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DISCRETIZATION OF PRIME COUNTING FUNCTIONS, CONVEXITY AND THE RIEMANN HYPOTHESIS

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Abstract. We study tails of prime counting functions. Our approach leads to representations with a main term and an error term for the asymptotic size of each tail. It is further shown that the main term is of a specific shape and can be written discretely as a sum involving probabilities of certain events belonging to a perturbed binomial distribution. The limitations of the error term in our representation give us equivalent conditions for various forms of the Riemann hypothesis, for classical type zero-free regions in the case of the Riemann zeta function and the size of semigroups of integers in the sense of Beurling. Inspired by the works of Panaitopol, asymptotic companions pertaining to the magnitude of specific prime counting functions are obtained in terms of harmonic numbers, hyperharmonic numbers and the number of indecomposable permutations. By introducing the notion of asymptotic convexity and fusing it with a nice generalization of an inequality of Ramanujan due to Hassani, we arrive at a curious asymptotic inequality for the classical prime counting function at any convex combination of its arguments and further show that quotients arising from prime counting functions of progressions furnish examples of asymptotically convex, but not convex functions.

Keywords: prime counting function; discretization; Riemann hypothesis; harmonic number; indecomposable permutation; asymptotic convexity

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1. Introduction

Let $\pi(x)$ be the number of prime numbers that are not greater than x. Finding the asymptotic behavior of $\pi(x)$ with enough precision has always been a central topic of interest in number theory and analysis. The celebrated prime number theorem with the classical error term asserts that

$$\pi(x) = \operatorname{li} x + O(x\mathrm{e}^{-c\sqrt{\log x}}) \tag{1.1}$$

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for some constant $c > 0$ as $x \to \infty$, where

$$\operatorname{li} x = \int_2^x \frac{1}{\log t}\,\mathrm{d}t$$

is the logarithmic integral. In the classical approach to prime number theory, a very fruitful idea, due to Chebyshev, is to linearize the distribution by introducing the Chebyshev functions

$$\vartheta(x) = \sum_{p \leqslant x} \log p, \quad \psi(x) = \sum_{n \leqslant x} \Lambda(n),$$

where p denotes a prime number throughout, and $\Lambda(n)$ is the von Mangoldt function. This offers the key option of connecting the distribution to its generating function via the Dirichlet series identity

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for $\Re(s) > 1$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1$$

is the Riemann zeta function. One then obtains an equivalent formulation of the prime number theorem in the linearized form

$$\psi(x) = x + O(x\mathrm{e}^{-c\sqrt{\log x}}) \tag{1.2}$$

for some constant $c > 0$ (throughout, c might represent a different constant in each occurrence) as $x \to \infty$. Another striking outcome of this approach is the equivalence of the Riemann hypothesis (abbreviated as RH from now on) to the formula

$$\psi(x) = x + O(x^{1/2+\varepsilon})$$

for every $\varepsilon > 0$. RH is also known to be equivalent to the estimate

$$M(x) = \sum_{n \leqslant x} \mu(n) = O(x^{1/2+\varepsilon})$$

for every $\varepsilon > 0$, where $\mu(n)$ is the Möbius function and $M(x)$ is the Mertens function. For a study of biases in the behavior of weighted Mertens sums, we refer to [4]. Constant components of $M(x)$ were recently discovered by Camargo and Martin (see [15]) who employed them to confirm a suspicion, first mentioned by Sylvester, about the difficulty of finding harmonic systems for the count of prime numbers. Other connections between RH and various sorts of oscillating sums were studied in [2], [3],

[5], [6]. Despite all of these, formula (1.1) can still claim a number of advantages over (1.2). First, the equivalence of formulas involving $\pi(x)$ and $\psi(x)$ breaks down in the case of Ω-type results for the detection of oscillations around the main term. Although it is relatively easier to show that $\psi(x)$ oscillates around x (see [35]), the analogous problem for the oscillation of $\pi(x)$ around $\operatorname{li} x$ turned out to be much more difficult. Overcoming a remarkable range of technical difficulties, Littlewood in [24] disproved a prediction, usually attributed to Gauss, that

$$\pi(x) < \operatorname{li} x$$

holds for all large x. Second, the main term $\operatorname{li} x$ in (1.1) is intrinsically loaded with a rich collection of function theoretic properties. Since it is known that $1/\log t$ does not have an elementary antiderivative, there is no closed form evaluation of $\operatorname{li} x$, and consequently it is a transcendental function. To compensate with this defect, $\operatorname{li} x$ allows us to see infinitely many phases of the distribution. Precisely, by repeated applications of partial integration, we have

