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L-FUZZY IDEAL DEGREES IN EFFECT ALGEBRAS

Xiaowei Wei and Fu-Gui Shi

In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees in an effect algebra E, in symbol $\mathfrak{D}_{ei}$. Further, we characterize L-fuzzy ideal degrees by cut sets. Then it is shown that an L-fuzzy subset A in E is an L-fuzzy ideal if and only if $\mathfrak{D}_{ei}(A) = \top$, which can be seen as a generalization of fuzzy ideals. Later, we discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

Keywords: effect algebra, L-fuzzy ideal degree, cut set, (L, L)-fuzzy convexity

Classification: 03B52, 03G27, 52A01

1. INTRODUCTION

In 1994, Foulis and Bennett [5] introduced effect algebras to model unsharp quantum logics. We know that the ideals of effect algebras (pseudo-effect algebras) have attracted a lot of attention [13, 39, 42]. Since Zadeh introduced the concept of fuzzy sets, many branches of mathematics were discussed in fuzzy cases [17, 23, 34, 35]. In particular, Liu and Wang [11] proposed the concept of fuzzy ideals for effect algebras in the unit interval $[0, 1]$. Later, Liu [10] introduced and investigated fuzzy ideals and fuzzy filters in pseudo-effect algebras. In order to better study fuzzy sets, cut sets were introduced, which can be seen as a bridge between fuzzy sets and classic sets. The reader is referred to [9, 37] for more information of cut sets.

Many branches of mathematics have the concept of convexities [31], such as vector spaces, metric spaces, lattices, graphs, matroids and so on. At present, for the convex theory, the research has formed a system, as follows: Rosa [24] first proposed the concept of fuzzy convexities, which are called L-convex structures nowadays [2, 22, 25, 30, 36, 40, 45]. Afterwards, Shi and Xiu [28] gave a new approach to fuzzification of convexity and proposed the concept of M-fuzzifying convex structures [19, 20, 33]. Later, Shi and Xiu [29] further introduced the definition of (L, M)-fuzzy convex structures, which provided a more general framework of fuzzy convex structures [21, 43, 44].

Groups, rings and fields are important parts of algebra. Williams, Latha and Chandrasekeran discussed the fuzzification of bi-Γ-ideals in Γ-semigroups and studied some properties [38]. Öztürk, Jun and Yazarli introduced a kind of fuzzy Γ-ring and discussed

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some properties [18]. Malik and Mordeson [14] introduced the concepts of the fuzzy weak direct sum and the fuzzy complete direct sum of fuzzy subrings of commutative rings. Later, Mehmood, Shi and Hayat [16] introduced a new approach to the fuzzification of rings. Further, Mehmood and Shi [15] discussed the M-hazy vector spaces over M-hazy field. Recently, Shi and Xin [27] gave the concept of L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees, which generalized the notion of degrees to which a fuzzy subset is a fuzzy subgroup to L-fuzzy setting. Later, Li and Shi [9], Wen et al. [37] characterized L-fuzzy convex structures by L-convex fuzzy sublattice degrees and L-convex degrees on vector spaces, respectively. The L-fuzzy convexity is also called the (L, L)-fuzzy convex structure.

In this paper, considering L being a completely distributive lattice, we first introduce the definition of L-fuzzy ideal degrees and will characterize (L, L)-fuzzy convexities by L-fuzzy ideal degrees on an effect algebra. If the L-fuzzy ideal degree of an L-fuzzy subset equals to the maximum element in a lattice, then the L-fuzzy subset is an L-fuzzy ideal, which can be seen as a generalization of the fuzzy ideal on effect algebras. We further characterize L-fuzzy ideal degrees by four types of cut sets. We also discuss the relations between L-fuzzy ideals and their cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convex structure. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

2. PRELIMINARIES

2.1. Effect algebras

Definition 2.1. (Foulis and Bennett [5]) An effect algebra is a partial algebra $\big(E, +, 0, 1\big)$, where $0, 1$ are two different constants and $+$ is a partial binary operation satisfying the following:

(E1) If $x + y$ is defined, then $y + x$ is also defined, and $x + y = y + x$;

(E2) $x + y$ and $\big(x + y\big) + z$ are defined if and only if $y + z$ and $x + \big(y + z\big)$ are defined, and $\big(x + y\big) + z = x + \big(y + z\big)$;

(E3) For any $x \in E$, there exists a unique $y \in E$ such that $x+y$ is defined and $x+y = 1$;

(E4) If $x + 1$ is defined, then $x = 0$.

We often denote the effect algebra $\big(E, +, 0, 1\big)$ briefly by E. For any $x \in E$, we denote the unique y in condition (**E3**) by x'. The operation $+$ of an effect algebra $\big(E, +, 0, 1\big)$ can induce a partial order $\leq$ as follows: $x \leq y$ if and only if there exists $z \in E$ such that $x + z$ is defined and $x + z = y$. If $x + y$ is defined, then it is denoted by $x \perp y$.

In order to better understand effect algebras, we give the following examples, which are the most important effect algebras.

Example 2.2. (1) Let $\mathcal{H}$ be a complete Hilbert space and $B\big(\mathcal{H}\big)$ the set of all bounded linear operators on $\mathcal{H}$, $E\big(\mathcal{H}\big) = \big\{ A \mid A \in B\big(\mathcal{H}\big), 0 \leq A \leq I \big\}$. For $A, B \in E\big(\mathcal{H}\big)$, if we define

$$A \perp B \Longleftrightarrow A + B \leq I,$$

then $\big(E(\mathcal{H}), +, 0, I\big)$ is an effect algebra.

(2) Let $E=[0,1]$. For any $x,y\in[0,1]$, $x\perp y$ if and only if $x+y\leq 1$, then $\big(E,+,0,1\big)$ is an effect algebra.

Definition 2.3. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A mapping $f: E\longrightarrow F$ is called a morphism provided that

(M1) $f(1_E)=1_F$;

(M2) If $x,y\in E$ and $x\perp y$, then $f(x)\perp f(y)$ and $f(x)+f(y)=f(x+y)$.

