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ROOT LOCATION FOR THE CHARACTERISTIC POLYNOMIAL OF A FIBONACCI TYPE SEQUENCE

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Abstract. We analyse the roots of the polynomial $x^n - px^{n-1} - qx - 1$ for $p \geqslant q \geqslant 1$. This is the characteristic polynomial of the recurrence relation $F_{k,p,q}(n) = pF_{k,p,q}(n-1) + qF_{k,p,q}(n-k+1) + F_{k,p,q}(n-k)$ for $n \geqslant k$, which includes the relations of several particular sequences recently defined. In the end, a matricial representation for such a recurrence relation is provided.

Keywords: Fibonacci number; root; characteristic polynomial

MSC 2020: 11A63, 11B39, 11J86

1. Introduction

For integers $k \geqslant 2$ and $n \geqslant 0$, and a rational number $p \geqslant 1$, the (k,p)-*Fibonacci numbers*, denoted by $F_{k,p}(n)$, are defined by the recursion relation

$$(1.1) \qquad F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \quad \text{for } n \geqslant k,$$

satisfying the initial conditions

$$F_{k,p}(0) = 0 \quad \text{and} \quad F_{k,p}(n) = p^{n-1} \quad \text{for } 1 \leqslant n \leqslant k-1.$$

The (k,p)-Fibonacci numbers were recently considered in [1]. They are of the Fibonacci type and clearly $F_{2,1}(n+1)$ is the nth Fibonacci number. Indeed, these numbers include many notorious sequences and their properties have been studied in [1], [7].

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Earlier, to define the so-called *generalized Pell numbers*, Włoch in [13] considered the recurrence relation

$$P_k(n) = P_k(n-1) + P_k(n-k+1) + P_k(n-k) \quad \text{for } n \geqslant k+3. \tag{1.2}$$

While the characteristic polynomial of (1.1) is

$$f_{n,p}(x) = x^n - px^{n-1} - (p-1)x - 1, \tag{1.3}$$

the one for (1.2) is

$$p_n(x) = x^n - x^{n-1} - x - 1. \tag{1.4}$$

The characterization of the roots of the polynomial $f_{n,1}(x) = x^n - x^{n-1} - 1$, which is a particular case of (1.3), was studied by Kilic and Stakhov and Rozin in [5], [8], [9]. For a general $p \geqslant 2$, Trojovský proved in [11] the next theorem.

Theorem 1.1 ([11]). *For the integer numbers $n \geqslant 3$ and $p \geqslant 2$, the polynomial $f_{n,p}(x)$ defined in* (1.3) *has:*

(i) *a unique positive root, say $a_{n,p}$, and*

$$p < a_{n,p} < p + \frac{2}{p^{n-3}}.$$

Moreover, $\lim_{n\to\infty} a_{n,p} = p$ and $\lim_{p\to\infty} a_{n,p} = \infty$.

(ii) *a unique negative root if n is even.*

(iii) *two negative roots if n is odd and*

  (a) *$p = 3$ and $n \geqslant 7$,*

  (b) *$p \in \{4, 5, 6\}$ and $n \geqslant 5$, or*

  (c) *$p \geqslant 7$ and $n \geqslant 3$.*

(iv) *only simple roots.*

For the polynomial $p_n(x)$, Trojovský in [10] also proved the following theorem.

Theorem 1.2 ([10]). *For a given integer number $n \geqslant 2$, the polynomial $p_n(x)$ defined in* (1.4) *has:*

(i) *a unique positive root, say a_n, and*

$$1 < a_n < 1 + \sqrt{\frac{2}{n-1}}.$$

Moreover, $\lim_{n\to\infty} a_n = 1$.

(ii) *a unique negative root if n is even.*

(iii) *only simple roots.*

Additionally, in Theorem 1.2 it is also proved that the sequence (a_n) is strictly decreasing.

The aim of this note is to bring both theorems and all particular cases into a common ground, providing a new type of recurrence relation extending both (1.1) and (1.2). In the last section, we provide a determinantal interpretation for these sequences.

