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Yaping Mao; Christopher Melekian; Eddie Cheng

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# A LOWER BOUND FOR THE 3-PENDANT TREE-CONNECTIVITY OF LEXICOGRAPHIC PRODUCT GRAPHS

Yaping Mao, Xining, Christopher Melekian, Fort Worth,
Eddie Cheng, Rochester

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*Abstract.* For a connected graph $G=(V,E)$ and a set $S \subseteq V(G)$ with at least two vertices, an $S$-Steiner tree is a subgraph $T=(V',E')$ of $G$ that is a tree with $S \subseteq V'$. If the degree of each vertex of $S$ in $T$ is equal to 1, then $T$ is called a pendant $S$-Steiner tree. Two $S$-Steiner trees are *internally disjoint* if they share no vertices other than $S$ and have no edges in common. For $S \subseteq V(G)$ and $|S| \geqslant 2$, the pendant tree-connectivity $\tau_G(S)$ is the maximum number of internally disjoint pendant $S$-Steiner trees in $G$, and for $k \geqslant 2$, the $k$-pendant tree-connectivity $\tau_k(G)$ is the minimum value of $\tau_G(S)$ over all sets $S$ of $k$ vertices. We derive a lower bound for $\tau_3(G \circ H)$, where $G$ and $H$ are connected graphs and $\circ$ denotes the lexicographic product.

*Keywords*: connectivity; Steiner tree; internally disjoint Steiner tree; packing; pendant tree-connectivity, lexicographic product

*MSC 2020*: 05C05, 05C40, 05C70, 05C76

## 1. Introduction

A classical theorem due to Menger states that a graph $G$ is $k$-vertex-connected if and only if for every pair of vertices $u, v \in V(G)$ there are at least $k$ paths joining $u$ and $v$ which have no edges or vertices other than the endpoints in common (see [5]); we say such paths are *internally disjoint*. This foundational result has led a number of authors to consider generalizations of connectivity which replace the $k$ internally disjoint paths with different structures, see [2]. One such generalization is due to Hager (see [1]), who uses the notion of *internally disjoint Steiner trees* to define the

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*tree-connectivity* of a graph. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an *$S$-Steiner tree* is a subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. Note that when $|S| = 2$, an $S$-Steiner tree is simply a path connecting the two vertices of $S$. Two $S$-Steiner trees $T$ and $T'$ are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geqslant 2$, the *local tree-connectivity* $\kappa_G(S)$ is the maximum number of pairwise internally disjoint $S$-Steiner trees which can be constructed in $G$. For an integer $k$ with $2 \leqslant k \leqslant n$, the *$k$-tree-connectivity* of $G$, $\kappa_k(G)$, is defined as the minimum value of $\kappa_G(S)$ when $S$ runs over all $k$-element subsets of $V(G)$. When $|S| = 2$, $\kappa_2(G)$ is simply the standard vertex connectivity $\kappa(G)$. For more information regarding the $k$-tree-connectivity and other generalizations of connectivity, we refer the reader to [4].

For some applications, we may want to consider only *pendant $S$-Steiner trees*, in which each vertex of $S$ is a leaf. Analogously, the *local pendant tree-connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant $S$-Steiner trees in $G$. For an integer $k$ with $2 \leqslant k \leqslant n$, *$k$-pendant tree-connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S)\colon\ S \subseteq V(G),\ |S| = k\}$. We let $\tau_k(G) = 0$ when $G$ is disconnected. When $k = 2$, we once again recover the connectivity, as $\tau_2(G) = \kappa(G)$. Thus, we can see that the $k$-pendant tree-connectivity generalizes the connectivity, and is generalized by the $k$-tree-connectivity.

We may observe that the $k$-tree-connectivity and $k$-pendant tree-connectivity of a graph are indeed different. The bound $\kappa_k(G) \geqslant \tau_k(G)$ is immediate, since any pendant $S$-Steiner tree is an $S$-Steiner tree, but the inequality may be strict. The *lexicographic product* of two graphs $G$ and $H$, written as $G \circ H$, is a graph with vertex set $V(G \circ H) = V(G) \times V(H)$, in which two vertices $(u, v)$ and $(u', v')$ of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. Note that the lexicographic product is not commutative.

The connectivity and tree-connectivity of graphs constructed using the lexicographic product have previously been investigated. Yang and Xu in [6] determined the connectivity of the lexicographic product of two graphs.