Lemma 2.4. (Dvurečenskij and Pulmannová [3]) Let $f: E\longrightarrow F$ be a morphism between two effect algebras. Then

(1) f is order-preserving, i. e., $x\leq y$ implies $f(x)\leq f(y)$ for all $x,y\in E$;

(2) $f(x')=f(x)'$ for all $x\in E$.

Definition 2.5. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A morphism $f: E\longrightarrow F$ is called a monomorphism provided that $f(x)\leq f(y)$ implies $x\leq y$ for all $x,y\in E$.

Definition 2.6. (Dvurečenskij and Pulmannová [3]) Let E be an effect algebra. A nonempty subset I of E is said to be an ideal provided that

(I1) If $x\in I$ and $y\in E$ with $y\leq x$, then $y\in I$;

(I2) If $x,y\in I$ and $x\perp y$, then $x+y\in I$.

2.2. Cut sets and (L,L)-fuzzy convexities

A partially ordered set $(L,\leq)$ [1] is said to be a lattice if any two elements λ and μ in L have a smallest upper bound, denoted by $\lambda\vee\mu$, as well as a greatest lower bound, denoted by $\lambda\wedge\mu$. Let $(L,\leq)$ be a partially ordered set and $\Lambda\subseteq L$ be a nonempty subset. If for any $\lambda,\mu\in\Lambda$, there always exists $\theta\in\Lambda$ such that $\lambda\leq\theta$ and $\mu\leq\theta$, then Λ is called upward directed.

Let L be a lattice. If for any $B\subseteq L$, $\bigvee B$ and $\bigwedge B$ exist, then L is called a complete lattice. An element λ in a complete lattice L is said to be a prime element if $\mu\wedge\theta\leq\lambda$ implies $\mu\leq\lambda$ or $\theta\leq\lambda$. An element λ is said to be co-prime if $\lambda\leq\mu\vee\theta$ implies $\lambda\leq\mu$ or $\lambda\leq\theta$ [6]. Every complete lattice is always a bounded lattice such that the unit is the top element and the zero is the bottom element. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in a complete lattice L is defined as follows: for $\lambda,\mu\in L$, $\lambda\prec\mu$ if and only if for any subset $A\subseteq L$, such that $\mu\leq\bigvee A$ implies $\lambda\leq\theta$ for some $\theta\in A$ [4]. The set $\big\{\,\lambda\,|\,\lambda\prec\mu\,\big\}$ is said to be the greatest minimal family of μ, denoted by $\beta(\mu)$ [32]. Moreover, for any $\mu\in L$, we define $\alpha(\mu)=\big\{\,\lambda\in L\,|\,\lambda\prec^{op}\mu\,\big\}$. A complete lattice L is a completely distributive lattice if and only if $\mu=\vee\beta(\mu)=\wedge\alpha(\mu)$ for all $\mu\in L$ [32]. In a completely distributive lattice L, α is an $\wedge$-$\cup$ mapping and β is a union-preserving mapping. There also exists an implication operator $\rightarrow: L \times L \longrightarrow L$ as the right adjoint for the meet operator $\wedge$, which is defined by

$$\lambda \rightarrow \mu = \bigvee \Big\{ \theta \in L \,|\, \lambda \wedge \theta \leq \mu \Big\},$$

for all $\lambda, \mu \in L$.

In this paper, if not otherwise specified, we always assume that L is a completely distributive lattice, the smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively.

Lemma 2.7. (Höhle and Šostak [8]) Let L be a completely distributive lattice and the operation $\rightarrow$ be the implication operator corresponding to $\wedge$. For any $\lambda, \mu, \theta \in L$ and $\{\lambda_i\}_{i \in I} \subseteq L$, then the following statements hold:

(1) $\top \rightarrow \lambda = \lambda$;

(2) $\lambda \leq \theta \rightarrow \mu \Longleftrightarrow \lambda \wedge \theta \leq \mu$;

(3) $\lambda \rightarrow \mu = \top \Longleftrightarrow \lambda \leq \mu$;

(4) $\lambda \rightarrow \big(\bigwedge\limits_{i \in I} \lambda_i \big) = \bigwedge\limits_{i \in I} \big(\lambda \rightarrow \lambda_i \big)$, hence $\lambda \rightarrow \mu \leq \lambda \rightarrow \theta$ whenever $\mu \leq \theta$;

(5) $\big(\bigvee\limits_{i \in I} \lambda_i \big) \rightarrow \mu = \bigwedge\limits_{i \in I} \big(\lambda_i \rightarrow \mu \big)$, hence $\lambda \rightarrow \mu \leq \theta \rightarrow \mu$ whenever $\theta \leq \lambda$;

(6) $(\lambda \rightarrow \mu) \wedge (\mu \rightarrow \theta) \leq \lambda \rightarrow \theta$.

Lemma 2.8. (Li and Shi [9], Wen et al. [37]) Let L be a completely distributive lattice and $\lambda, \mu \in L$. Then the following statements are equivalent:

(1) $\lambda \leq \mu$;

(2) for any $\delta \in J(L)$, $\delta \leq \lambda$ implies $\delta \leq \mu$;

(3) for any $\delta \in P(L)$, $\lambda \not\leq \delta$ implies $\mu \not\leq \delta$;

(4) for any $\delta \in \beta(\top)$, $\delta \in \beta(\lambda)$ implies $\delta \in \beta(\mu)$;

(5) for any $\delta \in \alpha(\perp)$, $\delta \notin \alpha(\lambda)$ implies $\delta \notin \alpha(\mu)$.

In what follows, we will recall some famous examples of t-norms on interval $[0, 1]$.

