Sujoy Majumder; Lata Mahato

On the meromorphic solutions of a certain type of nonlinear difference-differential equation

Mathematica Bohemica, Vol. 148 (2023), No. 1, 73–94

Persistent URL: http://dml.cz/dmlcz/151528

148 (2023) MATHEMATICA BOHEMICA No. 1, 73–94

ON THE MEROMORPHIC SOLUTIONS OF A CERTAIN TYPE OF NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATION

Sujoy Majumder, Raiganj, Lata Mahato, Kolkata

Received December 6, 2020. Published online March 30, 2022.
Communicated by Dagmar Medková

Abstract. The main objective of this paper is to give the specific forms of the meromorphic solutions of the nonlinear difference-differential equation

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z, f)$ is a difference-differential polynomial in $f(z)$ of degree $d \leqslant n-1$ with small functions of $f(z)$ as its coefficients, p_1, p_2 are nonzero rational functions and α_1, α_2 are non-constant polynomials. More precisely, we find out the conditions for ensuring the existence of meromorphic solutions of the above equation.

Keywords: nonlinear differential equation; differential polynomial; Nevanlinna's value distribution theory

MSC 2020: 34M05, 30D35, 33E30, 30D30

1. Introduction, definitions and results

In the paper, a meromorphic function means a function meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory, e.g., $N(r, f)$, $m(r, f)$, $T(r, f)$, $N(r, a; f)$, $\overline{N}(r, a; f)$, $m(r, a; f)$, etc. (see [2]). We denote by $S(r, f)$ a quantity, not necessarily the same at each of its occurrence, that satisfies the condition $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ except possibly a set of finite linear measure.

A meromorphic function $a = a(z)$ is called a small function of a meromorphic function f if $T(r, a) = S(r, f)$. Let us denote by $S(f)$ the class of all small functions of f. Clearly $\mathbb{C} \subset S(f)$ and if f is a transcendental function, then every rational function is a member of $S(f)$.

DOI: 10.21136/MB.2022.0186-20

The order and hyper-order of a meromorphic function $f(z)$ are denoted and defined by

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \varrho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

respectively. It is clear that if $\varrho(f) < \infty$, then $\varrho_2(f) = 0$.

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}\cup\{\infty\}$. We use the notations $N_{k)}(r,a;f)$ and $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity not greater than k and the counting function of a-points of f with multiplicity greater than k, respectively. Similarly, $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions, respectively.

By a differential polynomial $P_d(z,f)$ in $f(z)$ of degree d, we mean a polynomial in $f(z)$ and its derivatives of a total degree at most d with small functions of $f(z)$ as coefficients. When the coefficients are polynomials, we call $P_d(z,f)$ an algebraic differential polynomial.

By a difference-differential polynomial $P_d(z,f)$ in $f(z)$ of degree d, we mean a polynomial in $f(z)$, its shifts and their derivatives of a total degree at most d with small functions of $f(z)$ as coefficients.

It is always an interesting and quite difficult problem to prove the existence of the entire or meromorphic solutions $f(z)$ of a given differential equation and to find out the solutions if they exist. A special type of nonlinear differential equation

$$f^n(z) + P_d(z,f) = h(z),$$

where $h(z)$ is a given entire or meromorphic function and $P_d(z,f)$ is a differential polynomial in $f(z)$ of degree d, has become a matter of increasing interest among the researchers.

It is easy to show that the function $f_1(z) = \sin z$ is a solution of the nonlinear differential equation $4f^3(z) + 3f''(z) = -\sin 3z$. In [3], it was proved that $f_2(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ is also a solution of this equation. In 2004, Yang and Li (see [10]) proved that this equation admits exactly three entire solutions, namely $f_1(z)$, $f_2(z)$ and $f_3(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Since the function $-\sin 3z$ is a linear combination of $\mathrm{e}^{\mathrm{i}3z}$ and $\mathrm{e}^{-\mathrm{i}3z}$, so it is interesting to find all entire solutions of the general equation

$$f^n(z) + P_d(z,f) = p_1\mathrm{e}^{\lambda z} + p_2\mathrm{e}^{-\lambda z}, \tag{1.1}$$

where p_1, p_2 and λ are nonzero constants and $P_d(z,f)$ denotes a differential polynomial in $f(z)$ of degree $d \leqslant n-1$.

In 2004, Yang and Li (see [10]) answered the above question partially and obtained the following result.

Theorem A ([10]). *Let $n \in \mathbb{N} \setminus \{1,2\}$, $P_d(z,f)$ be a differential polynomial in f of degree $d \leqslant n-3$, $b \in S(f)$ and λ, p_1, p_2 be three nonzero constants. Then the differential equation*

$$f^n(z) + P_d(z,f) = b(z)(p_1 \mathrm{e}^{\lambda z} + p_2 \mathrm{e}^{-\lambda z})$$

has no transcendental entire solution $f(z)$.

In 2006, Li and Yang (see [6]) derived similar conclusion when the term on the right-hand side of equation (1.1) was replaced by $p_1(z)\mathrm{e}^{\alpha_1 z} + p_2(z)\mathrm{e}^{\alpha_2 z}$, where $p_1(z)$, $p_2(z)$ are nonzero polynomials, α_1, α_2 are two constants with $\alpha_1/\alpha_2 \notin \mathbb{Q}$, and presented their result as follows.

Theorem B ([6]). *Let $n \in \mathbb{N}\setminus\{1,2,3\}$ and $P_d(z,f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leqslant n-3$. Let $p_1(z)$, $p_2(z)$ be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \notin \mathbb{Q}$. Then the differential equation*

$$f^n(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1 z} + p_2(z)\mathrm{e}^{\alpha_2 z}$$

has no transcendental entire solutions.

In 2011, Li derived the possible forms of solutions of equation (1.1) when $d \leqslant n-2$, and obtained the following result (see [5]).

Theorem C ([5]). *Let $n \in \mathbb{N} \setminus \{1\}$, $P_d(z,f)$ be a differential polynomial in $f(z)$ of degree $d \leqslant n-2$ and p_1, p_2, α_1, α_2 be nonzero constants and $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the equation*

$$f^n(z) + P_d(z,f) = p_1 \mathrm{e}^{\alpha_1 z} + p_2 \mathrm{e}^{\alpha_2 z}$$

satisfying $N(r,\infty;f) = S(r,f)$, then one of the following holds:

(i) $f(z) = c_0(z) + c_1 \mathrm{e}^{\alpha_1/nz}$,
(ii) $f(z) = c_0(z) + c_2 \mathrm{e}^{\alpha_2/nz}$,
(iii) $f(z) = c_1 \mathrm{e}^{\alpha_1/nz} + c_2 \mathrm{e}^{\alpha_2/nz}$ *and* $\alpha_1 + \alpha_2 = 0$,

where $c_0 \in S(f)$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_i^n = p_i$, $i = 1,2$.

In 2013, Liao, Yang and Zhang (see [7]) extended the above results by considering that $h(z)$ is a meromorphic function of integer order and improved the results of Theorems B and C. Actually, they obtained the following result.

Theorem D ([7]). *Let $n \in \mathbb{N} \setminus \{1, 2\}$ and $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients. Suppose that p_1, p_2 are nonzero rational functions and α_1, α_2 are polynomials. If $d \leqslant n-2$, the differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}$$

admits a meromorphic function $f(z)$ with finitely many poles. Then α_1'/α_2' is a rational number. Furthermore, only one of the following four cases holds:

(1) *$f(z) = q(z)\mathrm{e}^{p(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = 1$, where $q(z)$ is a nonzero rational function and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z) = \alpha_2'(z)$;*

(2) *$f(z) = q(z)\mathrm{e}^{p(z)}$ and either $\alpha_1'(z)/\alpha_2'(z) = k/n$ or $\alpha_1'(z)/\alpha_2'(z) = n/k$, where $q(z)$ is a nonzero rational function, $k \in \mathbb{N}$ with $1 \leqslant k \leqslant d$ and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z)$ or $np'(z) = \alpha_2'(z)$;*

(3) *$f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_2'(z)/p_2(z) + \alpha_2'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = (n-1)/n$ or $f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_1'(z)/p_1(z) + \alpha_1'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = n/(n-1)$, where $\psi(z)$ is a rational function;*

(4) *$f(z) = \gamma_1(z)\mathrm{e}^{\beta_1(z)} + \gamma_2(z)\mathrm{e}^{-\beta_1(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = -1$, where $\gamma_1(z)$, $\gamma_2(z)$ are nonzero rational functions and $\beta_1(z)$ is a polynomial with $n\beta_1'(z) = \alpha_1'(z)$ or $n\beta_1'(z) = \alpha_2'(z)$.*

Now it is interesting to find out all the meromorphic solutions of the following nonlinear differential-difference equation:

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}, \tag{1.2}$$

where $P_d(z, f)$ is a differential-difference polynomial in $f(z)$ of degree $d \leqslant n-1$ with small functions of $f(z)$ as its coefficients, $p_1(z)$, $p_2(z)$ are nonzero rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials.