**Theorem 1.1** ([6]). *Let $G$ and $H$ be two graphs. If $G$ is nontrivial, noncomplete and connected, then $\kappa(G \circ H) = \kappa(G)|V(H)|$.*

Furthermore, Li and Mao in [3] gave a lower bound for the 3-tree-connectivity of the lexicographic product of two graphs.

**Theorem 1.2** ([3]). *Let $G$ and $H$ be two connected graphs. Then*

$$\kappa_3(G \circ H) \geqslant \kappa_3(G)|V(H)|,$$

*and the bound is sharp.*

In this paper, we derive the following lower bound for the 3-pendant tree-connectivity of the lexicographic product of two graphs.

**Theorem 1.3.** *Let $G$ and $H$ be two connected graphs. Then*

$$\tau_3(G \circ H) \geqslant \min\{(\tau_3(G)+1)(|V(H)|-1), \tau_3(G)|V(H)|\}.$$

*If* $\tau_3(G) = 0$,

$$\tau_3(G \circ H) \geqslant (\tau_3(G)+1)(|V(H)|-1),$$

*and the bound is sharp.*

The sharpness of this bound when $\tau_3(G) = 0$ is illustrated by the following example:

**Example 1.4.** Let $P_n$ be a path on $n$ vertices and $Q_m$ be a path on $m$ vertices, $n, m \geqslant 2$. Then $\tau_3(P_n) = 0$, $|V(Q_m)| = m$, and $\tau_3(P_n \circ Q_m) = m - 1 = (\tau_3(P_n)+1)(|V(Q_m)|-1)$.

If $V(P_n) = \{u_1, \ldots, u_n\}$ and $V(Q_m) = \{v_1, \ldots, v_m\}$, then let

$$S = \{(u_1, v_1), (u_1, v_2), (u_2, v_1)\} \subseteq V(P_n \circ Q_m).$$

The vertex $(u_1, v_1)$ has degree $m+1$ in the graph $P_n \circ Q_m$, but only $m-1$ of these edges can appear in pendant $S$-Steiner trees as it is adjacent to the two other vertices in $S$, so we have $\tau_3(P_n \circ Q_m) \leqslant m - 1$.

In graphs with $\tau_3(G) > 0$, we are not aware of an example, where this bound is sharp. This is due to the fact that the number of pendant Steiner trees in a graph constructed through the lexicographic product depends on the number of Steiner trees in its first factor, not only the number of pendant Steiner trees. Nevertheless, the method by which this bound is constructed illustrates the similarity between the 3-pendant tree-connectivity and other measures of connectivity in lexicographic product graphs.

## 2. Derivation of lower bound

In this section, let $G$ and $H$ be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G \circ H) = \{(u_i, v_j)\colon 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$. For $v \in V(H)$ we use $G(v)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u_i, v)\colon 1 \leqslant i \leqslant n\}$. Similarly, for $u \in V(G)$ we use $H(u)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u, v_j)\colon 1 \leqslant j \leqslant m\}$; we refer to the subgraphs $H(u)$ as *copies* of $H$.

We now introduce the general idea of the proof of Theorem 1.3. In Subsection 2.1, we first study the 3-pendant tree-connectivity of the lexicographic product of a path $P$ and a connected graph $H$ and show $\tau_3(P \circ H) \geqslant |V(H)| - 1$. Next, we investigate the 3-pendant tree-connectivity of the lexicographic product of a tree $T$ and a connected graph $H$ and show $\tau_3(T \circ H) \geqslant |V(H)| - 1$ in Subsection 2.2. After these preparations, we consider the graph $G \circ H$ and prove $\tau_3(G \circ H) \geqslant \min\{(\tau_3(G) + 1)(|V(H)| - 1), \tau_3(G)|V(H)|\}$ in Subsection 2.3.

Before proceeding, we mention two bounds derived by Hager (see [1]) relating the $k$-pendant tree-connectivity of a graph to its connectivity and minimum degree.

**Lemma 2.1** ([1]). *Let $G$ be a graph, $l > 0$, and $k \geqslant 3$. If $\tau_k(G) \geqslant l$, then $\delta(G) \geqslant k + l - 1$ and $\kappa(G) \geqslant k + l - 2$.*

**2.1. Lexicographic product of a path and a connected graph.** First, we consider the 3-pendant tree connectivity of the lexicographic product of a path and a connected graph.