Example 2.9. (1) The minimum t-norm $x * y = x \wedge y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1, & x \leq y; \\ y, & x > y. \end{cases}$$

(2) The product t-norm $x * y = x \cdot y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1\,, & x \leq y; \\ y/x\,, & x > y. \end{cases}$$

(3) The Łukasiewicz t-norm $x * y = \max\{x + y - 1, 0\}$. The corresponding implication is defined by $x \to y = \min\{1, 1 - x + y\}$.

An L-fuzzy subset [7] of a set X is a mapping from X to L, and the family of all L-fuzzy subsets on X will be denoted by L^X, called the L-power set of X. $\top_X$ and $\bot_X$ denote the largest element and the smallest element in L^X, respectively.

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets. Define $f_L^{\to} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$f_L^{\to}(A)(y) = \bigvee_{f(x)=y} A(x) \quad \text{and} \quad f_L^{\leftarrow}(B)(x) = B\big(f(x)\big),$$

for all $A \in L^X, B \in L^Y$, $x \in X$ and $y \in Y$. Then the L-fuzzy subset $f_L^{\to}(A)$ is called the image of A under f, and $f_L^{\leftarrow}(B)$ the preimage of B.

If L is a completely distributive lattice, then we can define

$$A_{[\lambda]} = \big\{\, x \in X \mid A(x) \geq \lambda \,\big\},\ A^{(\lambda)} = \big\{\, x \in X \mid A(x) \nleq \lambda \,\big\},$$

$$A_{(\lambda)} = \big\{x \in X \mid \lambda \in \beta\big(A(x)\big)\,\big\},\ A^{[\lambda]} = \big\{x \in X \mid \lambda \notin \alpha\big(A(x)\big)\,\big\},$$

for all $A \in L^X$ and $\lambda \in L$.

In [29], Shi and Xiu introduced the notion of (L, M)-fuzzy convexities. When $L = M$, we called it (L, L)-fuzzy convex structure. In what follows, we will recall it.

Definition 2.10. A mapping $\mathfrak{C} : L^X \longrightarrow L$ is said to be an (L, L)-fuzzy convex structure on X if it satisfies the following three conditions:

(C1) $\mathfrak{C}(\top_X) = \mathfrak{C}(\bot_X) = \top$;

(C2) If $\{A_i\}_{i \in I} \subseteq L^X$, then $\mathfrak{C}\big(\bigwedge_{i \in I} A_i\big) \geq \bigwedge_{i \in I} \mathfrak{C}(A_i)$;

(C3) If $\{A_i\}_{i \in I} \subseteq L^X$ is nonempty and upward directed, then $\mathfrak{C}\big(\bigvee_{i \in I} A_i\big) \geq \bigwedge_{i \in I} \mathfrak{C}(A_i)$.

If $\mathfrak{C}$ is an (L, L)-fuzzy convex structure on X, then $(X, \mathfrak{C})$ is said to be an (L, L)-fuzzy convexity space. Every (L, L)-fuzzy convex structure on X is also called an (L, L)-fuzzy convexity on X.

Definition 2.11. Let $(X, \mathfrak{C}_x)$ and $(Y, \mathfrak{C}_y)$ be two (L, L)-fuzzy convexity spaces. Then a mapping $f : X \longrightarrow Y$ is called

(1) an (L, L)-fuzzy convexity-preserving mapping if $\mathfrak{C}_x(f_L^{\leftarrow}(B)) \geq \mathfrak{C}_y(B)$ for all $B \in L^Y$;

(2) an (L, L)-fuzzy convex-to-convex mapping if $\mathfrak{C}_y(f_L^{\to}(A)) \geq \mathfrak{C}_x(A)$ for all $A \in L^X$.

3. L-FUZZY IDEAL DEGREES

In this section, we will introduce the concept of L-fuzzy ideal degrees and investigate it by cut sets, further discuss some properties of L-fuzzy ideal degrees from the perspective of convexity. If not otherwise specified, E denotes an effect algebra and L is a completely distributive lattice with $\to$ (implication operator) corresponding to $\wedge$ (lattice infimum).

3.1. L-fuzzy ideal degrees

Definition 3.1. Let E be an effect algebra and A be an L-fuzzy subset in E. Then the L-fuzzy ideal degree $\mathfrak{D}_{ei}(A)$ of A is defined by

$$\mathfrak{D}_{ei}(A) = \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

It is obvious that the $\mathfrak{D}_{ei}$ is a mapping from L^E to L. In [11], authors proposed the concept of L-fuzzy ideals with $L = [0, 1]$, which are said to be fuzzy ideals.

A mapping $A : E \longrightarrow [0, 1]$ is called a fuzzy ideal of an effect algebra E provided that

(II1) $A(y) \leq A(x)$ if $x \leq y$;

(II2) $A(x) \wedge A(y) \leq A(x + y)$ if $x \perp y$,

for all $x, y \in E$.

In what follows, we will generalize the concept of fuzzy ideals from [0,1] to a lattice.

Definition 3.2. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then the A is called an L-fuzzy ideal.

Remark 3.3. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then

$$\big(A(y) \to A(x)\big) \wedge \big(A(z) \wedge A(w) \to A(z + w)\big) = \top,$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. It follows that

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z + w),$$

for $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we could obtain that L-fuzzy ideals can be seen as generalizations of fuzzy ideals from [0,1] to a lattice L.

In the sequel, we will give some examples of L-fuzzy ideal degrees.

Example 3.4. Let $E = \{0, x, x', 1\}$ with $0 \leq x \leq x' \leq 1$, $x+x' = 1$ be an effect algebra.

(1) Let $A : E \longrightarrow [0, 1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0, 1]$. If A is a constant value mapping on E, then

$$\mathfrak{D}_{ei}(A) = \top.$$

In this case, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, the L-fuzzy subset A satisfies

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z + w).$$

Hence, the L-fuzzy subset A is an L-fuzzy ideal.