In 2018, Lü, Wu, Wang and Yang (see [8]) derived the possible forms of the solutions of equation (1.2) when $n = 3$, $d = 1$, and obtained the following result.

Theorem E ([8]). *Let $P_d(z, f)$ denote a difference-differential polynomial in $f(z)$ of degree one with small functions as its coefficients such that $P_d(z, 0) \equiv 0$ and let p_1, p_2, α_1, α_2 be nonzero constants such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is an entire solution with $\varrho_2(f) < 1$ to equation*

$$f^3(z) + P_d(z, f) = p_1\mathrm{e}^{\alpha_1 z} + p_2\mathrm{e}^{\alpha_2 z},$$

then one of the following relations holds:

(1) $f(z) = c_1 \exp(\frac{1}{3}\alpha_1 z) + c_2 \exp(\frac{1}{3}\alpha_2 z)$, *where* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *satisfying* $c_1^3 = p_1$, $c_2^3 = p_2$ *and* $\alpha_1 + \alpha_2 = 0$,

(2) $f^3(z) = (p_1 - c_1)\exp(\alpha_1 z)$ *and* $P_d(z, f) = c_1 \exp(\alpha_1 z) + p_2 \exp(\alpha_2 z)$, *where* c_1 *is a constant,*

(3) $f^3(z) = (p_2 - c_2)\exp(\alpha_2 z)$ *and* $P_d(z, f) = p_1 \exp(\alpha_1 z) + c_2 \exp(\alpha_2 z)$, *where* c_2 *is a constant.*

For further study, it is quite natural to ask the following questions.

Question 1. What happens if $f^3(z)$ is replaced by $f^n(z)$, where $n \in \mathbb{N}$, in Theorem E?

Question 2. What will happen if we delete the condition $P_d(z, 0) \equiv 0$ in Theorem E?

Question 3. How to find the solutions of equation (1.2) under the condition $n \geqslant d + 2$?

The main objective of this paper is to find out the possible answers to the above questions. The following theorem is the main result of the paper.

Theorem 1.1. *Let* $P_d(z, f)$ *be a difference-differential polynomial in* $f(z)$ *of degree* $d \in \mathbb{N} \cup \{0\}$ *with small functions of* $f(z)$ *as its coefficients and* $n \in \mathbb{N}$ *such that* $n \geqslant d+2$. *Suppose that* $p_1(z)$, $p_2(z)$ *are nonzero rational functions and* $\alpha_1(z)$, $\alpha_2(z)$ *are non-constant polynomials. If* $f(z)$ *is a meromorphic solution to the difference-differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)} \tag{1.3}$$

satisfying $\varrho_2(f) < 1$ *and* $N(r, \infty; f) = S(r, f)$, *then one of the following cases holds:*

(1) $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$ *and* $\alpha_1'(z) \equiv \alpha_2'(z)$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = c_0 p_2(z)$, *where* $c_0 \in \mathbb{C} \setminus \{0\}$;

(2) $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$ *and* $\alpha_1' \equiv \alpha_2'(z)$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_1(z) + c_1 p_2(z)$, *where* $c_1 \in \mathbb{C}$;

(3) $T(r, \mathrm{e}^{(k\alpha_1 - n\alpha_2)/(n+1)}) = S(r, f)$, *where* $k \in \{0, 1, 2, \ldots, d\}$. *In this case,* $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_1(z)$;

(4) $T(r, \mathrm{e}^{(k\alpha_2 - n\alpha_1)/(n+1)}) = S(r, f)$, *where* $k \in \{0, 1, 2, \ldots, d\}$. *In this case,* $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_2(z)$;

(5) $T(r, \mathrm{e}^{(n-1)\alpha_1 - n\alpha_2}) = S(r, f)$. *In this case,* $f(z) = u_1(z)\mathrm{e}^{\alpha_1(z)/n} - v_1(z)$, *where* $u_1(z)$ *and* $v_1(z)$ *are nonzero small functions of* $f(z)$ *such that* $u_1^n(z) = p_1(z)$;

(6) $T(r, \mathrm{e}^{(n-1)\alpha_2 - n\alpha_1}) = S(r, f)$. *In this case,* $f(z) = u_2(z)\mathrm{e}^{\alpha_2(z)/n} - v_2(z)$, *where* $u_2(z)$ *and* $v_2(z)$ *are nonzero small functions of* $f(z)$ *such that* $u_2^n(z) = p_2(z)$;

(7) $T(r, \mathrm{e}^{\alpha_1-\alpha_2}) = S(r,f)$. *In this case,* $f(z) = q(z)\mathrm{e}^{\alpha_1/n}$ *and* $P_d(z,f) \equiv 0$, *where* $q(z)$ *and* $\varphi(z)$ *are nonzero small functions of* $f(z)$ *such that* $q^n(z) = p_1(z) + \varphi(z)p_2(z)$*;*

(8) $T(r, \mathrm{e}^{\alpha_1+\alpha_2}) = S(r,f)$. *In this case,* $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$, *where* $\delta_1(z)$, $\delta_2(z)$ *are nonzero small functions of* $f(z)$ *and* $\gamma(z)$ *is a non-constant polynomial such that either* $\mathrm{e}^{n\gamma(z)+\alpha_1(z)}$ *is a small function of* $f(z)$ *or* $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ *is a small function of* $f(z)$.

From Theorem 1.1 we have the following corollary.

Corollary 1.1. *Equation* (1.2) *does not have any meromorphic solution* $f(z)$ *satisfying* $N(r,\infty;f) = S(r,f)$, $\varrho(f) = \infty$ *and* $\varrho_2(f) < 1$.

Remark 1.1. It is easy to see that conclusions (5) and (6) in Theorem 1.1 can not be removed by the following examples.

Example 1.1. Let us consider the difference-differential equation

$$f^3(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) = -\frac{1}{3}f'(z) - \frac{2}{27}$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 3z$ and $\alpha_2(z) = 2z$. Here $n = 3$ and $d = 1$. One can easily verify that $f(z) = u_1(z)\mathrm{e}^{\alpha_1(z)/3} - v_1(z)$, where $u_1(z) = 1$, $v_1(z) = \frac{1}{3}$ is a solution of the given difference-differential equation.

Example 1.2. Let us consider the difference-differential equation

$$f^4(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) = f^2(z+c) - 3(f'(z))^2 - 4f''(z)f(z) - 2f(z+c)$, $p_1(z) = 1$, $p_2(z) = 4$, $\alpha_1(z) = 4z$, $\alpha_2(z) = 3z$ and $c \in \mathbb{C}\setminus\{0\}$ such that $\mathrm{e}^c = 1$. Here $n = 4$ and $d = 2$. One can easily verify that $f(z) = u_2(z)\mathrm{e}^{\alpha_2(z)/4} - v_2(z)$, where $u_2(z) = 1$ and $v_2(z) = -1$ is a solution of the given difference-differential equation.

Remark 1.2. It is easy to see that conclusion (8) in Theorem 1.1 cannot be removed by the following examples.

Example 1.3. Let us consider the difference-differential equation

$$f^2(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) \equiv -2$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 2z$ and $\alpha_2(z) = -2z$. Here $n = 2$ and $d = 0$. One can easily verify that $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$ is a solution of the given difference-differential equation, where $\delta_1(z) = \delta_2(z) = 1$ and $\gamma(z) = z$. Also we see that $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$.

E x a m p l e 1.4. Let us consider the difference-differential equation

$$f^3(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)},$$

where $P_d(z, f) = zf''(z) - f'(z) - (4z^3 + 3)f(z)$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 3z^2$ and $\alpha_2(z) = -3z^3$. Here $n = 3$ and $d = 1$. One can easily verify that $f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}$ is a solution of the given difference-differential equation, where $\delta_1(z) = \delta_2(z) = 1$ and $\gamma(z) = z^2$. Also we see that $e^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$.