$$\operatorname{li} x = \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} + O\Big(\frac{x}{(\log x)^{m+1}}\Big) \tag{1.3}$$

for every nonnegative integer m (the case $m = 0$ refers to no application of partial integration, so that the sum over k in (1.3) is taken to be zero). Therefore, using (1.3), (1.1) becomes

$$\pi(x) = \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} + O\Big(\frac{x}{(\log x)^{m+1}}\Big) \tag{1.4}$$

for every nonnegative integer m (again, the case $m = 0$ should be taken as to correspond to the Chebyshev type estimate $\pi(x) = O(x/\log x)$ using the elementary estimate $\operatorname{li} x \ll x/\log x$). Lastly, formulas like (1.1) become handy for providing a partial answer to the problem of finding an exact representation of the nth prime number p_n, since we may write

$$n = \pi(p_n) = \operatorname{li}(p_n + e_n),$$

where e_n represents a relatively small error that results from the O-term in (1.1). But then

$$p_n = \operatorname{li}^{-1} n - e_n \tag{1.5}$$

holds, where li^{-1} is the inverse function of li. Of course, we can only have a limited satisfaction with (1.5), since it is still a largely open problem to describe the nature of e_n and control its size with the help of an exact analytical formula even under the assumption of RH. In his famous memoir, Riemann in [34] made a fundamental breakthrough on this question which, at least from a heuristic standpoint, achieves this goal. We should remark here that (1.4) improves itself to an asymptotic expansion

$$\pi(x) \sim \sum_{k=0}^{\infty} \frac{k!\, x}{(\log x)^{k+1}}$$

which means that

$$\pi(x) - \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} = \frac{(1+o(1))m!\, x}{(\log x)^{m+1}} \tag{1.6}$$

holds when $x \to \infty$ for every $m \geqslant 0$. It is worth mentioning at this point that a serious weakness of (1.3) is its inadequacy for numerical approximations of $\operatorname{li} x$ for fixed x. For approximations of $\operatorname{li} x$ when x is fixed, one can either use the fact that the function series

$$\log\log t + \sum_{n=1}^{\infty} \frac{(\log t)^n}{n \cdot n!}$$

is an antiderivative for $1/\log t$ or directly refer to truncations of a rapidly converging function series representation discovered by Ramanujan (see [9], pages 126–131) in the form

$$\operatorname{Li}(x) := \int_0^x \frac{1}{\log t}\, \mathrm{d}t = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \right) \frac{1}{\log t}\, \mathrm{d}t = \gamma + \log\log x + \sqrt{x} \sum_{n=1}^{\infty} b_n (\log x)^n,$$

where the coefficients are defined by

$$b_n = \frac{(-1)^{n-1}}{n!\, 2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1}, \quad \text{and} \quad \gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.5772\ldots$$

is the Euler-Mascheroni constant. Formula (1.6) motivates us to asymptotically study the tails of the distribution

$$\Pi_m(x) := \pi(x) - \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}}$$

arising from the infinitely many phases indexed by m. In particular, our approach offers the possibility of getting various discrete forms for the main contributions to prime counting functions and their tails with the help of harmonic and hyperharmonic numbers which are combinatorially significant quantities. The nth harmonic number is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geqslant 1$, where by convention, $H_0 = 0$. Harmonic numbers arise naturally as probabilities of certain events resulting from perturbations of the binomial distribution. They also arise from expected values of specific random variables like as in the coupon collector's problem. The error term for the tail of the distributions comes in different guises giving us equivalent conditions at one stroke for RH, generalized Riemann hypothesis (GRH), quasi-Riemann hypothesis, zero-free regions and the distribution of Beurling type integers. Let us make a digression to give some of the ingredients necessary to formulate the statements in our first result. For any $\frac{1}{2} \leqslant \theta < 1$, quasi-Riemann hypothesis at θ, denoted as $\mathrm{RH}(\theta)$, claims that the real part of any zero of the Riemann zeta function is not greater than θ. The quasi-generalized Riemann hypothesis, denoted as $\mathrm{GRH}(\theta)$, claims the same inequality for the real part of any zero of Dirichlet L-functions $L(s, \chi)$ modulo q, where

