})_{n \geq 2-k}$  is called  $k$ -generalized Fibonacci

sequence  $(F_n^{(k)})_{n \geq 2-k}$  (see [6]). For  $r = 2$ , the sequence  $(G_n^{(k)})_{n \geq 2-k}$  is called  $k$ -generalized Pell sequence  $(P_n^{(k)})_{n \geq 2-k}$  (see [13]). The terms of these sequences are called  $k$ -generalized Fibonacci numbers and  $k$ -generalized Pell numbers, respectively. When  $k = 2$ , we have Fibonacci and Pell sequences  $(F_n)_{n \geq 0}$  and  $(P_n)_{n \geq 0}$ , respectively.

There has been much interest in the question, when the terms of a linear recurrence sequence are perfect powers. For instance, in [14], Ljunggren showed that for  $n \geq 2$ ,  $P_n$  is a perfect square precisely for  $P_7 = 13^2$  and  $P_n = 2x^2$  precisely for  $P_2 = 2$ . In [9], Cohn solved the same equations for Fibonacci numbers. Later, these problems were extended by Bugeaud, Mignotte and Siksek (see [8]) for Fibonacci numbers and by Pethő (see [16]) for Pell numbers. Pethő [16] and Cohn [10] independently found all perfect powers in the Pell sequence. They proved:

**Theorem 1.** *The only positive integer solution  $(n, y, m)$  with  $m \geq 2$  and  $y \geq 2$  of the Diophantine equation  $P_n = y^m$  is given by  $(n, m, y) = (7, 2, 13)$ .*

Bugeaud, Mignotte and Siksek (see [8]) solved the Diophantine equation  $F_n = y^p$  for  $p \geq 2$  using modular approach and classical linear forms in logarithms. Lastly, Bravo and Luca handled this problem with  $y = 2$ , for  $k$ -generalized Fibonacci numbers. They showed in [6] that the Diophantine equation  $F_n^{(k)} = 2^m$  in positive integers  $(n, m)$  has the solutions  $(n, m) = (6, 3)$  for  $k = 2$  and  $(n, m) = (t, t - 2)$  for all  $2 \leq t \leq k + 1$ .

In this paper, we deal with the Diophantine equation

$$(1) \quad P_n^{(k)} = y^m$$

in positive integers  $n, m$  with  $k, y \geq 2$ . Our main result is the following.

**Theorem 2.** *Let  $2 \leq y \leq 1000$ . Then Diophantine equation (1) has only the solution  $(n, m, k, y) = (7, 2, 2, 13)$ .*

## 2. PRELIMINARIES

The characteristic polynomial of the sequence  $(P_n^{(k)})_{n \geq 2-k}$  is

$$(2) \quad \Psi_k(x) = x^k - 2x^{k-1} - \dots - x - 1.$$

We know from Lemma 1 of [19] that this polynomial has exactly one positive real root located between 2 and 3. We denote the roots of the polynomial in (2) by

$\alpha_1, \alpha_2, \dots, \alpha_k$ . Particularly, let  $\alpha = \alpha_1$  denote the positive real root of  $\Psi_k(x)$ . The positive real root  $\alpha = \alpha(k)$  is called dominant root of  $\Psi_k(x)$ . The other roots are strictly inside the unit circle. In [5], the Binet-like formula for  $k$ -generalized Pell numbers is given by

$$(3) \quad P_n^{(k)} = \sum_{j=1}^k \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n.$$

It was also shown in [5] that the contribution of the roots inside the unit circle to formula (2) is very small, more precisely the approximation

$$(4) \quad |P_n^{(k)} - g_k(\alpha)\alpha^n| < \frac{1}{2}$$

holds for all  $n \geq 2 - k$ , where

$$(5) \quad g_k(z) = \frac{z - 1}{(k + 1)z^2 - 3kz + k - 1}.$$

From [3], we can give the inequality, which will be used in the proof of Lemma 8,

$$(6) \quad \left| \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \right| < 1$$

for  $k \geq 2$ , where  $\alpha_j$ 's for  $j = 1, 2, \dots, k$  are the roots of the polynomial in (2).