**Proposition 2.2.** *Let $H$ be a connected graph and $P$ be a path with $n$ vertices, $n \geqslant 2$. Then $\tau_3(P \circ H) \geqslant |V(H)| - 1$. Moreover, the bound is sharp.*

Let $V(H) = \{v_1, v_2, \ldots, v_m\}$ and $V(P) = \{u_1, u_2, \ldots, u_n\}$. Without loss of generality, let $u_i$ and $u_j$ be adjacent if and only if $|i - j| = 1$, where $1 \leqslant i \neq j \leqslant n$. It suffices to show that $\tau_{P \circ H}(S) \geqslant m - 1$ for any $S = \{x, y, z\} \subseteq V(P \circ H)$, that is, there exist $m - 1$ internally disjoint pendant $S$-Steiner trees in $P \circ H$. We prove this using the following three lemmas.

**Lemma 2.3.** *If $x$, $y$ and $z$ belong to the same copy $H(u_i)$ $(1 \leqslant i \leqslant n)$, then there exist $m$ internally disjoint pendant $S$-Steiner trees.*

P r o o f. Without loss of generality, we assume $x, y, z \in V(H(u_1))$. Then the trees $T_j$ on $\{x, y, z, (u_2, v_j)\}$ with edges $\{x(u_2, v_j), y(u_2, v_j), z(u_2, v_j)\}$ for $1 \leqslant j \leqslant m$ are $m$ internally disjoint pendant $S$-Steiner trees, as desired. □

**Lemma 2.4.** *If only two of $x$, $y$ and $z$ belong to the same copy $H(u_i)$ $(1 \leqslant i \leqslant n)$, then there exist $m - 1$ internally disjoint pendant $S$-Steiner trees.*

P r o o f. Without loss of generality, we may assume that $x, y \in V(H(u_\alpha))$ and $z \in V(H(u_\beta))$, where $1 \leqslant \alpha < \beta \leqslant n$. Additionally, we may write $x = (u_\alpha, v_{m-1})$, $y = (u_\alpha, v_m)$, $z = (u_\beta, v_m)$, and let $w = (u_\beta, v_{m-1})$ be adjacent to $z$ in $H(u_\beta)$.

If $\beta = \alpha + 1$, then the trees $T_j$ on $\{x, y, z, (u_\alpha, v_j), (u_\beta, v_j)\}$ with edges $\{x(u_\beta, v_j), y(u_\beta, v_j), (u_\alpha, v_j)(u_\beta, v_j), (u_\alpha, v_j)z\}$ for $1 \leqslant j \leqslant m - 2$ together with the tree $T_w$ on

$\{w, x, y, z\}$ with edges $\{xw, yw, zw\}$ are $m-1$ internally disjoint pendant $S$-Steiner trees, as desired.

If $\beta > \alpha + 1$, then let $W_j$ be a path with $V(W_j) = \{(u_k, v_j)\colon\ \alpha + 1 \leqslant k \leqslant \beta - 1\}$ and $E(W_j) = \{(u_k, v_j)(u_{k+1}, v_j)\colon\ \alpha + 1 \leqslant k < \beta - 1\}$, and let $w_{j,\alpha+1}$ and $w_{j,\beta-1}$ refer to the initial and terminal vertices, respectively, of $W_j$. Note that $W_j$ may consist of a single vertex and no edges, in which case we have $w_{j,\alpha+1} = w_{j,\beta-1}$. Then the trees $T_j$ on $\{x, y, z, (u_{\alpha+1}, v_j), \ldots, (u_{\beta-1}, v_j)\}$ with edges $E(W_j) \cup \{xw_{j,\alpha+1}, yw_{j,\alpha+1}, zw_{j,\beta-1}\}$ for $1 \leqslant j \leqslant m$ are $m$ internally disjoint pendant $S$-Steiner trees, as desired. □

**Lemma 2.5.** *If $x$, $y$ and $z$ are contained in distinct copies $H(u_i)$, then there exist $m-1$ internally disjoint pendant $S$-Steiner trees.*

P r o o f. We may assume without loss of generality that $x \in V(H(u_\alpha))$, $y \in V(H(u_\beta))$ and $z \in V(H(u_\gamma))$, where $1 \leqslant \alpha < \beta < \gamma \leqslant n$. Additionally, by the structure of $P \circ H$ we may suppose that $x = (u_\alpha, v_m)$, $y = (u_\beta, v_m)$ and $z = (u_\gamma, v_m)$.