2. Lemmas

The following lemmas are needful in the proof of our main result.

Lemma 2.1 ([4]). *Let $f(z)$ be a transcendental meromorphic function and $f^n(z)P(z, f) = Q(z, f)$, where $P(z, f)$ and $Q(z, f)$ are polynomials in $f(z)$ and its derivatives with meromorphic coefficients, say $\{a_\lambda(z)\colon \lambda \in I\}$ such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in $f(z)$ and its derivatives is less than or equal to n, then $m(r, P(z, f)) = S(r, f)$.*

Lemma 2.2 ([2]). *Let $f(z)$ be a non-constant meromorphic function and let $a_i \in S(f)$, $i = 1, 2$. Then $T(r, f) \leqslant \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f)$.*

Lemma 2.3 ([9]). *Let $f(z)$ be a non-constant meromorphic function and let $a_n\,(\not\equiv 0), a_{n-1}, \ldots, a_0 \in S(f)$. Then $T\Bigl(r, \sum\limits_{i=0}^{n} a_i f^i\Bigr) = nT(r, f) + S(r, f)$.*

Lemma 2.4 ([11]). *Let f be a non-constant meromorphic function and $k \in \mathbb{N}$. Then*

$$m\Bigl(r, \frac{f^{(k)}}{f}\Bigr) = S(r, f)$$

and if f is of finite order of growth, then

$$m\Bigl(r, \frac{f^{(k)}}{f}\Bigr) = O(\log r).$$

Lemma 2.5 ([1]). *Let $c \in \mathbb{C}\setminus\{0\}$, $\varepsilon > 0$ and $f(z)$ be a non-constant meromorphic function such that $\varrho_2(f) < 1$. Then*

$$m\Bigl(r, \frac{f(z+c)}{f(z)}\Bigr) = o\Bigl(\frac{T(r, f)}{r^{1-\varrho_2(f)-\varepsilon}}\Bigr)$$

outside of an exceptional set of finite logarithmic measure.

Lemma 2.6. *Let $n \in \mathbb{N}$ and $P_d(z,f)$ be a difference-differential polynomial in $f(z)$ of degree $d \leqslant n-1$ with small functions of $f(z)$ as its coefficients. Suppose that $p_1(z)$, $p_2(z)$ are nonzero rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials. If $f(z)$ is a meromorphic solution to the nonlinear difference-differential equation*

$$f^n(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)} \tag{2.1}$$

satisfying $\varrho_2(f) < 1$ and $N(r,\infty;f) = S(r,f)$, then $f(z)$ is a transcendental meromorphic function of finite order.

Proof. Let $f(z)$ be a rational function satisfying the differential-difference equation (2.1). Then clearly $p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}$ is a rational function, say $R_1(z)$, and so $-p_1(z)\mathrm{e}^{\alpha_1(z)} = p_2(z)\mathrm{e}^{\alpha_2(z)} - R_1(z)$. This shows that $p_2(z)\mathrm{e}^{\alpha_2(z)} - R_1(z)$ has finitely many zeros. But from Lemma 2.2, one can easily conclude that $p_2(z)\mathrm{e}^{\alpha_2(z)} - R_1(z)$ has infinitely many zeros. Therefore we arrive at a contradiction. Consequently, any non-constant meromorphic solution of the difference-differential equation (2.1) must be transcendental.

A difference-differential polynomial $P_d(z,f)$ in $f(z)$ can be expressed as

$$P_d(z,f) = \sum_{\mu} b_\mu(z) G_\mu(z,f),$$

where $b_\mu \in S(f)$ and

$$\begin{aligned} G_\mu(z,f) = {} & (f(z))^{p_0^\mu}(f'(z))^{p_1^\mu}\dots(f^{(k)}(z))^{p_k^\mu}(f(z+c_0))^{q_0^\mu}(f(z+c_1))^{q_1^\mu}\dots(f(z+c_k))^{q_k^\mu} \\ & \times (f(z+c_\mu))^{l_0^\mu}(f'(z+c_\mu))^{l_1^\mu}\dots(f^{(k)}(z+c_\mu))^{l_k^\mu}, \end{aligned}$$

$p_0^\mu, p_1^\mu, \dots, p_k^\mu, q_0^\mu, q_1^\mu, \dots, q_k^\mu, l_0^\mu, l_1^\mu, \dots, l_k^\mu \in \mathbb{N} \cup \{0\}$ such that $\sum\limits_{j=0}^{k} p_j^\mu + \sum\limits_{j=0}^{k} q_j^\mu + \sum\limits_{j=0}^{k} l_j^\mu = \mu \leqslant d$. Therefore we have

$$P_d(z,f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z,f)}{f^\mu(z)} f^\mu(z). \tag{2.2}$$

Now by Lemmas 2.4 and 2.5, we derive

$$\begin{aligned} & m\Big(r, b_\mu(z)\frac{G_\mu(z,f)}{f^\mu(z)}\Big) \\ & \quad = m\Big(r, b_\mu(z)\Big(\frac{f'(z)}{f(z)}\Big)^{p_1^\mu}\dots\Big(\frac{f^{(k)}(z)}{f(z)}\Big)^{p_k^\mu}\dots\Big(\frac{f(z+c_\mu)}{f(z)}\Big)^{l_0^\mu}\dots\Big(\frac{f^{(k)}(z+c_\mu)}{f(z)}\Big)^{l_k^\mu}\Big) \\ & \quad = S(r,f). \end{aligned}$$

Therefore (2.2) takes the form

$$P_d(z,f) = c_d(z)f^d(z) + c_{d-1}(z)f^{d-1}(z) + \ldots + c_0(z),$$

where $c_d(z) \not\equiv 0$ and $m(r, c_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, d$. Now by using the mathematical induction, it follows that $m(r, P_d(z, f)) \leqslant dm(r, f) + S(r, f)$. Since $N(r, \infty; f) = S(r, f)$, it follows that

$$T(r, P_d(z, f)) \leqslant dT(r, f) + S(r, f). \tag{2.3}$$

Now from (2.1) and (2.3) we have

$$T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) = T(r, f^n(z) + P_d(z, f)) = nT(r, f) + S(r, f) \tag{2.4}$$

and

$$\begin{aligned} T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) &= T(r, f^n(z) + P_d(z, f)) \\ &\geqslant T(r, f^n(z)) - T(r, P_d(z, f)) \\ &\geqslant (n-d)T(r, f) + S(r, f). \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$(n-d)T(r, f) + S(r, f) \leqslant T(p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}) \leqslant nT(r, f) + S(r, f),$$

which implies that $\varrho(f) < \infty$. This completes the proof. □

Lemma 2.7 ([5]). *Suppose that $f(z)$ is a transcendental meromorphic function and $q_1, q_2, q_3, a \in S(f)$ such that $q_3 a \not\equiv 0$. If*

$$q_1 f^2 + q_2 f f' + q_3 (f')^2 = a,$$

then

$$q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q_3' \equiv 0.$$

Lemma 2.8 ([2]). *Let $f(z)$ be a non-constant meromorphic function and $n \in \mathbb{N}$. Suppose that*

$$g(z) = f^n(z) + P_{n-1}(z, f),$$

where $P_{n-1}(z, f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients and

$$N(r, f) + N\Big(r, \frac{1}{g}\Big) = S(r, f).$$

Then $g(z) = (f(z) + \gamma(z))^n$, where $\gamma \in S(f)$.

Lemma 2.9. *Let $f(z)$ be a non-constant meromorphic function and $n \in \mathbb{N}$. Suppose that*

$$g(z) = f^{n+1}(z) + P_{n-1}(z, f), \tag{2.6}$$

where $P_{n-1}(z, f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients and

$$N(r, f) + N\Big(r, \frac{1}{g}\Big) = S(r, f).$$

Then $g(z) = f^{n+1}(z)$ and $P_{n-1}(z, f) \equiv 0$.