Throughout this paper,  $\alpha$  denotes the positive real root of the polynomial given in (2). The following relation between  $\alpha$  and  $P_n^{(k)}$  given in [5] is valid for all  $n \geq 1$ .

$$(7) \quad \alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1}.$$

Furthermore, Kılıç in [13] proved that

$$(8) \quad P_n^{(k)} = F_{2n-1}$$

for all  $1 \leq n \leq k + 1$ .

**Lemma 3** ([5], Lemma 3.2). *Let  $k, l \geq 2$  be integers. Then:*

- (a) *If  $k > l$ , then  $\alpha(k) > \alpha(l)$ , where  $\alpha(k)$  and  $\alpha(l)$  are the values of  $\alpha$  relative to  $k$  and  $l$ , respectively.*
- (b)  *$\varphi^2(1 - \varphi^{-k}) < \alpha < \varphi^2$ , where  $\varphi = \frac{1}{2}(1 + \sqrt{5})$  is the golden section.*
- (c)  *$g_k(\varphi^2) = 1/(\varphi + 2)$ .*
- (d)  *$0.276 < g_k(\alpha) < \frac{1}{2}$ .*

For solving equation (1), we use linear forms in logarithms and Baker's theory. For this, we give some notations, lemmas and a theorem.

Let  $\eta$  be an algebraic number of degree  $d$  with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \dots + a_d = a_0 \prod_{i=1}^d (x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the  $a_i$ 's are integers with  $\gcd(a_0, \dots, a_n) = 1$  and  $a_0 > 0$  and the  $\eta^{(i)}$ 's are conjugates of  $\eta$ . Then

$$(9) \quad h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^d \log(\max\{|\eta^{(i)}|, 1\}) \right)$$

is called the logarithmic height of  $\eta$ . In particular, if  $\eta = a/b$  is a rational number with  $\gcd(a, b) = 1$  and  $b \geq 1$ , then  $h(\eta) = \log(\max\{|a|, b\})$ .

We give some properties of the logarithmic height whose proofs can be found in [7]:

$$(10) \quad h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$

$$(11) \quad h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$

$$(12) \quad h(\eta^m) = |m|h(\eta).$$

Now, from Lemma 6 given in [4], we can deduce the estimation

$$(13) \quad h(g_k(\alpha)) < 5 \log k \quad \text{for } k \geq 2,$$

which will be used in the proof of Lemma 8.

We give a theorem deduced from Corollary 2.3 of Matveev [15], which provides a large upper bound for the subscript  $n$  in equation (1) (also see Theorem 9.4 in [8]).

**Theorem 4.** *Assume that  $\gamma_1, \gamma_2, \dots, \gamma_t$  are positive real algebraic numbers in a real algebraic number field  $\mathbb{K}$  of degree  $D$ ,  $b_1, b_2, \dots, b_t$  are rational integers, and  $\Lambda := \gamma_1^{b_1} \dots \gamma_t^{b_t} - 1$  is not zero. Then*

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} t^{9/2} D^2 (1 + \log D)(1 + \log B) A_1 A_2 \dots A_t),$$

where  $B \geq \max\{|b_1|, \dots, |b_t|\}$ , and  $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$  for all  $i = 1, \dots, t$ .

In [12], Dujella and Pethő gave a version of the reduction method based on the Baker and Davenport (see [1]). Then, in [2], the authors proved the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [12]. This lemma is based on the theory of continued fractions and will be used to lower the upper bound obtained by Theorem 4 for the subscript  $n$  in (1).

**Lemma 5.** Let  $M$  be a positive integer, let  $p/q$  be a convergent of the continued fraction expansion of the irrational number  $\gamma$  such that  $q > 6M$ , and let  $A, B, \mu$  be some real numbers with  $A > 0$  and  $B > 1$ . Let  $\varepsilon := \|\mu q\| - M\|\gamma q\|$ , where  $\|\cdot\|$  denotes the distance from  $x$  to the nearest integer. If  $\varepsilon > 0$ , then there exists no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers  $u, v$ , and  $w$  with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following lemma can be found in [11].