Let $W_j$ be a path with $V(W_j) = \{(u_k, v_j)\colon\ \alpha + 1 \leqslant k \leqslant \gamma - 1\}$ and $E(W_j) = \{(u_k, v_j)(u_{k+1}, v_j)\colon\ \alpha + 1 \leqslant k < \gamma - 1\}$, and let $w_{j,\alpha+1}$ and $w_{j,\gamma-1}$ refer to the two endpoints of $W_j$. Note that $W_j$ may consist of a single vertex and no edges, in which case we have $w_{j,\alpha+1} = w_{j,\gamma-1} = (u_\beta, v_j)$. If $\gamma = \beta + 1$, then the trees $T_j$ on $\{x, y, z, (u_{\alpha+1}, v_j), \ldots, (u_\gamma, v_j)\}$ with edges

$$E(W_j) \cup \{xw_{j,\alpha+1}, zw_{j,\gamma-1}, w_{j,\gamma-1}(u_\gamma, v_j), y(u_\gamma, v_j)\}$$

for $1 \leqslant j \leqslant m-1$ are $m-1$ internally disjoint pendant $S$-Steiner trees, as desired.

Otherwise, we have $\gamma > \beta+1$, and the trees $T_j$ on $\{x, y, z, (u_{\alpha+1}, v_j), \ldots, (u_{\gamma-1}, v_j)\}$ with edges $E(W_j) \cup \{xw_{j,\alpha+1}, yw_{j,\beta+1}, zw_{j,\gamma-1}\}$ for $1 \leqslant j \leqslant m-1$ are $m-1$ internally disjoint pendant $S$-Steiner trees, as desired. □

From Lemmas 2.3, 2.4 and 2.5, we conclude that for any $S \subseteq V(P \circ H)$, there exist $m-1$ internally disjoint pendant $S$-Steiner trees, which implies that $\tau_{P \circ H}(S) \geqslant m-1$. From the arbitrariness of $S$ we have $\tau_3(P \circ H) \geqslant m-1$. This completes the proof of Proposition 2.2. □

**2.2. Lexicographic product of a tree and a connected graph.** Next, we consider the 3-pendant tree-connectivity of the lexicographic product of a tree and a connected graph, which can be seen as a generalization of the result in Subsection 2.1.

**Proposition 2.6.** *Let $H$ be a graph and let $T$ be a tree with $n$ vertices. For any $S = \{x, y, z\} \subseteq V(T \circ H)$, if $x \in V(H(u_\alpha))$, $y \in V(H(u_\beta))$ and $z \in V(H(u_\gamma))$, then $\tau(S) \geqslant |V(H)| - 1$, where $u_\alpha$, $u_\beta$, $u_\gamma$ are three vertices in $T$. If $u_\alpha$, $u_\beta$, $u_\gamma$ do not lie on a path in $T$, then $\tau(S) \geqslant |V(H)|$.*

P r o o f. Set $V(T)=\{u_1,u_2,\ldots,u_n\}$ and $V(H)=\{v_1,v_2,\ldots,v_m\}$. It suffices to prove that for any $S=\{x,y,z\}\subseteq V(T\circ H)$, where $x\in V(H(u_\alpha))$, $y\in V(H(u_\beta))$ and $z\in V(H(u_\gamma))$, $\tau_{T\circ H}(S)\geqslant m-1$ when $u_\alpha$, $u_\beta$ and $u_\gamma$ lie on a path, and that $\tau_{T\circ H}(S)\geqslant m$ otherwise. So for any $S=\{x,y,z\}\subseteq V(T\circ H)$ we need to show there exist $m-1$ (or $m$) internally disjoint pendant $S$-Steiner trees in $T\circ H$. Recall that $V(T\circ H)=\bigcup\limits_{i=1}^{n} V(H(u_i))$.

If all three of $u_\alpha$, $u_\beta$ and $u_\gamma$ lie on a path $P$ in $T$ (including cases, where two or three of them are the same vertex), then by Proposition 2.2 we have $\tau(S)\geqslant m-1$, since $P\circ H$ is a subgraph of $T\circ H$.