2. The roots

Our aim is to analyse the roots of the characteristic polynomial of the recurrence relation defined by

$$F_{k,p,q}(n) = pF_{k,p,q}(n-1) + qF_{k,p,q}(n-k+1) + F_{k,p,q}(n-k) \tag{2.1}$$

for $n \geqslant k$ and $p \geqslant q \geqslant 1$, which is

$$f(x) = x^n - px^{n-1} - qx - 1. \tag{2.2}$$

It contains both (1.3) and (1.4). In general, finding explicit solutions is a difficult task, as we can see in some related polynomials in [3].

We split our main result into several propositions, providing a common framework for future developments.

Proposition 2.1. *The polynomial $f(x)$ defined in (2.2) has only one positive root, say $a_{n,p,q}$, which satisfies $p < a_{n,p,q} < p+1$ for $n \geqslant 3$. Moreover, $\lim_{n\to\infty} a_{n,p,q} = p$ for any $p \geqslant q \geqslant 1$.*

Proof. First we claim that there is no root in $(0,p)$. The solution of the equation $f(x) = 0$ is equivalent to that of $x^{n-1} = (qx+1)/(x-p)$. When $x \in (0,p)$, we have $x^{n-1} > 0$, but $(qx+1)/(x-p) < 0$. So no $x \in (0,p)$ would lead to $x^{n-1} = (qx+1)/(x-p)$.

Next we claim that there is exactly one root in $[p,\infty)$. Since the derivative of f, defined by

$$f'(x) = x^{n-2}(nx - np + p) - q,$$

is an increasing function and $f'(p) = p^{n-1} - q \geqslant 0$ for $p \geqslant q \geqslant 1$ and $n \geqslant 3$, we conclude that $f'(x) > 0$ for $x > p$. This implies that there is at most one root in $[p,\infty)$. Actually, a root in $[p,\infty)$ does exist, by noting that $f(p) = -pq-1 < 0$ and

$$f(p+1) = (p+1)^{n-1} - (p+1)q - 1 \geqslant (p+1)^2 - (p+1)p - 1 = p > 0$$

for $n \geqslant 3$ and $p \geqslant q \geqslant 0$. So the unique root in $[p,\infty)$, say $a_{n,p,q}$, lies in $(p, p+1)$.

Furthermore, the unique positive root $a_{n,p,q}$ is infinitely close to p. In fact, for any sufficiently small positive constant $\delta > 0$, we have

$$f(p+\delta) = (p+\delta)^{n-1}\delta - (p+\delta)q - 1,$$

taking into account that $f(p+\delta) > 0$ is equivalent to

$$n > 1 + \frac{\ln((p+\delta)q+1) - \ln\delta}{\ln(p+\delta)},$$

which is clearly true for a sufficiently large n. This means that the unique positive root $a_{n,p,q}$ approaches to p as $n \to \infty$, independently of $q \geqslant 1$. □

In the next proposition, we analyse the roots in terms of the parity of n.

Proposition 2.2. *If n is even, then the polynomial $f(x)$ defined in* (2.2) *has only one negative root, which is in the interval* $(-1, 0)$.

If n is odd, the polynomial $f(x)$ has either none or exactly two negative roots, which are both in the interval $(-1, 0)$ (if they exist). In particular, when n is sufficiently large, $f(x)$ must have two negative roots in the interval $(-1, 0)$ when $q > 1$.

P r o o f. First assume that n is even. Set $g(x) = f(-x)$. We have

$$g(x) = f(-x) = (-x)^n - p(-x)^{n-1} + qx - 1 = x^n + px^{n-1} + qx - 1.$$

Clearly $g(x)$ increases in $x > 0$, and $g(0) = -1 < 0$, but $\lim\limits_{x\to\infty} g(x) = \infty$. So $g(x)$ has exactly one positive root. Equivalently, $f(x)$ has exactly one negative root. Actually, the unique positive root of $g(x)$ lies in $(0, 1)$, since $g(1) = p + q > 0$. This means that the unique negative root of $f(x)$ is in $(-1, 0)$.

Next assume that n is odd. Set $h(x) = -f(-x)$. This time we have

$$h(x) = -f(-x) = -((-x)^n - p(-x)^{n-1} + qx - 1) = x^n + px^{n-1} - qx + 1.$$

Note that

$$h'(x) = x^{n-2}(nx + np - p) - q,$$

which increases in $x > 0$. Together with $h'(0) = -q < 0$ and $\lim\limits_{x\to\infty} h'(x) = \infty$, we can deduce that $h'(x) = 0$ has exactly one root in $(0, \infty)$. As a consequence, $h(x)$ has at most two roots in $(0, \infty)$ (or, equivalently, $f(x)$ has at most two negative roots), since $h(0) = 1 > 0$ and $\lim\limits_{x\to\infty} h(x) = \infty$.