**Lemma 6.** Let  $a, x \in \mathbb{R}$ . If  $0 < a < 1$  and  $|x| < a$ , then

$$|\log(1+x)| < \frac{-\log(1-a)}{a}|x| \quad \text{and} \quad |x| < \frac{a}{1-e^{-a}}|e^x - 1|.$$

Finally, we give the following lemma, which can be found in [17].

**Lemma 7.** If  $m \geq 1$ ,  $T \geq (4m^2)^m$  and  $x/(\log x)^m < T$ , then  $x < 2^m T (\log T)^m$ .

Before proving our result, we prove the following lemma, which gives an estimate on  $n$  in terms of  $k$  and  $y$ .

**Lemma 8.** All solutions  $(n, m, k, y)$  of equation (1) satisfy the inequality

$$(14) \quad n < 6.81 \cdot 10^{12} k^4 (\log k)^2 \log n \cdot \log y.$$

**Proof.** Assume that  $P_n^{(k)} = y^m$  with  $m, k, y \geq 2$ . If  $1 \leq n \leq k+1$ , then we have  $P_n^{(k)} = F_{2n-1} = y^m$  by (8).  $F_{2n-1} = y^m$  is not satisfied for any  $n \geq 1$  by Theorem 1 given in [8]. Then we suppose that  $n \geq k+2$ , which implies that  $n \geq 4$ . Let  $\alpha$  be the positive real root of  $\Psi_k(x)$  given in (2). Then  $2 < \alpha < \varphi^2 < 3$  by Lemma 3 (b). Using (7), we get  $\alpha^{n-2} < y^m < \alpha^{n-1}$ . Making necessary calculations, we obtain

$$(15) \quad m < (n-1) \frac{\log \alpha}{\log y} \leq (n-1) \frac{\log \varphi^2}{\log 2} < 1.4n$$

for  $n \geq 4$ . Now, let us rearrange (1) using inequality (4). Thus, we have

$$(16) \quad |y^m - g_k(\alpha)\alpha^n| < \frac{1}{2}.$$

If we divide both sides of inequality (16) by  $g_k(\alpha)\alpha^n$ , from Lemma 3, we get

$$(17) \quad \left| \frac{y^m}{\alpha^n g_k(\alpha)} - 1 \right| < \frac{1}{2g_k(\alpha)\alpha^n} < \frac{1}{0.552\alpha^n} < \frac{1.82}{\alpha^n}.$$

In order to use Theorem 4, we take

$$(\gamma_1, b_1) := (y, m), \quad (\gamma_2, b_2) := (\alpha, -n), \quad (\gamma_3, b_3) := (g_k(\alpha), -1).$$

The number field containing  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$  is  $\mathbb{K} = \mathbb{Q}(\alpha)$ , which has degree  $D = k$ . We show that the number

$$\Lambda_1 := \frac{y^m}{\alpha^n g_k(\alpha)} - 1$$

is nonzero. In contrast to this, assume that  $\Lambda_1 = 0$ . Then

$$y^m = \alpha^n g_k(\alpha) = \frac{\alpha - 1}{(k + 1)\alpha^2 - 3k\alpha + k - 1} \alpha^n.$$

Conjugating the above equality by some automorphism belonging to the Galois group of the splitting field of  $\Psi_k(x)$  over  $\mathbb{Q}$  and taking absolute values, we get

$$y^m = \left| \frac{\alpha_i - 1}{(k + 1)\alpha_i^2 - 3k\alpha_i + k - 1} \alpha_i^n \right|$$

for some  $i > 1$ . Using (6) and that  $|\alpha_i| < 1$ , we obtain from the last equality that

$$y^m = \left| \frac{\alpha_i - 1}{(k + 1)\alpha_i^2 - 3k\alpha_i + k - 1} \right| |\alpha_i|^n < 1,$$

which is impossible since  $y \geq 2$ . Therefore  $\Lambda_1 \neq 0$ .