Proof. Firstly, from Lemma 2.8 we have $g(z) = (f(z) + \gamma(z))^{n+1}$, where $\gamma \in S(f)$. If possible, suppose that $\gamma \not\equiv 0$. Now from (2.6) we have

$$(f(z) + \gamma(z))^{n+1} = f^{n+1}(z) + P_{n-1}(z, f)$$

and so

$$(n+1)\gamma(z) f^n(z) + Q_{n-1}(z, f) = P_{n-1}(z, f),$$

where $Q_{n-1}(z, f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients. Therefore we have

$$f^{n-1}(z)(n+1)\gamma(z) f(z) = P_{n-1}(z, f) - Q_{n-1}(z, f).$$

Now by Lemma 2.1, we conclude that $m(r, f) = S(r, f)$. Since $N(r, \infty; f) = S(r, f)$, it follows that $T(r, f) = S(r, f)$, which is impossible. Hence $\gamma \equiv 0$. Consequently, $g(z) = f^{n+1}(z)$ and $P_{n-1}(z, f) \equiv 0$. This completes the proof. □

3. Proof of the theorem

Proof of Theorem 1.1. By the given condition, we have

$$f^n + P_d = p_1 e^{\alpha_1} + p_2 e^{\alpha_2}, \tag{3.1}$$

where $P_d = P_d(z, f)$. Let f be a meromorphic solution of equation (3.1). Then by Lemma 2.6, we can conclude that f is a transcendental meromorphic function of finite order. Now differentiating both sides of (3.1) once, we get

$$n f^{n-1} f' + P_d' = (p_1 \alpha_1' + p_1') e^{\alpha_1} + (p_2 \alpha_2' + p_2') e^{\alpha_2}. \tag{3.2}$$

Now by eliminating e^{α_2} from (3.1) and (3.2), we have

$$f^{n-1}(n p_2 f' - (p_2 \alpha_2' + p_2') f) + p_2 P_d' - (p_2 \alpha_2' + p_2') P_d = A_1 e^{\alpha_1}, \tag{3.3}$$

where $A_1 = p_2(p_1\alpha_1' + p_1') - p_1(p_2\alpha_2' + p_2')$. Again by eliminating e^{α_1} from (3.1) and (3.2), we have

$$f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f) + p_1P_d' - (p_1\alpha_1' + p_1')P_d = -A_1\mathrm{e}^{\alpha_2}. \tag{3.4}$$

Suppose that $A_1 \equiv 0$. Then we have $\alpha_1' - \alpha_2' = p_2'/p_2 - p_1'/p_1$ and so $\alpha_1' \equiv \alpha_2'$. Now from (3.3) we have

$$f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f) = (p_2\alpha_2' + p_2')P_d - p_2P_d'. \tag{3.5}$$

Suppose that $np_2f' - (p_2\alpha_2' + p_2')f \not\equiv 0$. Then by Lemma 2.1, we have

$$\begin{cases} m(r, np_2f' - (p_2\alpha_2' + p_2')f) = S(r, f), \\ m(r, np_2ff' - (p_2\alpha_2' + p_2')f^2) = S(r, f). \end{cases} \tag{3.6}$$

Since $N(r, \infty; f) = S(r, f)$, from (3.6) we conclude that

$$T(r, f) \leqslant T(r, np_2f'f - (p_2\alpha_2' + p_2')f^2) + T(r, np_2f' - (p_2\alpha_2' + p_2')f) + O(1) = S(r, f),$$

which is impossible. Therefore $np_2f' - (p_2\alpha_2' + p_2')f \equiv 0$ and so by integration, we get $f^n = c_0p_2\mathrm{e}^{\alpha_2}$, where $c_0 \in \mathbb{C} \setminus \{0\}$. Therefore we let $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = c_0p_2(z)$.

Next we suppose that $A_1(z) \not\equiv 0$. Now differentiating (3.3) once, we get

$$\begin{aligned} f^{n-2}(-(p_2\alpha_2' + p_2')'f^2 - np_2\alpha_2'ff' + (n-1)np_2(f')^2 + np_2ff'') + Q_d' \\ = (A_1' + A_1\alpha_1')\mathrm{e}^{\alpha_1}, \end{aligned} \tag{3.7}$$

where

$$Q_d = p_2P_d' - (p_2\alpha_2' + p_2')P_d. \tag{3.8}$$

Eliminating e^{α_1} from (3.3) and (3.7), we get

$$f^{n-2}(h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff'') = R_d, \tag{3.9}$$

where

$$\begin{cases} R_d = (A_1' + A_1\alpha_1')Q_d - A_1Q_d', \\ h_{21} = (p_2\alpha_2' + p_2')(A_1' + A_1\alpha_1') - A_1(p_2\alpha_2' + p_2')', \\ h_{22} = -n(\alpha_1' + \alpha_2')p_2A_1 - np_2A_1', \\ h_{23} = n(n-1)p_2A_1 \not\equiv 0, \\ h_{24} = np_2A_1 \not\equiv 0. \end{cases} \tag{3.10}$$

Clearly, h_{2j} are rational functions for $j = 1, 2, 3, 4$.

First we suppose that $h_{21} \equiv 0$. Then we have

$$\frac{(p_2\alpha_2' + p_2')'}{p_2\alpha_2' + p_2'} - \frac{A_1'}{A_1} \equiv \alpha_1'$$

and so by integration we have $p_2\alpha_2' + p_2' = c_1 A_1 e^{\alpha_1}$, where $c_1 \in \mathbb{C} \setminus \{0\}$. This shows that $A_1 e^{\alpha_1} \in S(f)$. Then from (3.3) we have

$$f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = (p_2\alpha_2' + p_2')P_d - p_2 P_d' + A_1 e^{\alpha_1}. \tag{3.11}$$

In this case, one can also easily conclude that $f(z) = q(z)e^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = c_1 p_2(z)$, where $c_1 \in \mathbb{C} \setminus \{0\}$.

Next we suppose that $h_{21} \not\equiv 0$. Let

$$h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff'' = a. \tag{3.12}$$

Now we consider the following two cases.

Case 1. Suppose that $a \equiv 0$. Then from (3.12) we have

$$-h_{21}f^2 \equiv h_{22}ff' + h_{23}(f')^2 + h_{24}ff''. \tag{3.13}$$

Let z_1 be a zero of f of order l_1 such that $h_{2i}(z_1) \neq 0, \infty$ for $i = 1, 2, 3, 4$. Clearly, z_1 is a zero with multiplicity $2l_1$ of the left-hand side of equation (3.13) and a zero with multiplicity $2l_1 - 2$ of the right-hand side of equation (3.13). Therefore we arrive at a contradiction from (3.13). Now from (3.13) we can easily conclude that $N(r, 0; f) = O(\log r)$. Since $a \equiv 0$, from (3.9) and (3.10) we have

$$R_d \equiv 0, \quad \text{i.e., } (A_1' + A_1\alpha_1')Q_d \equiv A_1 Q_d'. \tag{3.14}$$

First we suppose that $Q_d \equiv 0$. Then from (3.8) we have

$$(p_2\alpha_2' + p_2')P_d \equiv p_2 P_d'. \tag{3.15}$$

If $P_d \equiv 0$, then from (3.1) and (3.3) we have, respectively,

$$f^n = p_1 e^{\alpha_1} + p_2 e^{\alpha_2} \tag{3.16}$$

and

$$f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = A_1 e^{\alpha_1}. \tag{3.17}$$

Now (3.17) gives

$$np_2\frac{f'}{f} - (p_2\alpha_2' + p_2') = A_1\frac{e^{\alpha_1}}{f^n}. \tag{3.18}$$

Using Lemma 2.4, one can easily conclude from (3.18) that $m(r, \mathrm{e}^{\alpha_1}/f^n) = O(\log r)$. Since $N(r, 0; f) = O(\log r)$, we have $T(r, \mathrm{e}^{\alpha_1}/f^n) = O(\log r)$. Then by the first fundamental theorem, we have $T(r, f^n/\mathrm{e}^{\alpha_1}) = O(\log r)$. Also from (3.16) we have

$$f^n \mathrm{e}^{-\alpha_1} = p_1 + p_2 \mathrm{e}^{\alpha_2 - \alpha_1}.$$

This shows that $T(r, \mathrm{e}^{\alpha_2-\alpha_1}) = O(\log r)$ and so $\mathrm{e}^{\alpha_2-\alpha_1}$ is a nonzero constant. Let $\mathrm{e}^{\alpha_2-\alpha_1} = c_2 \in \mathbb{C}\setminus\{0\}$. Clearly $\alpha' \equiv \alpha_2'$. Now from (3.16) we have $f^n = \varphi_1 \mathrm{e}^{\alpha_1}$, where $\varphi_1 = p_1 + c_1 p_2$ is a rational function. In this case we also have $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_1(z) + c_1 p_2(z)$.