(2) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0.1}{0} + \frac{0.3}{x} + \frac{0.7}{x'} + \frac{1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ and $A(z) \wedge A(w) \leq A(z+w)$ if $z \perp w$. Since $x \leq x'$, but we have

$$A(x') = 0.7 \nleq 0.3 = A(x).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(3) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{1}{0} + \frac{0.7}{x} + \frac{0.3}{x'} + \frac{0.1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ with $A(y) \leq A(z)$ if $z \leq y$. Since $x + x' = 1$, but we have

$$A(x) \wedge A(x') = 0.7 \wedge 0.3 \nleq 0.1 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(4) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0}{0} + \frac{0.7}{x} + \frac{1}{x'} + \frac{0.3}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = \perp$. In this case, we have $0 \leq x$ and $x + x' = 1$. But

$$A(x) = 0.7 \nleq 0 = A(0) \text{ and } A(x) \wedge A(x') = 0.7 \wedge 1 \nleq 0.3 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

Lemma 3.5. Let E be an effect algebra and A be an L-fuzzy subset in E.

(1) If $A(x) = \top$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$;

(2) If $A(x) = \perp$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$.

Proof. It is easy and omitted. □

Lemma 3.6. Let E be an effect algebra and A an L-fuzzy subset in E. For any $\lambda \in L$, $\lambda \leq \mathfrak{D}_{ei}(A)$ if and only if for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, then

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Proof. Necessity: Take any $\lambda \in L$. If $\lambda \leq \mathfrak{D}_{ei}(A)$, then

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

Hence, it follows that

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

which means

$$\lambda \leq A(y) \to A(x) \ \text{ and } \ \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we obtain

$$\lambda \wedge A(y) \leq A(x) \ \text{ and } \ \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

Sufficiency: Take any $\lambda \in L$. For any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, by the assumption, we have

$$\lambda \wedge A(y) \leq A(x) \ \text{ and } \ \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it follows that

$$\lambda \leq A(y) \to A(x) \ \text{ and } \ \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

which means

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say, we obtain

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

as desired. □

Theorem 3.7. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y,\ z \perp w\Big\}.$$

Proof. Take any $t \in L$. Then it follows that

$$\begin{aligned} t \leq \mathfrak{D}_{ei}(A) &\Longleftrightarrow t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w), \text{ for any } x, y, z, w \in E \\ &\qquad \text{with } x \leq y \text{ and } z \perp w \text{ (by Lemma 3.6)} \\ &\Longrightarrow t \leq \bigvee \Big\{\lambda \in L \,| \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\, x \leq y, z \perp w\Big\}. \end{aligned}$$

On the other hand, assume that

$$t \leq \bigvee \Big\{\lambda \in L \,| \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\ x \leq y, z \perp w\Big\}.$$

For any $\alpha \prec t$, it follows from the definition of binary relation $\prec$ that $\alpha \leq \lambda$ for some $\lambda \in L$ satisfying

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x \leq y$ and $z \perp w$. Then we know

$$\alpha \leq \lambda \leq A(y) \to A(x) \text{ and } \alpha \leq \lambda \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $\alpha \in L$ with $\alpha \prec t$. By $t = \bigvee \{ \alpha \in L \mid \alpha \prec t \}$, we know

$$t \leq A(y) \to A(x) \text{ and } t \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say,

$$t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Then it follows from Lemma 3.6 that $t \leq \mathfrak{D}_{ei}(A)$. Hence, we obtain

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any}\, x \leq y,\ z \perp w \Big\},$$

as desired. $\square$

In the following, cut sets of L-fuzzy subset A may be empty. If cut sets of A are empty, then we still consider the empty set as a special ideal of E, when we discuss that cut sets of L-fuzzy subset A are ideals. That is to say, the empty set is a special ideal of E.

Theorem 3.8. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \leq \lambda,\ A_{[\mu]} \text{ is an ideal of E} \Big\}.$$

Proof. Assume that $A_{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \leq \lambda$. Take any $x, y \in E$ with $x \leq y$. Let $\theta = \lambda \wedge A(y)$. Then we have $\theta \leq \lambda$ and $\theta \leq A(y)$, which imply $y \in A_{[\theta]}$. By the assumption, we know that $A_{[\theta]}$ is an ideal of E. Then it shows that

$$x \in A_{[\theta]},$$

which means $\theta \leq A(x)$. It follows that

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Hence, it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L|\,\lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w),\text{ for any } x\leq y, z\perp w\Big\}\\ &\geq \bigvee\Big\{\,\lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]}\text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $\lambda \wedge A(y) \leq A(x)$ and $\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$ for all $x, y, z, w \in E$ with $z \perp w$ and $x \leq y$. For any $\mu \leq \lambda$, we need to prove $A_{[\mu]}$ is an ideal of E.

(**I1**) If $y \in A_{[\mu]}$ with $x \leq y$, then $\mu \leq A(y)$. It implies that

$$\mu \leq \lambda \wedge A(y) \leq A(x).$$

Then it follows that $x \in A_{[\mu]}$.

(**I2**) If $z, w \in A_{[\mu]}$ and $z \perp w$, then

$$\mu \leq \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Hence, it follows that

$$z + w \in A_{[\mu]}.$$

That is to say, $A_{[\mu]}$ is an ideal of E. Then it implies that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L\,|\,\lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w),\text{for any } x\leq y,\ z\perp w\Big\}\\ &\leq \bigvee\Big\{\,\lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]}\text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Theorem 3.9. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\,\lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\}.$$

Proof. Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x)$ and $\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$ for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. For $\mu \notin \alpha(\lambda)$, we need to prove that $A^{[\mu]}$ is an ideal of E.

(**I1**) If $x \leq y$ and $y \in A^{[\mu]}$, then $\mu \notin \alpha(A(y))$. It follows from

$$\lambda \wedge A(y) \leq A(x)$$

that

$$\alpha(A(x)) \subseteq \alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y)),$$

which means $\mu \notin \alpha(A(x))$. Hence, we obtain $x \in A^{[\mu]}$.