Moreover, since  $h(y) = \log y$ ,  $h(\gamma_2) = (\log \alpha)/k < (\log 3)/k$  by (9) and  $h(g_k(\alpha)) < 5 \log k$  by (13), we can take  $A_1 := k \log y$ ,  $A_2 := \log 3$  and  $A_3 := 5k \log k$ . Also, since  $m \leq 1.4n$ , it follows that  $B := 1.4n$ . Thus, taking into account inequality (17) and using Theorem 4, we obtain

$$\frac{1.82}{\alpha^n} > |\Lambda_1| > \exp(-Ck^2(1 + \log k)(1 + \log 1.4n)k \log y \cdot \log 3 \cdot 5k \log k)$$

and so

$$n \log \alpha - \log 1.82 < Ck^2 \cdot 3 \log k \cdot 2 \log n \cdot k \log y \cdot \log 3 \cdot 5k \log k,$$

where  $C = 1.4 \cdot 30^6 \cdot 3^{9/2}$  and we have used the fact that  $1 + \log k < 3 \log k$  for all  $k \geq 2$  and  $1 + \log 1.4n < 2 \log n$  for  $n \geq 4$ . From the last inequality, a quick computation with Mathematica yields

$$n \log \alpha < 4.72 \cdot 10^{12} k^4 (\log k)^2 \cdot \log n \cdot \log y$$

or

$$n < 6.81 \cdot 10^{12} k^4 (\log k)^2 \cdot \log n \cdot \log y.$$

Thus, the proof is completed. □

### 3. THE PROOF OF THEOREM 2

Assume that Diophantine equation (1) is satisfied for  $2 \leq y \leq 1000$ . If  $1 \leq n \leq k + 1$ , then we have  $P_n^{(k)} = F_{2n-1} = y^m$  by (8). The equation  $F_{2n-1} = y^m$  has no solutions by Theorem 1 given in [8]. Then we suppose that  $n \geq k + 2$ . If  $k = 2$ , then we have  $P_n^{(2)} = P_n = y^m$ , which implies that  $(n, m, k, y) = (7, 2, 2, 13)$  by Theorem 1. Now, assume that  $k \geq 3$ . So,  $n \geq 5$ . On the other hand, since  $y \leq 1000$ , it follows that

$$(18) \quad \frac{n}{\log n} < 4.71 \cdot 10^{13} k^4 (\log k)^2$$

by (14). By Lemma 7, inequality (18) yields that

$$n < 2T \log T,$$

where  $T := 4.71 \cdot 10^{13} k^4 (\log k)^2$ . Making necessary calculations, we get

$$(19) \quad n < 3.3 \cdot 10^{15} k^4 (\log k)^3$$

for all  $k \geq 3$ .

Let  $k \in [3, 555]$ . Then, we obtain  $n < 7.9 \cdot 10^{28}$  from (19). Now, let us try to reduce this upper bound on  $n$  by applying Lemma 5. Let

$$z_1 := m \log y - n \log \alpha + \log \frac{1}{g_k(\alpha)}$$

and  $x := e^{z_1} - 1$ . Then from (17), it is seen that

$$|x| = |e^{z_1} - 1| < \frac{1.82}{\alpha^n} < 0.12$$

for  $n \geq 5$ . Choosing  $a := 0.12$ , we get the inequality

$$|z_1| = |\log(x + 1)| < \frac{\log \frac{100}{88} \cdot 1.82}{0.12 \cdot \alpha^n} < \frac{1.94}{\alpha^n}$$

by Lemma 6. Thus, it follows that

$$0 < \left| m \log y - n \log \alpha + \log \frac{1}{g_k(\alpha)} \right| < \frac{1.94}{\alpha^n}.$$

Dividing this inequality by  $\log \alpha$ , we get

$$(20) \quad 0 < |m\gamma - n + \mu| < AB^{-w},$$

where

$$\gamma := \frac{\log y}{\log \alpha}, \quad \mu := \frac{1}{\log \alpha} \log \frac{1}{g_k(\alpha)}, \quad A := 2.8, \quad B := \alpha, \quad \text{and} \quad w := n.$$