Next we suppose that $P_d \not\equiv 0$. Then from (3.15) we have

$$\frac{P_d'}{P_d} \equiv \alpha_2' + \frac{p_2'}{p_2}. \tag{3.19}$$

Integrating, we get $P_d = c_3 p_2 \mathrm{e}^{\alpha_2}$, where $c_3 \in \mathbb{C} \setminus \{0\}$ and so from (3.1) we get

$$f^n + \Big(1 - \frac{1}{c_3}\Big) P_d = p_1 \mathrm{e}^{\alpha_1}.$$

If $c_3 \neq 1$, then by Lemma 2.9, we have $f^n = p_1 \mathrm{e}^{\alpha_1}$ and $P_d \equiv 0$, which contradicts the fact that $P_d \not\equiv 0$. Therefore $c_3 = 1$ and so $f^n = p_1 \mathrm{e}^{\alpha_1}$ and $P_d = p_2 \mathrm{e}^{\alpha_2} \not\equiv 0$. In this case also, we have $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_1(z)$. Note that

$$P_d(z, f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z, f)}{f^\mu(z)} f^\mu(z), \tag{3.20}$$

where $b_\mu \in S(f)$ and

$$\begin{aligned} G_\mu(z, f) = {} & (f(z))^{p_0^\mu} (f'(z))^{p_1^\mu} \dots (f^{(k)}(z))^{p_k^\mu} \\ & \times (f(z + c_\mu))^{q_0^\mu} (f'(z + c_\mu))^{q_1^\mu} \dots (f^{(k)}(z + c_\mu))^{q_k^\mu}, \end{aligned}$$

$p_0^\mu, p_1^\mu, \dots, p_k^\mu, q_0^\mu, q_1^\mu, \dots, q_k^\mu \in \mathbb{N} \cup \{0\}$ such that $\sum\limits_{j=0}^{k} p_j^\mu + \sum\limits_{j=0}^{k} q_j^\mu = \mu \leqslant d$. Now by Lemmas 2.4 and 2.5, we derive $m(r, G_\mu(z, f)/f^\mu(z)) = S(r, f)$. Since $N(r, \infty; f) + N(r, 0; f) = S(r, f)$, it follows that $T(r, G_\mu(z, f)/f^\mu(z)) = S(r, f)$. Therefore (3.20) takes the form $P_d(z, f) = c_d(z) f^d(z) + c_{d-1}(z) f^{d-1}(z) + \dots + c_0(z)$, where $c_d(z) \not\equiv 0$ and $c_i \in S(f)$ for $i = 0, 1, 2, \dots, d$. Now substituting $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$ into $P_d(z, f) = p_2(z)\mathrm{e}^{\alpha_2(z)}$, we get

$$\sum_{k=0}^{d} a_{2k}(z) \mathrm{e}^{k\alpha_1(z)/n} = p_2(z) \mathrm{e}^{\alpha_2(z)}, \tag{3.21}$$

where $a_{2k}(z)$ $(k = 0, 1, \dots, d)$ are small functions of $f(z)$.

Since $T(r,f)=T(r,\mathrm{e}^{\alpha_1/n})+S(r,f)$, it follows that $a_{2k}(z)$, $k=0,1,\ldots,d$, are small functions of $\mathrm{e}^{\alpha_1/n}$ and so $a_{2k}(z)$, $k=0,1,\ldots,d$, are small functions of $\mathrm{e}^{k\alpha_1/n}$, where $k\in\{1,2,\ldots,d\}$. Since $p_2\not\equiv 0$, from (3.21) we conclude that there exists at least one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. We now claim that there exists exactly one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. If $d=0$, then our claim is true. Next we suppose that $d\geqslant 1$. If possible, suppose that there exist at least two values of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. For the sake of simplicity we may assume that $a_{2k}\not\equiv 0$ for $k\in\{0,1,2,\ldots,d\}$. Now by Lemma 2.3 we have

$$T\Big(r,\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)=dT(r,\mathrm{e}^{\alpha_1/n})+S(r,\mathrm{e}^{\alpha_1/n}). \tag{3.22}$$

Also from (3.21) we have

$$N\Big(r,-a_{20};\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)=N(r,0;p_2)\leqslant S(r,\mathrm{e}^{\alpha_1/n}). \tag{3.23}$$

Now from Lemmas 2.2, 2.3, (3.22) and (3.23) we have

$$\begin{aligned}
dT(r,\mathrm{e}^{\alpha_1/n})&\leqslant \overline{N}\Big(r,0;\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+\overline{N}\Big(r,\infty;\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)\\
&\quad+\overline{N}\Big(r,-a_{20};\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&\leqslant \overline{N}\Big(r,0;\sum_{k=0}^{d-1}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&\leqslant T\Big(r,\sum_{k=0}^{d-1}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&=(d-1)T(r,\mathrm{e}^{\alpha_1/n})+S(r,\mathrm{e}^{\alpha_1/n}),
\end{aligned}$$

which is impossible. Therefore there exists exactly one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$ and so from (3.21) we conclude that there must exist exactly one value of $k\in\{0,1,2,\ldots,d\}$ such that $\mathrm{e}^{(k\alpha_1-n\alpha_2)/n}$ is a small function of f.

Next we suppose that $Q_d\not\equiv 0$. Then from (3.14) we have

$$\frac{Q_d'}{Q_d}\equiv\frac{A_1'}{A_1}+\alpha_1'. \tag{3.24}$$

Integrating, we get $Q_d=c_4A_1\mathrm{e}^{\alpha_1}$, where $c_4\in\mathbb{C}\setminus\{0\}$ and so from (3.3) we get

$$f^{n-1}(np_2f'-(p_2\alpha_2'+p_2')f)\equiv\Big(\frac{1}{c_4}-1\Big)Q_d.$$

Let $\varphi_3 = np_2f' - (p_2\alpha_2' + p_2')f$. If $c_4 \neq 1$, then by Lemma 2.1, we have $m(r,\varphi_3) = S(r,f)$ and $m(r,\varphi_3 f) = S(r,f)$. Since $N(r,\infty;f) = S(r,f)$, it follows that $T(r,\varphi_3) = S(r,f)$ and $T(r,\varphi_3 f) = S(r,f)$. Note that

$$T(r,f) \leqslant T(r,\varphi_3 f) + T\Big(r,\frac{1}{\varphi_3}\Big) + S(r,f) = S(r,f),$$

which is impossible. Hence $c_4 = 1$ and so $\varphi_3 \equiv 0$. Then we have

$$n\frac{f'}{f} = \frac{p_2'}{p_2} + \alpha_2'.$$

On integration, we get $f^n = c_5 p_2 \mathrm{e}^{\alpha_2}$, where $c_5 \in \mathbb{C}\setminus\{0\}$. If $c_5 \neq 1$, then from (3.1) we have

$$\Big(1 - \frac{1}{c_5}\Big)f^n + P_d = p_1\mathrm{e}^{\alpha_1}.$$

Now by Lemma 2.9, we conclude that $P_d \equiv 0$ and so $Q_d \equiv 0$, which contradicts the fact that $Q_d \not\equiv 0$. Hence $c_5 = 1$ and so $f^n = p_2\mathrm{e}^{\alpha_2}$. Also from (3.1) we have $P_d = p_1\mathrm{e}^{\alpha_1}$. In this case we have $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_2(z)$. Also there must exist exactly one $k \in \{0,1,2,\ldots,d\}$ such that $\mathrm{e}^{(k\alpha_2 - n\alpha_1)/n}$ is a small function of f.