(**I2**) If $z, w \in A^{[\mu]}$ and $z \perp w$, then

$$\mu \notin \alpha(A(z)) \cup \alpha(A(w)) \cup \alpha(\lambda) = \alpha(\lambda \wedge A(z) \wedge A(w)).$$

It follows from

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$$

that

$$\alpha(A(z+w)) \subseteq \alpha(\lambda \wedge A(z) \wedge A(w)).$$

Hence, we obtain

$$\mu \notin \alpha(A(z+w)).$$

Then it follows that $z+w \in A^{[\mu]}$, which means

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any}\, x \leq y, z \perp w\Big\} \\ &\leq \bigvee \Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A^{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \notin \alpha(\lambda)$. In the sequel, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Suppose that $\mu \notin \alpha(\lambda \wedge A(y))$. It follows from

$$\alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y))$$

that

$$\mu \notin \alpha(\lambda) \text{ and } \mu \notin \alpha(A(y)).$$

It implies that $y \in A^{[\mu]}$. By the assumption, we know that $A^{[\mu]}$ is an ideal of E, which means $x \in A^{[\mu]}$. Then it follows that

$$\mu \notin \alpha(A(x)).$$

Hence, we obtain

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it shows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\ \lambda \in L \mid \lambda \wedge A(y) \leq A(x),\ \ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any}\, x \leq y, z \perp w\Big\} \\ &\geq \bigvee \Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Theorem 3.10. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E} \Big\}.$$

P r o o f. Assume that $\lambda \in L$ with $\lambda \wedge A(y) \leq A(x)$ and $\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$ for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. If $\mu \in P(L)$ and $\lambda \nleq \mu$, then we need to prove that $A^{(\mu)}$ is an ideal of E.

Assume that $y \in A^{(\mu)}$. If $x \notin A^{(\mu)}$, then $A(x) \leq \mu$. It follows from

$$\lambda \wedge A(y) \leq A(x)$$

that

$$\lambda \wedge A(y) \leq \mu.$$

By $\mu \in P(L)$ and $y \in A^{(\mu)}$, i.e., $A(y) \nleq \mu$, we have $\lambda \leq \mu$. This is a contradiction. Hence, it follows that

$$x \in A^{(\mu)}.$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$z, w \in A^{(\mu)} \text{ implies } z + w \in A^{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\} \\ &\leq \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\}.\end{aligned}$$

Conversely, assume that $A^{(\mu)}$ is an ideal of E for $\lambda \in L$ and $\mu \in P(L)$ with $\lambda \nleq \mu$. In what follows, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

For any $x, y \in E$ with $x \leq y$, let $\mu \in P(L)$ and $\lambda \wedge A(y) \nleq \mu$. Then we have

$$\lambda \nleq \mu \text{ and } A(y) \nleq \mu.$$

It follows that $y \in A^{(\mu)}$. By the assumption, we know $A^{(\mu)}$ is an ideal of E, then $x \in A^{(\mu)}$. Further, it implies that

$$A(x) \nleq \mu.$$

Hence, we have

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{ for any } x \leq y, z \perp w\Big\} \\ &\geq \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\},\end{aligned}$$

as desired. □

Theorem 3.11. Let E be an effect algebra and A an L-fuzzy subset in E. If $\beta(\lambda\wedge\mu)=\beta(\lambda)\cap\beta(\mu)$ for all $\lambda,\mu\in L$, then

$$\mathfrak{D}_{ei}(A)=\bigvee\Big\{\lambda\in L\mid\forall\,\mu\in\beta(\lambda),\ A_{(\mu)}\text{ is an ideal of E}\Big\}.$$

P r o o f. Assume that $\lambda\in L$ such that $\lambda\wedge A(y)\le A(x)$ and $\lambda\wedge A(z)\wedge A(w)\le A(z+w)$ for all $x,y,z,w\in E$ with $x\le y$ and $z\perp w$. For any $\mu\in\beta(\lambda)$, we need to prove that $A_{(\mu)}$ is an ideal of E.

(**I1**) If $y\in A_{(\mu)}$ and $x\le y$, then

$$\mu\in\beta(A(y))\cap\beta(\lambda)=\beta(A(y)\wedge\lambda)\subseteq\beta(A(x)).$$

It follows that $x\in A_{(\mu)}$.

(**I2**) If $z,w\in A_{(\mu)}$ and $z\perp w$, then

$$\mu\in\beta(A(z))\cap\beta(A(w))\cap\beta(\lambda)=\beta(A(z)\wedge A(w)\wedge\lambda)\subseteq\beta(A(z+w)).$$

Hence, we obtain

$$z+w\in A_{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A)&=\bigvee\Big\{\lambda\in L\mid\lambda\wedge A(y)\le A(x),\ \lambda\wedge A(z)\wedge A(w)\le A(z+w),\text{ for any }x\le y,z\perp w\Big\}\\&\le\bigvee\Big\{\lambda\in L\mid\forall\,\mu\in\beta(\lambda),\ A_{(\mu)}\text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A_{(\mu)}$ is an ideal of E for $\lambda\in L$ with $\mu\in\beta(\lambda)$. For any $x,y,z,w\in E$ with $x\le y$ and $z\perp w$, we need to prove

$$\lambda\wedge A(y)\le A(x)\text{ and }\lambda\wedge A(z)\wedge A(w)\le A(z+w).$$

(i) Assume that $x,y\in E$ with $x\le y$. Let $\mu\in\beta(\lambda\wedge A(y))$. Then it follows from

$$\beta(\lambda\wedge A(y))=\beta(A(y))\cap\beta(\lambda)$$

that

$$\mu\in\beta(\lambda)\text{ and }\mu\in\beta(A(y)).$$

It implies $y\in A_{(\mu)}$. By the assumption, we know that $A_{(\mu)}$ is an ideal of E. Then it shows $x\in A_{(\mu)}$. Hence, we have

$$\mu\in\beta(A(x)).$$

It follows that $\lambda\wedge A(y)\le A(x)$.

