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Abstract. Ramsey theory, to quote an American mathematician Theodore S. Motzkin, implies that complete disorder is impossible. We will demonstrate key concepts of Ramsey theory on the famous SIM pencil game.

The SIM pencil game was invented in 1969 by a mathematician at the Sandia Corporation laboratories, Gustavus Simmons, in Albuquerque, who described the game in *The Journal of Recreational Mathematics* [5]. In his note he says that one of his colleagues picked the name as short for SIMple SIMmons, and because the game **bears resemblance to**[*] the familiar game of Nim [6]. Although Simmons was not the first to think of it (the idea actually occurred independently to a number of mathematicians), he was the first to publish it and to analyze it completely with a computer program.

Rules of the game:

1. Place six **vertices** in a hexagonal arrangement.
2. The first player has a red crayon, the second player has a blue crayon.
3. The first player draws a straight line between any two vertices.
4. The second player draws a straight line between any two vertices.
5. Players take turns drawing lines until one player has completed a triangle in which all sides are of his colour and loses the game.

This combinatorial game is a zero-sum perfect-information[2)] game without chance moves[3)], such as Chess or Checkers. The only possible outcomes are restricted to win-loss, loss-win, tie-tie, or draw-draw[4)] (the

[*] **Překlad slov, která jsou v textu vyznačena tučně, naleznete ve slovníčku pod článkem.**

[2)] A game has perfect information if each player is perfectly informed of all the events that have previously occurred; there is no bluffing for example.

[3)] A chance move is a situation in which transitions from state to state depend on other chance factors, such as the weather, earthquakes, or the stock market. These games involve no rolling of dice, hidden information, or bluffing.

[4)] Tie is an end position such as in a 3x3 tic-tac-toe, where no player wins, whereas draw is a dynamic tie: any position from which a player has a nonlosing move, but cannot force a win.

zero-sum property). In case of a game of SIM on six vertices, a game ending in tie-tie or draw-draw is also not possible, we will see in the rest of this article that it is a consequence of Ramsey theory.

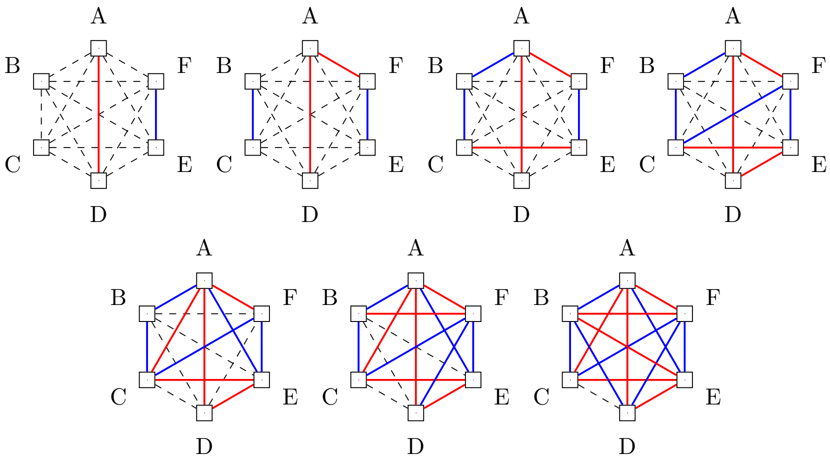

Fig. 1: A possible game of SIM where the red player loses

In his 1930 paper *On a Problem of Formal Logic* [4], Frank P. Ramsey introduced and proved both an infinite and finite version of what is now known as Ramsey's theorem. His interest was to solve what he called "one of the leading problems of mathematical logic", namely to determine whether a logic formula is true or false. The modern graph version of Ramsey's theorem follows.

Theorem 1 (Ramsey's theorem)**.** *For all k, r there exists n such that for every k-colouring of the edges of the complete graph K_n, there is a subset of r vertices, where all edges connecting them are of the same colour.*

By graph (denoted $G = (V, E)$), we mean a non-empty set of vertices V connected by a set of edges E. A complete graph K_n is a graph where every pair of distinct vertices is connected via an edge, meaning it has

$$\sum_{i=1}^{n} i = \frac{n(n-1)}{2}$$

edges in total. If we colour all the edges with k colours, we get a k-colouring of a graph.

Ramsey theorem essentially asks: Can we always find a complete monochromatic subgraph of size r in a coloured K_n? And if not, how many vertices must the initial graph have to find such a subgraph?

We would like to prove why in a game of SIM on six vertices a tie is impossible. Suppose a graph has six vertices. Let us take an **arbitrary** vertex, A. There are exactly five edges incident to A, of which, by the **pigeonhole principle**, at least three are of the same colour. Without loss of generality, we can assume the three edges are red and A is joined with C, D, E, respectively. See Fig. 2. In order to avoid creating a red monochromatic triangle, we colour edges $\{C, D\}$, $\{D, E\}$, $\{C, E\}$ blue, ergo creating a blue monochromatic triangle.