Case 2. Suppose that $a \not\equiv 0$. Then by Lemma 2.1, we can conclude that a is a small function of f. Now from (3.12) we have

$$\frac{1}{f^2} = \frac{h_{21}}{a} + \frac{h_{22}}{a}\frac{f'}{f} + \frac{h_{23}}{a}\Big(\frac{f'}{f}\Big)^2 + \frac{h_{24}}{a}\frac{f''}{f}. \tag{3.25}$$

Therefore from Lemma 2.4 and (3.25) we conclude that $m(r,1/f^2) = S(r,f)$, i.e., $m(r,1/f) = S(r,f)$. Consequently, by the first fundamental theorem, we have $T(r,f) = N(r,0;f) + S(r,f)$. This shows that f has infinitely many zeros. Let z_2 be a multiple zero of f such that $h_{2i}(z_2) \neq 0,\infty$ for $i = 1,2,3,4$. Then from (3.12) we conclude that z_2 is a zero of a. Therefore $N_{(2}(r,0;f) \leqslant T(r,a) = S(r,f)$, i.e., $N_{(2}(r,0;f) = S(r,f)$. Consequently, f has infinitely many simple zeros. Differentiating (3.12) once, we have

$$\begin{aligned} a' = {} & h_{21}'f^2 + (2h_{21} + h_{22}')ff' + (h_{22} + h_{23}')(f')^2 + (h_{22} + h_{24}')ff'' \\ & + (2h_{23} + h_{24})f'f'' + h_{24}ff'''. \end{aligned} \tag{3.26}$$

Now from (3.12) and (3.26) we have

$$\begin{aligned} & (ah_{21}' - a'h_{21})f^2 + (2ah_{21} + ah_{22}' - a'h_{22})ff' + (ah_{22} + ah_{23}' - a'h_{23})(f')^2 \\ & + (ah_{22} + ah_{24}' - a'h_{24})ff'' + a(2h_{23} + h_{24})f'f'' + ah_{24}ff''' \equiv 0. \end{aligned} \tag{3.27}$$

Let z_3 be a simple zero of f which is not a zero or pole of the coefficients in (3.27). Now from (3.27) we see that z_3 is a zero of $(2ah_{23}+ah_{24})f''-(a'h_{23}-ah_{22}-ah_{23}')f'$. Let

$$\alpha=\frac{(2ah_{23}+ah_{24})f''-(a'h_{23}-ah_{22}-ah_{23}')f'}{f}. \tag{3.28}$$

Since $N(r,\infty;f)+N_{(2}(r,0;f)=S(r,f)$, from (3.28) we see that $N(r,\infty;\alpha)=S(r,f)$. Also by Lemma 2.4, we have $m(r,\alpha)=S(r,f)$ and so $T(r,\alpha)=S(r,f)$. This shows that α is a small function of f. Therefore from (3.28) we have

$$f''=\frac{a'h_{23}-ah_{22}-ah_{23}'}{2ah_{23}+ah_{24}}f'+\frac{\alpha}{2ah_{23}+ah_{24}}f. \tag{3.29}$$

Now from (3.12) and (3.29) we have

$$a=q_1f^2+q_2ff'+q_3(f')^2, \tag{3.30}$$

where

$$q_1=h_{21}-\frac{\beta}{2ah_{23}+ah_{24}},\quad q_2=h_{22}+\frac{a'h_{23}-ah_{22}-ah_{23}'}{2ah_{23}+ah_{24}}h_{24}\quad\text{and}\quad q_3=h_{23}$$

are small functions of f. Also from (3.10) we see that

$$\frac{q_2}{q_3}=-\frac{2}{2n-1}(\alpha_1'+\alpha_2')-\frac{3}{2n-1}\frac{A_1'}{A_1}+\frac{1}{2n-1}\frac{a'}{a}-\frac{1}{2n-1}\frac{p_2'}{p_2}. \tag{3.31}$$

By Lemma 2.7, we have

$$q_3(q_2^2-4q_1q_3)\frac{a'}{a}+q_2(q_2^2-4q_1q_3)-q_3(q_2^2-4q_1q_3)'+(q_2^2-4q_1q_3)q_3'\equiv 0. \tag{3.32}$$

Let $\delta=q_2^2-4q_1q_3$. Clearly δ is a small function of f. Now we consider the following two sub-cases.

Sub-case 2.1. Suppose that $\delta=q_2^2-4q_1q_3\equiv 0$. Then from (3.30) we have

$$q_3\Big(f'+\frac{q_2}{2q_3}f\Big)^2=a.$$

This shows that $f'+q_2f/(2q_3)$ is a small function of f. Let $b=f'+q_2f/(2q_3)$. Since $a\not\equiv 0$, it follows that $b\not\equiv 0$. By substituting $f'=b-q_2f/(2q_3)$ into (3.3) and (3.4), we have, respectively,

$$f^n\Big(p_2\alpha_2'+p_2'+np_2\frac{q_2}{2q_3}\Big)-np_2bf^{n-1}+R_{1d}=A_1e^{\alpha_1} \tag{3.33}$$

and

$$f^n\Big(p_1\alpha_1'+p_1'+np_1\frac{q_2}{2q_3}\Big)-np_1bf^{n-1}+R_{2d}=-A_1e^{\alpha_2}, \tag{3.34}$$

where $R_{1d}=p_2P_d'-(p_2\alpha_2'+p_2')P_d$ and $R_{2d}=p_1P_d'-(p_1\alpha_1'+p_1')P_d$.

Let
$$\gamma_1 = p_2\alpha_2' + p_2' + np_2\frac{q_2}{2q_3} \quad \text{and} \quad \gamma_2 = p_1\alpha_1' + p_1' + np_1\frac{q_2}{2q_3}.$$
First we suppose that $\gamma_1 \equiv 0$. Then using (3.31), we get
$$\frac{p_2'}{p_2} + \alpha_2' = \frac{n}{2n-1}\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_2'}{p_2}\Big).$$
Therefore by integrating, we get
$$(p_2 e^{\alpha_2})^{2n-1} = c_6 \frac{A_1^{3n/2} p_2^{n/2}}{a^{n/2}} e^{n(\alpha_1+\alpha_2)},$$
where $c_6 \in \mathbb{C} \setminus \{0\}$. This shows that $e^{(n-1)\alpha_2 - n\alpha_1}$ is a small function of f. Next we suppose that $\gamma_2 \equiv 0$. Then using (3.31), we get
$$\frac{p_1'}{p_1} + \alpha_1' = \frac{n}{2n-1}\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_2'}{p_2}\Big).$$
Therefore by integrating, we get
$$(p_1 e^{\alpha_1})^{2n-1} = c_7 \frac{A_1^{3n/2} p_2^{n/2}}{a^{n/2}} e^{n(\alpha_1+\alpha_2)},$$
where $c_7 \in \mathbb{C} \setminus \{0\}$. This shows that $e^{(n-1)\alpha_1 - n\alpha_2}$ is a small function. Next we discuss the following four sub-cases.

Sub-case 2.1.1. Suppose that $\gamma_1 \equiv 0$ and $\gamma_2 \equiv 0$. Then both $e^{(n-1)\alpha_2 - n\alpha_1}$ and $e^{(n-1)\alpha_1 - n\alpha_2}$ are small functions of f. Clearly $e^{\alpha_1+\alpha_2}$ is a small function of f and so $e^{\alpha_2} = \varphi_4 e^{-\alpha_1}$, where φ_4 is a small function of f. Now from (3.33) and (3.34) we have, respectively,

(3.35) $$-np_2 b f^{n-1} + R_{1d} = A_1 e^{\alpha_1}$$

and

(3.36) $$-np_1 b f^{n-1} + R_{2d} = -A_1 \varphi_4 e^{-\alpha_1}.$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.35) and (3.36) we have

(3.37) $$f^{2n-3}(n^2 b^2 p_1 p_2 f) + R_{3d} = -A_1^2 \varphi_4,$$

where $R_{3d} = -np_2 b R_{2d} f^{n-1} - np_1 b R_{1d} f^{n-1} + R_{1d}R_{2d}$ is a differential polynomial in f of degree $\leqslant 2n-3$ with small functions as its coefficients. Then by applying Lemma 2.1, we get from (3.37) that $m(r, f) = S(r, f)$. Since $N(r, \infty; f) = S(r, f)$, it follows that $T(r, f) = S(r, f)$, which is impossible.