(ii) Assume that $z,w\in E$ and $z\perp w$. Let $\mu\in\beta(\lambda\wedge A(z)\wedge A(w))$. It follows from

$$\beta(\lambda\wedge A(z)\wedge A(w))=\beta(\lambda)\cap\beta(A(z))\cap\beta(A(w))$$

that

$$\mu\in\beta(\lambda),\mu\in\beta(A(z))\text{ and }\mu\in\beta(A(w)).$$

It implies that

$$z, w \in A_{(\mu)}.$$

By the assumption, we know that $A_{(\mu)}$ is an ideal of E and $z \perp w$. Then it shows that

$$z + w \in A_{(\mu)},$$

which means $\mu \in \beta(A(z+w))$. It follows that

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Hence, we have

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\} \\ &\geq \bigvee \Big\{\lambda \in L | \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Zhang [41] discussed the relations between fuzzy ideals and fuzzy filters in dual effect algebras. Liu [10], Liu and Wang [11] studied the connections between a fuzzy filter $\mathcal{F}$ and its cut sets $\mathcal{F}_{[\lambda]}$ for all $\lambda \in [0,1]$ in effect algebras and pseudo-effect algebras, respectively. In the sequel, on one hand, we investigate L-fuzzy ideals by cut sets $A_{[\lambda]}$ for all $\lambda \in L$, which generalizes the unit interval $[0,1]$ to a lattice L. On the other hand, we characterize L-fuzzy ideals by another three kinds of cut sets. In particular, we think that the empty set is a special ideal of an effect algebra E. By [9, 37], we obtain the following corollaries immediately.

Corollary 3.12. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent:

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in L$, $A_{[\lambda]}$ is an ideal;

(3) for every $\lambda \in J(L)$, $A_{[\lambda]}$ is an ideal;

(4) for every $\lambda \in L$, $A^{[\lambda]}$ is an ideal;

(5) for every $\lambda \in P(L)$, $A^{[\lambda]}$ is an ideal;

(6) for every $\lambda \in P(L)$, $A^{(\lambda)}$ is an ideal.

Corollary 3.13. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent when $\beta(\lambda \wedge \mu) = \beta(\lambda) \cap \beta(\mu)$ for all $\lambda, \mu \in L$.

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in J(L)$, $A_{(\lambda)}$ is an ideal;

(3) for every $\lambda \in L$, $A_{(\lambda)}$ is an ideal.

In what follows, we will characterize L-fuzzy ideals by L_β-nested sets and L_α-nested sets. We also can refer to [12, 26] for more information on nested sets. By [26], we can immediately obtain the following Theorems 3.14 and 3.15.

Theorem 3.14. Let E be an effect algebra and $\{A(\lambda) \mid \lambda \in L\}$ be an L_β-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A_{(\lambda)} \subseteq A(\lambda) \subseteq A_{[\lambda]}$ for all $\lambda \in L$;

(2) $A_{(\lambda)} = \bigcup\limits_{\lambda \in \beta(\nu)} A(\nu)$ for all $\lambda \in L$;

(3) $A_{[\nu]} = \bigcap\limits_{\lambda \in \beta(\nu)} A(\lambda)$ for all $\nu \in L$.

Theorem 3.15. Let E be an effect algebra and $\{A(\lambda) \mid \lambda \in L\}$ be an L_α-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A^{(\lambda)} \subseteq A(\lambda) \subseteq A^{[\lambda]}$ for all $\lambda \in L$;

(2) $A^{(\lambda)} = \bigcup\limits_{\nu \in \alpha(\lambda)} A(\nu)$ for all $\lambda \in P(L)$;

(3) $A^{[\lambda]} = \bigcap\limits_{\lambda \in \alpha(\nu)} A(\nu)$ for all $\lambda \in P(L)$.

3.2. (L, L)-fuzzy convexities are induced by L-fuzzy ideal degrees

In this subsection, we will characterize convex properties of L-fuzzy ideal degrees. By morphisms between effect algebras, we obtain one kind of mappings between convexity spaces. Firstly, we will investigate the structural properties of convexity spaces by L-fuzzy ideal degrees.

Theorem 3.16. Let E be an effect algebra and $\mathfrak{D}_{ei}$ an L-fuzzy ideal degree. Then $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity on E.

Proof. By Lemma 3.5, we only need to prove (**C2**) and (**C3**).

(**C2**) Let $\{A_i\}_{i\in I}$ be a family of L-fuzzy subsets in E. Then it follows that

$$\mathfrak{D}_{ei}\Big(\bigwedge_{i\in I} A_i\Big) = \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(\bigwedge_{i\in I} A_i(y) \to \bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(z) \wedge \bigwedge_{i\in I} A_i(w) \to \bigwedge_{i\in I} A_i(z+w)\Big)$$

$$= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)$$

$$= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)$$

$$\geq \bigwedge_{i\in I} \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A_i(y) \to A_i(x)\Big) \wedge \Big(A_i(z) \wedge A_i(w) \to A_i(z+w)\Big)$$

$$= \bigwedge_{i\in I} \mathfrak{D}_{ei}(A_i).$$

(**C3**) Let $\{A_i\}_{i\in I}$ be a upward directed family of L-fuzzy subsets in E. Then we need to prove

$$\mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big).$$

Take any $\lambda \in L$ with $\lambda \leq \bigwedge\limits_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Then it follows that $\lambda \leq \mathfrak{D}_{ei}\big(A_i\big)$ for all $i \in I$. By Lemma 3.6, we know

$$\lambda \wedge A_i(y) \leq A_i(x) \text{ and } \lambda \wedge A_i(z) \wedge A_i(w) \leq A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$, $z \perp w$ and $i \in I$. In what follows, we need to prove

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x) \text{ and } \lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