It can be easily seen that  $\log y / \log \alpha$  is irrational. If it were not, then we could write  $\log y / \log \alpha = b/a$  for some positive integers  $a$  and  $b$ . This implies that  $y^a = \alpha^b$ . Conjugating this equality by an automorphism belonging to the Galois group of the splitting field of  $\Psi_k(x)$  over  $\mathbb{Q}$  and taking absolute values, we get  $y^a = |\alpha_i|^b$  for some  $i > 1$ . This is impossible since  $|\alpha_i| < 1$  and  $y \geq 2$ . Put

$$M := 1.106 \cdot 10^{29},$$

which is an upper bound on  $m$  since  $m < 1.4n < 1.106 \cdot 10^{29}$ . Thus, we find that  $q_{91}$ , the denominator of the 91th convergent of  $\gamma$ , exceeds  $6M$ . Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log(Aq_{91}/\varepsilon)}{\log B}$$

is less than 164.9 for all  $k \in [3, 555]$ . So, if (20) has a solution, then

$$n < \frac{\log(Aq_{91}/\varepsilon)}{\log B} \leq 164.9,$$

that is,  $n \leq 164$ . In this case,  $m < 229$  by (15). A quick computation with Mathematica gives us that the equation  $P_n^{(k)} = y^m$  has no solutions for  $n \in [5, 164]$ ,  $m \in [2, 229]$  and  $k \in [3, 555]$ . Thus, this completes the analysis in the case  $k \in [3, 555]$ .

From now on, we can assume that  $k > 555$ . Then we can see from (19) that the inequality

$$(21) \quad n < 3.3 \cdot 10^{15} k^4 (\log k)^3 < \varphi^{k/2-2} < \varphi^{k/2}$$

holds for  $k > 555$ .

Now, let  $\lambda > 0$  be such that  $\alpha + \lambda = \varphi^2$ . By Lemma 3 (b), we obtain

$$\lambda = \varphi^2 - \alpha < \varphi^2 - \varphi^2(1 - \varphi^{-k}) = \varphi^{2-k},$$

i.e.,

$$(22) \quad \lambda < \frac{1}{\varphi^{k-2}}.$$

On the other hand,

$$\begin{aligned} \alpha^n &= (\varphi^2 - \lambda)^n = \varphi^{2n} \left(1 - \frac{\lambda}{\varphi^2}\right)^n = \varphi^{2n} e^{n \log(1 - \lambda/\varphi^2)} \\ &\geq \varphi^{2n} e^{-n\lambda} \geq \varphi^{2n} (1 - n\lambda) > \varphi^{2n} \left(1 - \frac{n}{\varphi^{k-2}}\right), \end{aligned}$$

where we have used the facts that  $\log(1 - x) \geq -\varphi^2 x$  for  $0 < x < 0.907$  and  $e^{-x} > 1 - x$  for all  $x \in \mathbb{R} \setminus \{0\}$ . Thus,

$$\alpha^n > \varphi^{2n} - \frac{n\varphi^{2n}}{\varphi^{k-2}} > \varphi^{2n} - \frac{\varphi^{2n}}{\varphi^{k/2}}$$

by (21). Since  $\alpha < \varphi^2$ , it follows that

$$\alpha^n < \varphi^{2n} + \frac{\varphi^{2n}}{\varphi^{k/2}}$$

and so we have

$$(23) \quad |\alpha^n - \varphi^{2n}| < \frac{\varphi^{2n}}{\varphi^{k/2}}.$$

Thus, we can write  $\alpha^n = \varphi^{2n} + \delta$  with  $|\delta| < \varphi^{2n}/\varphi^{k/2}$ . Also, the equality

$$(24) \quad g_k(\alpha) = g_k(\varphi^2) + \eta, \quad |\eta| < \frac{4k}{\varphi^k}$$

is given in Lemma 13 of [18]. Since  $g_k(\varphi^2) = 1/(\varphi + 2)$  by Lemma 3 (c), it follows that

$$g_k(\alpha) = \frac{1}{\varphi + 2} + \eta.$$

Now we can give the following result.