Sub-case 2.1.2. Suppose that $\gamma_1 \not\equiv 0$ and $\gamma_2 \equiv 0$. Since $\gamma_2 \equiv 0$, we have that $e^{(n-1)\alpha_1 - n\alpha_2}$ is a small function of f and so

$$e^{\alpha_2} = \varphi_5 e^{(n-1)\alpha_1/n}, \quad \text{where } \varphi_5 \in S(f). \tag{3.38}$$

Now from (3.33) and Lemma 2.8, there exists a small function v_1 of f such that

$$(f + v_1)^n = \frac{A_1}{\gamma_1} e^{\alpha_1}, \text{ i.e., } f = u_1 e^{\alpha_1/n} - v_1, \tag{3.39}$$

where u_1 is a nonzero small function of f. Since f has infinitely many zeros, it follows that $v_1 \not\equiv 0$. Now from (3.1), (3.38) and (3.39) we have

$$(u_1 e^{\alpha_1/n} - v_1)^n + P_d = p_1 e^{\alpha_1} + c_5 p_2 e^{(n-1)/n\alpha_1}.$$

Therefore by applying Lemma 2.4, we can conclude that $u_1^n(z) = p_1(z)$.

Sub-case 2.1.3. Suppose that $\gamma_1 \equiv 0$ and $\gamma_2 \not\equiv 0$. Since $\gamma_1 \equiv 0$, we have that $e^{(n-1)\alpha_2 - n\alpha_1}$ is a small function of f and so $e^{\alpha_1} = \varphi_6 e^{(n-1)/n\alpha_2}$, where $\varphi_6 \in S(f)$. Now proceeding in the same way as in Sub-case 2.1.2, one can easily conclude that $f = u_2 e^{\alpha_2/n} - v_2$, where u_2 and v_2 are nonzero small functions of f such that $u_2^n(z) = p_2(z)$.

Sub-case 2.1.4. Suppose that $\gamma_1 \not\equiv 0$ and $\gamma_2 \not\equiv 0$. Now from (3.33) and (3.34) and Lemma 2.8, there exist two small functions v_3 and v_4 of f such that

$$(f + v_3)^n = \frac{A_1}{\gamma_1} e^{\alpha_1} \quad \text{and} \quad (f + v_4)^n = -\frac{A_1}{\gamma_2} e^{\alpha_2}.$$

From these we have, respectively,

$$f = u_3 e^{\alpha_1/n} - v_3 \quad \text{and} \quad f = u_4 e^{\alpha_2/n} - v_4, \tag{3.40}$$

where $u_3^n = A_1/\gamma_1 \not\equiv 0$ and $u_4^n = -A_1/\gamma_2 \not\equiv 0$. Since f has infinitely many zeros, it follows that $v_3 \not\equiv 0$ and $v_4 \not\equiv 0$.

First we suppose that $e^{\alpha_1 - \alpha_2}$ is a small function of f. Then clearly $e^{\alpha_2} = \varphi_7 e^{\alpha_1}$, where $\varphi_7 \in S(f)$. Now from (3.1) we have

$$f^n + P_d = p_5 e^{\alpha_1}, \tag{3.41}$$

where $p_5 = p_1 + \varphi_7 p_2$. If $p_5 \equiv 0$, then from (3.41) we have $f^{n-1} f = -P_d$ and so by Lemma 2.1, we conclude that $m(r, f) = S(r, f)$. This shows that $T(r, f) = S(r, f)$, which is impossible. Next we suppose that $p_5 \not\equiv 0$. Then by Lemma 2.9, we conclude that $f^n = p_5 e^{\alpha_1}$ and $P_d \equiv 0$. In this case we have $f(z) = q(z) e^{\alpha_1/n}$, where $q(z)$ is a nonzero small function of $f(z)$ such that $q^n(z) = p_1(z) + \varphi_7(z) p_2(z)$.

Next we suppose that $e^{\alpha_1-\alpha_2}$ is not a small function of f. Note that $T(r,f) \leqslant T(r, e^{\alpha_1/n}) + S(r,f)$. Also

$$T(r, e^{\alpha_1/n}) \leqslant T(r, u_3e^{\alpha_1/n})+S(r,f) \leqslant T(r, u_3e^{\alpha_1/n}-v_3)+S(r,f) = T(r,f)+S(r,f).$$

Combining these, we get $T(r,f) = T(r, u_3e^{\alpha_1/n}) + S(r,f)$. Similarly, we have $T(r,f) = T(r, u_4e^{\alpha_2/n}) + S(r,f)$. These show that $S(r,f) = S(r, u_3e^{\alpha_1/n}) = S(r, u_4e^{\alpha_2/n})$. Clearly u_3, u_4, v_3 and v_4 are small functions of both $e^{\alpha_1/n}$ and $e^{\alpha_2/n}$. On the other hand, from (3.40) we have

$$u_3e^{\alpha_1/n} - u_4e^{\alpha_2/n} = v_3 - v_4. \tag{3.42}$$

We claim that $v_3 \equiv v_4$. If not, suppose that $v_3 \not\equiv v_4$. Now by Lemma 2.2, we get

$$\begin{aligned}T(r,f) = T(r, u_3e^{\alpha_1/n}) + S(r,f) &\leqslant \overline{N}(r,0; u_3e^{\alpha_1/n}) + \overline{N}(r,\infty; u_3e^{\alpha_1/n})\\ &\quad + \overline{N}(r, v_3-v_4; u_3e^{\alpha_1/n}) + S(r, u_3e^{\alpha_1/n}) + S(r,f)\\ &= S(r,f),\end{aligned}$$

which is a contradiction. Hence, $v_3 \equiv v_4$ and so from (3.42) we have

$$u_3e^{\alpha_1/n} \equiv u_4e^{\alpha_2/n}.$$

This shows that $e^{(\alpha_1-\alpha_2)/n} = u_4/u_3$ and so $e^{\alpha_1-\alpha_2} = (u_4/u_3)^n$. Consequently, $e^{\alpha_1-\alpha_2}$ is a small function of f, which contradicts our assumption.

Sub-case 2.2. Suppose that $\delta = q_2^2 - 4q_1q_3 \not\equiv 0$. Then from (3.32) we have

$$\frac{q_2}{q_3} \equiv \frac{\delta'}{\delta} - \frac{q_3'}{q_3} - \frac{a'}{a}.$$

Therefore from (3.10) and (3.31) we have

$$2(\alpha_1' + \alpha_2') \equiv (2n-4)\frac{A_1'}{A_1} + (2n-2)\frac{a'}{a} + (2n-2)\frac{p_2'}{p_2} - (2n-1)\frac{\delta'}{\delta}.$$

Integrating, we get

$$e^{2(\alpha_1+\alpha_2)} = c_8\frac{A_1^{2n-4}a^{2n-2}p_2^{2n-2}}{\delta^{2n-1}},$$

where $c_8 \in \mathbb{C}$. This shows that $e^{\alpha_1+\alpha_2}$ is a small function of f and so $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, where $\varphi_8 \in S(f)$. Now from (3.3) and (3.4), we have, respectively,

$$f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f) + R_{1d} = A_1e^{\alpha_1} \tag{3.43}$$

and

$$f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f) + R_{2d} = -\varphi_8A_1e^{-\alpha_1}. \tag{3.44}$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.43) and (3.44) we have

$$f^{2n-2}(np_2f' - (p_2\alpha_2' + p_2')f)(np_1f' - (p_1\alpha_1' + p_1')f) + \mathcal{Q}_d^* = -\varphi_8 A_1^2, \tag{3.45}$$

where

$$\mathcal{Q}_d^* = f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f)R_{2d} + f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f)R_{1d} + R_{1d}R_{2d}$$

is a differential polynomial in f of degree $\leqslant 2n-2$ with small functions of f as its coefficients. Now by Lemma 2.1, we conclude that $((p_1\alpha_1' + p_1')f - np_1f') \times ((p_2\alpha_2' + p_2')f - np_2f') = b_{11}$, where b_{11} is a small function of f. If $b_{11} \equiv 0$, then we have either $(p_1\alpha_1' + p_1')f - np_1f' \equiv 0$ or $(p_2\alpha_2' + p_2')f - np_2f' \equiv 0$. Thus, in either case one can easily conclude that $N(r, 0; f) = S(r, f)$, which is impossible here. Hence $b_{11} \not\equiv 0$. Therefore we can assume that

$$(p_2\alpha_2' + p_2')f - np_2f' = b_1 e^{\gamma} \quad \text{and} \quad (p_1\alpha_1' + p_1')f - np_1f' = b_2 e^{-\gamma}, \tag{3.46}$$

where b_1, b_2 are small functions of f such that $b_1b_2 = b_{11}$ and γ is an entire function. Since f is of finite order, it follows that γ is a polynomial.