For any $\eta \prec \lambda \wedge \big(\bigvee\limits_{i\in I} A_i(z)\big) \wedge \big(\bigvee\limits_{i\in I} A_i(w)\big)$, there exist $i \in I$ and $j \in I$ such that

$$\eta \leq A_i(z),\ \eta \leq A_j(w) \text{ and } \eta \leq \lambda.$$

Since $\{A_i\}_{i\in I}$ is upward directed, there exists $k \in I$ such that $A_i \leq A_k$ and $A_j \leq A_k$. Then it follows that

$$A_i(z) \leq A_k(z) \text{ and } A_j(w) \leq A_k(w),$$

which means that

$$\eta \leq \lambda \wedge A_k(z) \wedge A_k(w) \leq A_k(z+w) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Hence, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Similarly, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x),$$

for all $x, y \in E$ with $x \leq y$. Then it follows from Lemma 3.6 that

$$\lambda \leq \mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big),$$

which implies $\mathfrak{D}_{ei}\big(\bigvee\limits_{i\in I} A_i\big) \geq \bigwedge\limits_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Hence, we obtain that $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity, as desired. □

Theorem 3.17. Let E and F be two effect algebras and $f : E \longrightarrow F$ be an effect algebra morphism. Then $f : (E, \mathfrak{D}_{ei}) \longrightarrow (F, \mathfrak{D}_{fi})$ is an (L, L)-fuzzy convexity-preserving mapping.

Proof. Take any L-fuzzy subset A in F. Then

$$\begin{aligned}
&\mathfrak{D}_{ei}\big(f_L^{\leftarrow}(A)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \big(f_L^{\leftarrow}(A)(y) \to f_L^{\leftarrow}(A)(x)\big) \wedge \big(f_L^{\leftarrow}(A)(z) \wedge f_L^{\leftarrow}(A)(w) \to f_L^{\leftarrow}(A)(z+w)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A(f(y)) \to A(f(x))\Big) \wedge \Big(A(f(z)) \wedge A(f(w)) \to A(f(z)+f(w))\Big)\\
&\geq \bigwedge_{\substack{x_1,y_1,z_1,w_1\in F\\ z_1\perp w_1, x_1\leq y_1}} \big(A(y_1) \to A(x_1)\big) \wedge \big(A(z_1) \wedge A(w_1) \to A(z_1+w_1)\big)\\
&= \mathfrak{D}_{fi}(A).
\end{aligned}$$

Hence, we obtain that f is an (L, L)-fuzzy convexity-preserving mapping, as desired. □

In the sequel, we will discuss the relations between L-fuzzy ideals and their inverse images by L-fuzzy ideal degrees.

Theorem 3.18. Let E and F be two effect algebras and $f : E \longrightarrow F$ a monomorphism. If B is an L-fuzzy ideal of F, then $f_L^{\leftarrow}(B)$ is an L-fuzzy ideal of E.

Proof. It can be obtained from Theorem 3.17. □

Remark 3.19. In this paper, we first introduce the concept of L-fuzzy ideal degrees and further investigate it. In order to highlight the idea of fuzzy mathematics, we discuss L-fuzzy ideal degrees, which emphasize the ideal of many-valued logics. The concept can reveal essential characterizations of different mathematical structures. There are some papers for different mathematical structures on degrees of mathematical structures, such as [9, 27, 37, 44] and (Y.-Y. Dong, F.-G. Shi, L-fuzzy Sub-Effect Algebras).

4. CONCLUSIONS

In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees. Then, we characterize L-fuzzy ideal degrees by four types of cut sets. By L-fuzzy ideal degrees, we could give the concept of L-fuzzy ideals, which can be seen as generalizations of fuzzy ideals. We also discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. These morphisms between effect algebras are (L, L)-fuzzy convexity-preserving mappings.

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REFERENCES

[1] B. A. Davey and H. A. Priestley: Introduction to Lattice and Order. Cambridge University Press, Cambridge 2002.

[2] Y. Y. Dong and J. Li: Fuzzy convex structures and prime fuzzy ideal space on residuated lattices. J. Nonlinear Convex Analysis *21* (2020), 12, 2725–2735.

[3] A. Dvurečenskij and S. Pulmannová: New trends in Quantum Structures. Springer-Science Business Media, B.V. 2000.

[4] P. Dwinger: Characterizations of the complete homomorphic images of a completely distributive lattice I. Mathematics *85* (1982), 4, 403–414. DOI:10.1016/1385-7258(82)90034-8

[5] D. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logice. Foundat. Physics *24* (1994), 1331–1352. DOI:10.1007/BF02283036

[6] G. Gierz et al.: Continuous lattices and domains, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge 2003.

[7] J. A. Goguan: *L*-fuzzy sets. J. Math. Analysis Appl. *18* (1967), 145–174. DOI:10.1016/0022-247X(67)90189-8

[8] U. Höhle and A. P. Šostak: Axiomatic foundations of fixed basis fuzzy topology. Math. Fuzzy Sets *3* (1999), 123–272. DOI:10.1007/978-1-4615-5079-2_5

[9] J. Li and F.-G. Shi: *L*-fuzzy convexity induced by *L*-convex fuzzy sublattice degree. Iranian J. Fuzzy Systems *14* (2017), 5, 83–102.

[10] D. L. Liu: Fuzzy filters and fuzzy ideals in pseudo-effect algebras. Computer Engrg. Appl. (2011), 50–52. DOI:10.1111/j.1740-9713.2011.00484.x

[11] D. L. Liu and G. J. Wang: Fuzzy filters in effect algebras. Fuzzy Systems Math. *23* (2009), 6–17.

[12] C. Z. Luo: Fuzzy sets and nested sets. Fuzzy Math. *4* (1983), 113–126.

[13] Z. H. Ma: Note on ideals of effect algebras. Inform. Sci. *179* (2009), 505–507. DOI:10.1016/j.ins.2008.07.018

[14] D. S. Malik and J. N. Mordeson: Fuzzy direct sums of fuzzy rings. Fuzzy Sets Systems *45* (1992), 83–91. DOI:10.1016/0165-0114(92)90094-K

[15] F. Mehmood and F.-G. Shi: M-hazy vector spaces over M-hazy field. Mathematics *9* (2021), 1118. DOI:10.3390/math9101118

[16] F. Mehmood, F.-G. Shi, and K. Hayat: A new approach to the fuzzification of rings. J. Nonlinear Convex Analysis *21* (12) (2020), 2637–2646.