Otherwise, $u_\alpha$, $u_\beta$ and $u_\gamma$ are distinct vertices of $T$ that do not lie on a path. This means that $u_\alpha$, $u_\beta$ and $u_\gamma$ are pairwise not adjacent in $T$. Let $P$ be the unique path in $T$ between $u_\alpha$ and $u_\beta$ and $Q$ be the unique path between $u_\alpha$ and $u_\gamma$. Since $u_\alpha$, $u_\beta$ and $u_\gamma$ are pairwise not adjacent, $P$ and $Q$ each contain at least one vertex other than $u_\alpha$, $u_\beta$ or $u_\gamma$. If $P=\{u_\alpha,p_1,\ldots,p_r,u_\beta\}$ and $Q=\{u_\alpha,q_1,\ldots,q_s,u_\gamma\}$, let $i$ be the maximal subscript such that $p_i=q_i$; since $T$ is a tree, there is only one such $i$, and $p_k=q_k$ for all $1\leqslant k\leqslant i$. Note that we may have $r=1$, $s=1$ or $i=s$, and that $p_i$ cannot be $u_\alpha$, $u_\beta$ or $u_\gamma$, otherwise these three vertices would lie on a path. Let $P_j$ be a path in $T\circ H$ with $V(P_j)=\{x,(p_1,v_j),\ldots,(p_r,v_j),y\}$ and $E(P_j)=\{x(p_1,v_j),(p_r,v_j)y\}\cup\{(p_k,v_j)(p_{k+1},v_j)\colon\ 1\leqslant k<r\}$. Similarly, let $Q_j$ be a path with $V(Q_j)=\{(q_i,v_j),\ldots,(q_s,v_j),z\}$ and

$$E(Q_j)=\{(q_s,v_j)z\}\cup\{(q_k,v_j)(q_{k+1},v_j)\colon\ i\leqslant k<s\}.$$

We can see that $P_j$ and $Q_j$ have the vertex $(p_i,v_j)$ in common, and the trees $T_j$ with $V(T_j)=V(P_j)\cup V(Q_j)$ and $E(T_j)=E(P_j)\cup E(Q_j)$ for $1\leqslant j\leqslant m$ are $m$ internally disjoint pendant $S$-Steiner trees, as desired.

Thus, for any $S=\{x,y,z\}\subseteq V(T\circ H)$ there exist $m-1$ internally disjoint pendant $S$-Steiner trees, which implies that $\tau_{T\circ H}(S)\geqslant m-1$. The proof is complete. □

**2.3. Lexicographic product of two general graphs.** Finally, we use the results of Subsections 2.1 and 2.2 to establish Theorem 1.3, regarding the 3-pendant tree-connectivity of the lexicographic product of two connected graphs.

P r o o f of Theorem 1.3. Let $\tau_3(G)=k$ and $\tau_3(H)=l$. Set $V(G)=\{u_1,u_2,\ldots,u_n\}$, $V(H)=\{v_1,v_2,\ldots,v_m\}$. From the definition of $\tau_3(G\circ H)$, for any $S=\{x,y,z\}\subseteq V(G\circ H)$, we need to prove that $\tau_{G\circ H}(S)\geqslant\min\{(k+1)(m-1),km\}$. Clearly, $V(G\circ H)=\bigcup\limits_{i=1}^{n} V(H(u_i))$.

*Case 1*: The vertices $x$, $y$ and $z$ belong to the same copy of $H$. Without loss of generality, let $x, y, z \in V(H(u_1))$. If $k = 0$, by the connectedness of $G$ we know that $u_1$ has a neighbor $u_2$ in $G$. Then the trees $T_j$ on $\{x, y, z, (u_2, v_j)\}$ with edges $\{x(u_2, v_j), y(u_2, v_j), z(u_2, v_j), (u_2, v_j)\}$ for $1 \leqslant i \leqslant m$ are $m = (k+1)m$ internally disjoint pendant $S$-Steiner trees in $G \circ H$. Otherwise, by Lemma 2.1, $\delta(G) \geqslant \tau_3(G) + 2 = k + 2$ and the vertex $u_1$ has at least $k + 2$ neighbors in $G$. Select $k + 2$ neighbors from them, say $u_2, u_3, \ldots, u_{k+3}$. Then the trees $T_{i,j}$ on $\{x, y, z, (u_i, v_j)\}$ with edges $\{x(u_i, v_j), y(u_i, v_j), z(u_i, v_j), (u_i, v_j)\}$ for $2 \leqslant i \leqslant k + 3$ and $1 \leqslant j \leqslant m$ are $(k + 2)m$ internally disjoint pendant $S$-Steiner trees in $G \circ H$.