It is worth mentioning that $f(x)$ having only one negative root is impossible. Recall that $f(x)$ has only one positive root; from Proposition 2.1, the complex roots and their conjugates occur in pairs, thus an odd n implies that the number of negative roots must be even.

Furthermore, $h(x)$ has exactly two roots in $(0,\infty)$ if and only if $h(r) < 0$ for some $r \in (0,\infty)$. When n is large enough, such an r must exist. If $r \in (1/q, 1)$, then $qr - 1 > 0$ holds. Moreover, $r^n + pr^{n-1}$ approaches to 0 as n tends to infinity, which means that

$$qr - 1 > r^n + pr^{n-1},$$

or, equivalently, $h(r) < 0$ for n large enough. Actually, the two positive roots of $h(x)$ are in $(0,1)$ or, equivalently, $f(x)$ has two negative roots in $(-1,0)$, because $h(0) = 1 > 0$ and $h(1) = p - q + 2 > 0$, and $h(0) < h(1)$ in particular. □

Remark 2.1. Notice that how large n should be in Proposition 2.2 is determined by p and q.

Finally, we show that all roots are simple with an eventual exception.

Proposition 2.3. *All roots are simple, except whenever*

$$f\Big(-\frac{n + 2pq - npq + \sqrt{(n + 2pq - npq)^2 + 4pq(n-1)^2}}{2(n-1)q}\Big) = 0$$

for some odd n. In that eventuality,

$$-\frac{n + 2pq - npq + \sqrt{(n + 2pq - npq)^2 + 4pq(n-1)^2}}{2(n-1)q}$$

is of multiplicity 2 and all other roots remain simple.

Proof. Suppose to the contrary that $f(x)$ has a root, say ε, with multiplicity at least 2. In this sense, $f(\varepsilon) = f'(\varepsilon) = 0$, i.e.,

$$f(\varepsilon) = \varepsilon^n - p\varepsilon^{n-1} - q\varepsilon - 1 = \varepsilon^{n-1}(\varepsilon - p) - q\varepsilon - 1 = 0$$

and

$$f'(\varepsilon) = \varepsilon^{n-2}(n\varepsilon - np + p) - q = 0.$$

This means that ε is a (nonzero) root of the quadratic equation on x:

$$x(x-p)q = (nx - np + p)(qx + 1),$$

or, equivalently,

$$(n-1)qx^2 - (npq - 2pq - n)x - p(n-1) = 0.$$

Now, let

$$l(x) = (n-1)qx^2 - (npq - 2pq - n)x - p(n-1).$$

From $l(-1/q) = -p - 1/q < 0$, we know that $l(x)$ has two real roots (which means that ε must be a real number); the smaller one is just ε (from Propositions 2.1 and 2.2), i.e.,

$$\varepsilon = -\frac{n + 2pq - npq + \sqrt{(n + 2pq - npq)^2 + 4pq(n-1)^2}}{2(n-1)q},$$

for some odd n. □

3. A matricial interpretation

Is it interesting that the recurrence (2.1) and, consequently, all particular cases can be interpreted in terms of Hessenberg matrices. In fact, it is known (cf., e.g., [4], [6]) that if $a_1, a_2, \dots$ is a sequence such that

$$a_{n+1} = p_{1,n}a_1 + \dots + p_{n,n}a_n,$$

then

$$a_{n+1} = a_1 \det \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 & p_{n,n} \end{pmatrix}.$$

For example, setting $k = 4$ in (2.1), we have

$$F_{4,p,q}(n) = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & & & \\ -1 & 0 & 0 & 0 & 1 & & \\ & -1 & 0 & 0 & q & 1 & \\ & & -1 & 0 & 0 & q & \ddots \\ & & & -1 & p & 0 & \ddots \\ & & & & -1 & p & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}_{(n+1)\times(n+1)}.$$

We believe that this representation might be useful in distinct settings by using the determinant properties. Of course, $F_{k,p,q}(n)$ can be interpreted in terms of the permanent (for more details on this and other interpretations, the reader is referred to [2], [12]).

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