[17] J. Močkoř: α-cuts and models of fuzzy logic. Int. J. General Systems *42* (2013), 67–78. DOI:10.1080/03081079.2012.710438

[18] M. A. Öztürk, Y. B. Jun, and H. Yazarli: A new view of fuzzy gamma rings. Hacettepe J. Math. Statist. *39* (2010), 3, 365–378.

[19] B. Pang: Convergence structures in M-fuzzifying convex spaces. Quaest. Math. *43* (2020), 11, 1541–1561. DOI:10.2989/16073606.2019.1637379

[20] B. Pang: L-fuzzifying convex structures as L-convex structures. J. Nonlinear Convex Anal. *21* (2020), 12, 2831–2841.

[21] B. Pang: Bases and subbases in (L, M)-fuzzy convex spaces. Computat. Appl. Math. (2020), 39–41. DOI:10.1007/s40314-020-1065-4

[22] B. Pang and F.-G. Shi: Strong inclusion orders between L-subsets and its applications in L-convex spaces. Quaest. Math. *41* (2018), 8, 1021–1043. DOI:10.2989/16073606.2018.1436613

[23] D. Pei and T. Fan: On generalized fuzzy rough sets. Int. J. General Systems *38* (2009), 3, 255–271. DOI:10.1080/03081070701553678

[24] M. V. Rosa: On fuzzy topology, fuzzy convexity spaces and fuzzy local convexity. Fuzzy Sets Systems *62* (1994), 97–100. DOI:10.1016/0165-0114(94)90076-0

[25] C. Shen and F.-G. Shi: L-convex systems and the categorical isomorphism to Scott-hull operators. Iranian J. Fuzzy Systems *15* (2018), 2, 23–40.

[26] F.-G. Shi: Theory of L_β-nested set and L_α-nested sets and applications. Fuzzy Systems Math. (1995), 65–72.

[27] F.-G. Shi and X. Xin: L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees. J. Advanced Res. Pure Math. *3* (2011), 4, 92–108. DOI:10.5373/jarpm.762.020211

[28] F.-G. Shi and Z. Y. Xiu: A new approach to the fuzzification of convex structures. J. Appl. Math. *3* (2014), 1–12. DOI:10.1155/2014/249183

[29] F.-G. Shi and Z. Y. Xiu: (L, M)-fuzzy convex structures. J. Nonlinear Sci. Appl. *10* (2017), 3655–3669. DOI:10.22436/jnsa.010.07.25

[30] Y. Shi and H. L. Huang: A characterization of strong Q-concave spaces. J. Nonlinear Convex Anal. *21* (2020), 12, 2771–2781.

[31] M. L. J. van de Vel: Theory of Vonvex Structures. North Holland, N. Y. 1993.

[32] G. J. Wang: Theory of topological molecular lattices. Fuzzy Sets Systems *47* (1992), 351–376. DOI:10.1016/0165-0114(92)90301-J

[33] K. Wang and F.-G. Shi: M-fuzzifying topological convex spaces. Iranian J. Fuzzy Systems *15*, (2018), 6, 159–174.

[34] X. W. Wei, B. Pang, and J. S. Mi: Axiomatic characterizations of L-valued rough sets using a single axiom. Inform, Sci. *580* (2021), 283–310. DOI:10.1016/j.ins.2021.08.078

[35] X. W. Wei, B. Pang, and J. S. Mi: Axiomatic characterizations of L-fuzzy rough sets by L-fuzzy unions and L-fuzzy intersections. Int. J. General Systems *51* (2022), 3, 277–312. DOI:10.1080/03081079.2021.2024178

[36] X. W. Wei and B. Wang: Fuzzy (restricted) hull operators and fuzzy convex structures on L-sets. J. Nonlinear Convex Anal. *21* (2020), 12, 2805–2815.

[37] Y. F. Wen, Y. Zhong, and F.-G. Shi: L-fuzzy convexity induced by L-convex degree on vector spaces. J. Intell. Fuzzy Systems *33* (2017), 4031–4041. DOI:10.3233/JIFS-17912

[38] D. R. P. Williams, K. B. Latha, and E. Chandrasekeran: Fuzzy bi-Γ-ideals in Γ-semigroups. Hacettepe J. Math. Statist. *38* (2009), 1, 1–15.

[39] J. Wu: Ideals, filters and supports in pseudo-effect algebras. Int. J. Theoret. Physics *43* (2004), 349–358. DOI:10.1023/B:IJTP.0000028868.66790.0c

[40] H. Yang and E. Q. Li: A new approach to interval operators in L-convex spaces. J. Nonlinear Convex Anal. *21* (2020), 12, 2705–2714.

[41] Q. W. Zhang: Ideals and filters in dual effect algebras. Hennan Sci. *34* (2016), 8, 1211–1214. DOI:10.1038/nbt1216-1214

[42] Q. W. Zhang: Note on ideals in effect algebras. Hennan Sci. *35* (2017), 10, 1567–1569. DOI:10.1016/j.ajem.2017.04.002

[43] F. F. Zhao and H. L. Huang: The relationships among L-ordered hull operators, restricted L-hull operators and strong L-fuzzy convex structures. J. Nonlinear Convex Anal. *21* (2020), 12, 2817–2829.

[44] Y. Zhong and F.-G. Shi: Characterizations of (L,M)-fuzzy topological degrees. Iranian J. Fuzzy Syst. *15* (2018), 4, 129–149.

[45] X. W. Zhou and F.-G. Shi: Some new results on six types mappings between L-convex spaces. Filomat. *34* (2020), 4767–4781. DOI:10.2298/FIL2014767Z

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