**Lemma 9.** *Let  $k > 555$  and let  $\alpha$  be the dominant root of the polynomial  $\Psi_k(x)$ . Let us consider  $g_k(x)$  defined in (5) as a function of a real variable. Then*

$$(25) \quad g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi + 2} + \frac{\delta}{\varphi + 2} + \eta\varphi^{2n} + \eta\delta,$$

where  $\delta$  and  $\eta$  are real numbers such that

$$(26) \quad |\delta| < \frac{\varphi^{2n}}{\varphi^{k/2}} \quad \text{and} \quad |\eta| < \frac{4k}{\varphi^k}.$$

So, using (16), (25) and (26), we obtain

$$(27) \quad \begin{aligned} \left| y^m - \frac{\varphi^{2n}}{\varphi + 2} \right| &= \left| (y^m - g_k(\alpha)\alpha^n) + \frac{\delta}{\varphi + 2} + \eta\varphi^{2n} + \eta\delta \right| \\ &\leq |y^m - g_k(\alpha)\alpha^n| + \frac{|\delta|}{\varphi + 2} + |\eta|\varphi^{2n} + |\eta||\delta| \\ &< \frac{1}{2} + \frac{\varphi^{2n}}{\varphi^{k/2}(\varphi + 2)} + \frac{4k\varphi^{2n}}{\varphi^k} + \frac{4k\varphi^{2n}}{\varphi^{3k/2}}. \end{aligned}$$

Dividing both sides of the above inequality by  $\varphi^{2n}/(\varphi + 2)$ , we get

$$(28) \quad \begin{aligned} |y^m \varphi^{-2n}(\varphi + 2) - 1| &< \frac{\varphi + 2}{2\varphi^{2n}} + \frac{1}{\varphi^{k/2}} + \frac{4k(\varphi + 2)}{\varphi^k} + \frac{4k(\varphi + 2)}{\varphi^{3k/2}} \\ &< \frac{0.05}{\varphi^{k/2}} + \frac{1}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} = \frac{1.06}{\varphi^{k/2}}, \end{aligned}$$

where we have used the facts that  $n \geq k + 2$  and

$$\frac{4k(\varphi + 2)}{\varphi^k} < \frac{0.005}{\varphi^{k/2}} \quad \text{and} \quad \frac{4k(\varphi + 2)}{\varphi^{3k/2}} < \frac{0.005}{\varphi^{k/2}} \quad \text{for } k > 555.$$

In order to use the result of Theorem 4, we take

$$(\gamma_1, b_1) := (y, m), \quad (\gamma_2, b_2) := (\varphi, -2n), \quad (\gamma_3, b_3) := (\varphi + 2, 1).$$

The number field containing  $\gamma_1, \gamma_2$ , and  $\gamma_3$  is  $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ , which has degree  $D = 2$ . We show that the number

$$\Lambda_1 := y^m \varphi^{-2n} (\varphi + 2) - 1$$

is nonzero. In contrast to this, assume that  $\Lambda_1 = 0$ . Then  $y^m (\varphi + 2) = \varphi^{2n}$  and conjugating this relation in  $\mathbb{Q}(\sqrt{5})$ , we get  $y^m (\beta + 2) = \beta^{2n}$ , where  $\beta = \frac{1}{2}(1 - \sqrt{5}) = \bar{\varphi}$ . The left-hand side of the last equality is always greater than 1, while the right-hand side is smaller than 1 because  $n \geq k + 2 > 512$ . This is a contradiction. Therefore  $\Lambda_1 \neq 0$ . Moreover, since

$$h(\gamma_1) = h(y) = \log y, \quad h(\gamma_2) = h(\varphi) \leq \frac{\log \varphi}{2}$$

and

$$h(\gamma_3) \leq h(\varphi) + h(2) + \log 2 \leq \frac{\log \varphi}{2} + \log 4$$

by (11), we can take  $A_1 := 2 \log y$ ,  $A_2 := \log \varphi$  and  $A_3 := \log 16\varphi$ . Also, since  $m < 1.4n$  by (15), we can take  $B := 2n$ . Thus, taking into account inequality (28) and using Theorem 4, we obtain