First we suppose that γ is a constant. Then from (3.46) we have

$$f' = \frac{1}{n}\Big(\alpha_2' + \frac{p_2'}{p_2}\Big)f - \frac{b_1 e^{\gamma}}{np_2} \quad \text{and} \quad f' = \frac{1}{n}\Big(\alpha_1' + \frac{p_1'}{p_1}\Big)f - \frac{b_2 e^{-\gamma}}{np_1}.$$

These imply that

$$\Big(\alpha_1' - \alpha_2' + \frac{p_1'}{p_1} - \frac{p_2'}{p_2}\Big)f = \frac{b_2 e^{-\gamma}}{p_1} - \frac{b_1 e^{\gamma}}{p_2}. \tag{3.47}$$

If $\alpha_1' - \alpha_2' + p_1'/p_1 - p_2'/p_2 \equiv 0$, then by integration, we have $e^{\alpha_1 - \alpha_2} = c_9 p_2/p_1$, where $c_9 \in \mathbb{C} \setminus \{0\}$ and so $\alpha_1 - \alpha_2$ is a constant. Since $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, it follows that e^{α_2} is a small function of f. Certainly e^{α_1} is also a small function of f. Now from (3.1) and Lemma 2.1, we conclude that $m(r, f) = S(r, f)$ and so $T(r, f) = S(r, f)$, which is impossible here. Therefore $\alpha_1' - \alpha_2' + p_1'/p_1 - p_2'/p_2 \not\equiv 0$. Now from (3.47), it follows that f is a small function of f, which is absurd.

Next we suppose that γ is a non-constant polynomial. Now solving for f, we get from (3.46) that

$$(p_1p_2(\alpha_2' - \alpha_1') + p_1p_2' - p_1'p_2)f = p_1b_1e^{\gamma} - p_2b_2e^{-\gamma}. \tag{3.48}$$

Using a similar argument, one can easily prove that $p_1p_2(\alpha_2' - \alpha_1') + p_1p_2' - p_1'p_2 \not\equiv 0$. Now from (3.48) we get $f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}$, where

$$\delta_1 = \frac{p_1b_1}{p_1p_2' - p_1'p_2 - p_1p_2(\alpha_1' - \alpha_2')} \quad \text{and} \quad \delta_2 = \frac{-p_2b_2}{p_1p_2' - p_1'p_2 - p_1p_2(\alpha_1' - \alpha_2')}.$$

Equation (3.46) can be rewritten as

$$A_2 f - np_2 f' = b_1 e^{\gamma}, \tag{3.49}$$

where $A_2 = p_2\alpha_2' + p_2'$. Differentiating (3.49) once, we get

$$A_2' f + (A_2 - np_2')f' - np_2 f'' = (b_1' + b_1\gamma')e^{\gamma}. \tag{3.50}$$

Using (3.29), we get from (3.50) that

(3.51)
$$\Big(A_2' - n\frac{p_2\alpha}{2ah_{23} + ah_{24}}\Big)f + \Big(A_2 - np_2' - n\frac{a'h_{23} - ah_{22} - ah_{23}'}{2ah_{23} + ah_{24}}p_2\Big)f' = (b_1' + b_1\gamma')e^{\gamma}.$$

Now from (3.10) and (3.51) we get

$$\begin{aligned}\Big(A_2' - \frac{1}{2n-1}\frac{\alpha}{aA_1}\Big)f + \Big(A_2 - np_2' - \frac{1}{2n-1}(\alpha_1' + \alpha_2')p_2 \\ - \frac{n(n-1)}{2n-1}\frac{a'}{a}p_2 + \frac{n(n-1)}{2n-1}p_2' + \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1}p_2\Big)f' = (b_1' + b_1\gamma')e^{\gamma}.\end{aligned} \tag{3.52}$$

Dividing (3.52) by (3.49), we get

$$\zeta_1 f + \zeta_2 f' \equiv 0, \tag{3.53}$$

where

$$\zeta_1 = A_2' - \frac{1}{2n-1}\frac{\alpha}{A_1} - A_2\Big(\frac{b_1'}{b_1} + \gamma'\Big)$$

and

$$\begin{aligned}\zeta_2 = A_2 - np_2' - \frac{1}{2n-1}(\alpha_1' + \alpha_2')p_2 - \frac{n(n-1)}{2n-1}\frac{a'}{a}p_2 \\ + \frac{n(n-1)}{2n-1}p_2' + \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1}p_2 + n\Big(\frac{b_1'}{b_1} + \gamma'\Big)p_2.\end{aligned}$$

Since $ff' \not\equiv 0$, it follows from (3.53) that either $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$ or $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. First we suppose that $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$. Then from (3.53), one can easily conclude that $N(r,0;f) = S(r,f)$, which is a contradiction. Next we suppose that $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. Now $\zeta_2 \equiv 0$ yields

$$\alpha_2' - \frac{(n-1)^2}{2n-1}\frac{p_2'}{p_2} - \frac{1}{2n-1}(\alpha_1' + \alpha_2') - \frac{n(n-1)}{2n-1}\frac{a'}{a} - \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1} + n\frac{b_1'}{b_1} + n\gamma' \equiv 0,$$

which implies that $e^{(2n-1)(n\gamma+\alpha_2)} = c_{10}p_2^{(n-1)^2}e^{\alpha_1+\alpha_2}(aA_1)^{n(n-1)}b_1^{-n}$, where $c_{10} \in \mathbb{C}\setminus\{0\}$. Consequently, $e^{n\gamma+\alpha_2}$ is a small function of f. Therefore $f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}$ and $e^{\alpha_1(z)+\alpha_2(z)}$ is a small function of $f(z)$, where $\delta_1(z)$, $\delta_2(z)$ are nonzero small functions of $f(z)$ and $\gamma(z)$ is a non-constant polynomial such that either $e^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$ or $e^{n\gamma(z)+\alpha_1(z)}$ is a small function of $f(z)$. □

4. An open problem

For further study, one may raise the following question as an open problem:

Open Problem. What will happen if we remove the condition $\varrho_2(f) < 1$ from Theorem 1.1?

References

[1] *R. Halburd, R. Korhonen, K. Tohge*: Holomorphic curves with shift-invariant hyperplane preimages. Trans. Am. Math. Soc. *366* (2014), 4267–4298. zbl MR doi

[2] *W. K. Hayman*: Meromorphic Functions. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964. zbl MR

[3] *J. Heittokangas, R. Korhonen, I. Laine*: On meromorphic solutions of certain nonlinear differential equations. Bull. Aust. Math. Soc. *66* (2002), 331–343. zbl MR doi

[4] *I. Laine*: Nevanlinna Theory and Complex Differential Equations. de Gruyter Studies in Mathematics 15. Walter de Gruyter, Berlin, 1993. zbl MR doi

[5] *P. Li*: Entire solutions of certain type of differential equations. II. J. Math. Anal. Appl. *375* (2011), 310–319. zbl MR doi

[6] *P. Li, C.-C. Yang*: On the nonexistence of entire solutions of certain type of nonlinear differential equations. J. Math. Anal. Appl. *320* (2006), 827–835. zbl MR doi

[7] *L.-W. Liao, C.-C. Yang, J.-J. Zhang*: On meromorphic solutions of certain type of nonlinear differential equations. Ann. Acad. Sci. Fenn., Math. *38* (2013), 581–593. zbl MR doi

[8] *W. Lü, L. Wu, D. Wang, C.-C. Yang*: The existence of solutions of certain type of nonlinear difference-differential equations. Open Math. *16* (2018), 806–815. zbl MR doi

[9] *C.-C. Yang*: On deficiencies of differential polynomials. II. Math. Z. *125* (1972), 107–112. zbl MR doi

[0] *C.-C. Yang, P. Li*: On the transcendental solutions of a certain type of nonlinear differential equations. Arch. Math. *82* (2004), 442–448. zbl MR doi

[11] *C.-C. Yang, H.-X. Yi*: Uniqueness Theory of Meromorphic Functions. Mathematics and its Applications (Dordrecht) 557. Kluwer Academic, Dordrecht, 2003. zbl MR doi

Authors' addresses: *Sujoy Majumder*, Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India, e-mail: sm05math@gmail.com, *Lata Mahato*, Department of Mathematics, Mahadevananda Mahavidyalaya, SN Banerjee Rd., Monirampore, Kolkata, West Bengal-700120, India, e-mail: himangshulata@gmail.com.