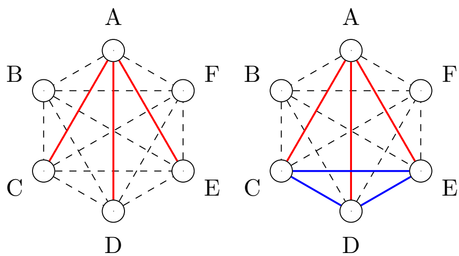

Fig. 2: Proof that a monochromatic triangle must always occur in a 2-coloured K_6

A game on six vertices determines a clear winner, but are we sure that six is the least number of vertices that avoids a tie? The 2-coloured complete graph in Fig. 3 serves as a counter-example, showing that five vertices are insufficient to guarantee a monochromatic triangle. Thus, six is the smallest number of vertices that cannot lead to a tie.

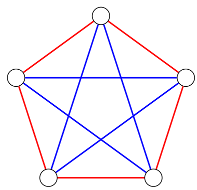

Fig. 3: A 2-coloured K_5 that does not contain any monochromatic triangles

Of course, SIM can be played on graphs with a number of vertices other than six. But how many vertices do we have to start with to guarantee that one of the players actually wins? For that we **coin the term**

Ramsey numbers. Ramsey number $R(r,b)$ is the smallest possible integer $R \in \mathbb{N}$ such that all red and blue coloured graphs on R vertices either contain a red monochromatic subgraph of size r or a blue monochromatic subgraph of size b.

Very little is known about these numbers:

$$R(3,3) = 6, \quad R(4,4) = 18,$$

but all we know about $R(5,5)$ is that it lies between 43 and 48. As for $R(6,6)$, we can only dream of the result. Even the **humble** fact that

$$R(4,5) = 25$$

(thanks to McKay, Radziszkowski, and their 1995 computer experiment) took a year or so to verify. Only very rough general lower and upper bounds on Ramsey numbers were established:

$$2^{\frac{k}{2}} < R(k,k) < \binom{2k-2}{k-1}.$$

For instance, if we want to play SIM where we avoid a complete subgraph on four vertices, the bounds give us 4 as the lower bound and 20 as the upper bound on the minimal number of vertices avoiding a tie. However, $R(4,4)$ has been proven to be equal to 18, therefore all graphs with fewer vertices will allow a tie.

	$b=3$	$b=4$	$b=5$	$b=6$
$r=3$	6	9	14	18
$r=4$	9	18	25	35–41
$r=5$	14	25	43–48	58–87

Tab. 1: A table of Ramsey numbers (or their bounds) for $r, b \leq 6$.

Paul Erdős, a famous Hungarian mathematician [2], developed and popularized Ramsey theory significantly. Ramsey numbers were reformulated as the Party problem. $R(3,3) = 6$ can be rephrased in the following way: Six is the least number of guests at a party to guarantee that at least three of them are **mutual** friends (they have already met before the party) or at least three of them are mutual strangers (they have not met before the party). To illustrate how difficult it is to determine $R(6,6)$, Erdős **fondly** told an anecdote about an alien invasion.

Aliens invade Earth and get ready to destroy the humankind—unless we can correctly answer their question. If the aliens ask us what is the least number of people at a party to guarantee that five of them are either mutual friends or mutual strangers, we should gather all mathematicians and try to test every possibility via brute force. If the aliens ask for six friends or strangers, it would be easier to prepare for a war than to even attempt to calculate it.

Concepts from Ramsey theory that we used on graphs can be applied on other mathematical structures. One of those structures are number sequences, which evolved into a stand-alone theorem called Van der Waerden's theorem (named after the Dutch mathematician Bartel L. van der Waerden). Here is a little puzzle for the reader to introduce van der Waerden's theorem.

Problem 1. Find the smallest number N, for which applies that in a sequence $1, 2, 3, \ldots, N$ for any 2-colouring an arithmetic progression of three terms could be found.

People have continued to work on Ramsey theory for over a century (see [1] for other **comprehensible** problems from this field): from van der Waerden and Ramsey in the 1920s, to Erdős and the Anonymous group in the 1930s, to Greenwood and Gleason and their 1955 proof of $R(4, 4)$, and to decades of work done by Radzikowski and Graham that has continued into the 21st century. This demonstrates a strong commitment to continuously produce new discoveries, which is **bolstered** by real-world applications of Ramsey theory—finding patterns in randomness quickly and efficiently is one of the most popular research topics in computer science nowadays.

It could even provide insight into P vs. NP problems, such as the Traveling Salesman Problem. The Traveling Salesman Problem asks, given a list of cities and the distance between the cities, what is the shortest route that visits each city only once and returns back to the original city. These problems seek to identify how "difficult" a problem is to solve with a computer, and have challenged researchers for several decades. The Traveling Salesman problem itself has many applications in fields other than mathematics, such as studying DNA sequences, building modern microchips, or general planning. The more that is known about Ramsey theory the more tools we have to identify the complexity of such problems.

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[6] Vopravil, V.: *Hry Nim. Rozhledy matematicko-fyzikální*, 95 (2020), 2, pp. 16–31.

Slovníček: Game of SIM and Ramsey theory

arbitrary = libovolný

to bear resemblance to = podobat se

to bolster = podpořit

to coin the term = zavést termín

comprehensible = srozumitelný, pochopitelný

fondly = s oblibou

humble = skromný, pokorný

mutual = společný, vzájemný

pigeonhole principle = přihrádkový princip, princip holubníku

vertex (pl. vertices) = vrchol