$$(1.06) \cdot \varphi^{-k/2} > |\Lambda_1| > \exp(-C(1 + \log 2n)2 \log y \cdot \log \varphi \cdot \log 16\varphi),$$

where  $C = 1.4 \cdot 30^6 3^{9/2} 2^2 (1 + \log 2)$ . This implies that

$$(29) \quad k < 4.2 \cdot 10^{13} \log n,$$

where we have used the fact that  $(1 + \log 2n) < 2.1 \log n$  for  $n \geq k + 2 > 557$ . On the other hand, from (19) we get

$$\log n < \log(3.3 \cdot 10^{15} k^4 (\log k)^3) < 35.8 + 4 \log k + 3 \log(\log k) < 43 \log k$$

for  $k \geq 3$ . So, from (29) we obtain

$$k < 4.2 \cdot 10^{13} \cdot 43 \log k,$$

which implies that

$$(30) \quad k < 7.1 \cdot 10^{16}.$$

Substituting this bound of  $k$  into (19), we get  $n < 4.9 \cdot 10^{87}$ , which implies that  $m < 6.86 \cdot 10^{87}$  by (15).

Now, let

$$z_2 := m \log y - 2n \log \varphi + \log(\varphi + 2)$$

and  $x := 1 - e^{z_2}$ . Then

$$|x| = |1 - e^{z_2}| < \frac{1.06}{\varphi^{k/2}} < 0.1$$

by (28) since  $k > 555$ . Choosing  $a := 0.1$ , we obtain the inequality

$$|z_2| = |\log(x + 1)| < \frac{\log \frac{10}{9}}{0.1} \frac{1.06}{\varphi^{k/2}} < \frac{1.12}{\varphi^{k/2}}$$

by Lemma 6. That is,

$$0 < |m \log y - 2n \log \varphi + \log(\varphi + 2)| < \frac{1.12}{\varphi^{k/2}}.$$

Dividing both sides of the above inequality by  $\log \varphi$ , it is seen that

$$(31) \quad 0 < |m\gamma - 2n + \mu| < AB^{-w},$$

where

$$\gamma := \frac{\log y}{\log \varphi}, \quad \mu := \frac{\log(\varphi + 2)}{\log \varphi}, \quad A := 2.33, \quad B := \varphi \quad \text{and} \quad w := \frac{1}{2}k.$$

It is clear that  $\log y / \log \varphi$  is irrational. If it were not, then  $\log y / \log \varphi = a/b$  for some positive integers  $a$  and  $b$  with. Thus, we get  $y^b = \varphi^a$ . Conjugating this equality in  $\mathbb{Q}(\sqrt{5})$ , we get  $y^b = \beta^a$ , which is impossible since  $\beta^a < 1$ , where  $\beta = \frac{1}{2}(1 - \sqrt{5}) = \bar{\varphi}$ . Besides, if we take  $M := 6.86 \cdot 10^{87}$ , which is an upper bound on  $m$ , we find that  $q_{212}$ , the denominator of the 212th convergent of  $\gamma$ , exceeds  $6M$ . Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log(Aq_{212}/\varepsilon)}{\log B}$$

is less than 614.4. So, if (31) has a solution, then

$$\frac{k}{2} < \frac{\log(Aq_{212}/\varepsilon)}{\log B} \leq 614.4,$$

that is,  $k \leq 1228$ . Hence, from (19), we get  $n < 2.71 \cdot 10^{30}$ , which implies that  $m < 3.8 \cdot 10^{30}$  by (15). If we apply again Lemma 5 to inequality (31) with  $M := 3.8 \cdot 10^{30}$ , we find that  $q_{84}$ , the denominator of the 84th convergent of  $\gamma$ , exceeds  $6M$ . After doing this, a quick computation with Mathematica shows that inequality (31) has solutions only for  $k \leq 552$ . This contradicts the fact that  $k > 555$ . Thus, the proof is completed.  $\square$

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