Since $\tau_3(H) = l$, it follows that there are $l$ internally disjoint pendant $S$-Steiner trees in $H(u_1)$. Observe that these $l$ pendant $S$-Steiner trees and the trees $T_j$ or $T_{i,j}$ are internally disjoint. So in either case, we have $\tau_{G \circ H}(S) \geqslant (k + 1)m + l \geqslant \min\{(k + 1)(m - 1), km\}$, as desired.

*Case 2*: Only two of $x, y$ and $z$ belong to the same copy of $H$. Without loss of generality, let $x, y \in H(u_1)$ and $z \in H(u_2)$. If $k = 0$, by the connectedness of $G$ we have at least one path $P$ joining $u_1$ and $u_2$ in $G$. From Proposition 2.2 we know there exist $m - 1$ internally disjoint pendant $S$-Steiner trees in $P \circ H$. So the total number of internally disjoint pendant $S$-Steiner trees in $G \circ H$ is at least $m - 1 = (k + 1)(m - 1)$. Otherwise, from Lemma 2.1, $\kappa(G) \geqslant \tau_3(G) + 1 = k + 1$ and hence, there exist $k + 1$ internally disjoint paths connecting $u_1$ and $u_2$ in $G$, say $P_1, P_2, \ldots, P_{k+1}$. From Proposition 2.2, there exist $m - 1$ internally disjoint pendant $S$-Steiner trees in $P_i \circ H$ $(1 \leqslant i \leqslant k + 1)$. So the total number of internally disjoint pendant $S$-Steiner trees in $G \circ H$ is at least $(k + 1)(m - 1)$. In either case, we have $\tau_{G \circ H}(S) \geqslant (k + 1)(m - 1) \geqslant \min\{(k + 1)(m - 1), km\}$, as desired.

*Case 3*: The vertices $x$, $y$ and $z$ are contained in distinct copies of $H$. Without loss of generality, let $x \in V(H(u_1))$, $y \in V(H(u_2))$ and $z \in V(H(u_3))$. If $k = 0$, then by the connectedness of $G$ there is at least one $S$-Steiner tree $T$ connecting $\{u_1, u_2, u_3\}$ in $G$. From Proposition 2.6, there exist $m - 1$ internally disjoint pendant $S$-Steiner trees in $T \circ H$. So the total number of internally disjoint pendant $S$-Steiner trees in $G \circ H$ is at least $m - 1 = (k+1)(m-1)$. Thus, we have $\tau_{G \circ H}(S) \geqslant (k+1)(m-1) \geqslant \min\{(k + 1)(m - 1), km\}$, as desired.

Otherwise, since $\tau_3(G) = k > 0$, it follows that there exist $k$ internally disjoint pendant Steiner trees connecting $\{u_1, u_2, u_3\}$ in $G$, say $T_1, T_2, \ldots, T_k$. From Proposition 2.6, since the trees $T_i$ are pendant $S$-Steiner trees, there exist $m$ internally disjoint pendant $S$-Steiner trees in $T_i \circ H$ $(1 \leqslant i \leqslant k)$. So the total number of internally disjoint pendant $S$-Steiner trees in $G \circ H$ is at least $km$. In either case, we have $\tau_{G \circ H}(S) \geqslant km \geqslant \min\{(k + 1)(m - 1), km\}$, as desired.

Considering each case above, we conclude that for any $S \subseteq V(G \circ H)$ with $|S| = 3$, we have

$$\tau_{G \circ H}(S) \geqslant \min\{(k+1)(m-1), km\},$$

which implies that

$$\tau_3(G \circ H) \geqslant \min\{(k+1)(m-1), km\} = \min\{(\tau_3(G)+1)(|V(H)|-1), \tau_3(G)|V(H)|\}.$$

□

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*Authors' addresses*: Yaping Mao, Academy of Plateau Science and Sustainability, Qinghai Normal University, Xining, Qinghai 810008, P. R. China, e-mail: `yapingmao@outlook.com`; Christopher Melekian (corresponding author), General Motors Financial Company, Fort Worth, Texas 76102, USA, e-mail: `chris.melekian@gmail.com`; Eddie Cheng, Department of Mathematics and Statistics, Oakland University, Rochester, Michigan 48309, USA, e-mail: `echeng@oakland.edu`.