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\Re(s) > 1$, and χ is any character modulo q. For a generalization of Dirichlet L-functions to the setting with periodic coefficients, see [1]. If $s = \sigma + \mathrm{i}t$ is a zero of $\zeta(s)$, then a classical zero-free region for $\zeta(s)$ is of the type

$$\sigma > 1 - \frac{c}{(\log |t|)^{\alpha}} \tag{1.7}$$

for some $\alpha \leqslant 1$ and $c > 0$ when $|t| > c$. By the Vinogradov-Korobov type zero-free region, we know that any $\alpha > \frac{2}{3}$ is available in (1.7), see [38]. If P is any set of prime numbers, then the Beurling type integers generated by P are defined to be the semigroup $\langle P \rangle$, where the prime factorization of any number in $\langle P \rangle$ consists of only primes from P. The Euler-Mascheroni constant γ is subject to the asymptotic formula

$$H_n = \log n + \gamma + O\Big(\frac{1}{n}\Big)$$

as $n \to \infty$. We can now state our first contribution.

Theorem 1. *Consider n independent trials of an experiment, where each trial results in success or failure. The probability of failure at the jth trial is assumed to be $1/(j+1)$ for $j \geqslant 1$. Let β_n denote the probability of the event that there is only one failure among the n attempts of the experiment. Then the following statements hold.*

(i) *For any given nonnegative integer m,* RH *is equivalent to the formula*

$$\Pi_m(x) = \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x^{1/2} \log x) \tag{1.8}$$

as $x \to \infty$.

(ii) *For any given nonnegative integer* m *and* $\frac{1}{2} < \theta < 1$, $\mathrm{RH}(\theta)$ *is equivalent to the formula*

$$\Pi_m(x) = \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O\Big(\frac{x^{\theta}}{\log x}\Big). \tag{1.9}$$

(iii) *For any given nonnegative integer* m *and* $\frac{1}{2} \leqslant \theta < 1$, $\mathrm{GRH}(\theta)$ *for all Dirichlet* L*-functions modulo* q *is equivalent to the formula*

$$\Pi_m(x, q, a) = \frac{\mathrm{e}^{\gamma} m!}{\varphi(q)} \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x^{\theta} \log x) \tag{1.10}$$

for all $1 \leqslant a \leqslant q$ *with* $(a, q) = 1$, *where*

$$\Pi_m(x, q, a) := \pi(x, q, a) - \frac{1}{\varphi(q)} \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}},$$

where $\varphi(q)$ *is Euler's function and* $\pi(x, q, a)$ *is the number of primes not greater than* x *that are equivalent* $a \pmod q$.

(iv) *For any given nonnegative integer* m *and* $0 < \alpha \leqslant 1$, *the existence of a zero-free region for* $\zeta(s)$ *in the form*

$$\sigma > 1 - \frac{c}{(\log |t|)^{\alpha}}$$

for large $|t|$ *is equivalent to the formula*

$$\Pi_m(x) = \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x\mathrm{e}^{-c(\log x)^{1/(1+\alpha)}}) \tag{1.11}$$

for some constant $c > 0$.

(v) *Let* P *be a set of prime numbers and let* $\pi_P(x)$ *and* $N_P(x)$ *be the counting functions of* P *and* $\langle P \rangle$, *respectively. If*

$$N_P(x) = bx + O(x\mathrm{e}^{-c(\log x)^{a}}) \tag{1.12}$$

holds for some constants $0 < b \leqslant 1$, $0 < a \leqslant 1$ *and* $c > 0$ *as* $x \to \infty$, *then for any given nonnegative integer* m,

$$\pi_P(x) - \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} = \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x\mathrm{e}^{-c(\log x)^{a'}}) \tag{1.13}$$

holds for some $c > 0$, *where* $a' = a/10$. *Conversely, if* (1.13) *holds for some nonnegative integer* m, $c > 0$ *and* $a' > 0$, *then* (1.12) *holds with* $a = a'/(a' + 2)$.

(vi) *For any given nonnegative integer* m, *we have*

$$(1.14)\quad \Pi_m(x) = \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x\mathrm{e}^{-(0.2098)(\log x)^{3/5}/(\log\log x)^{1/5}}).$$

For any given nonnegative integer m *and* $A > 0$, *if* $1 \leqslant a \leqslant q \leqslant (\log x)^A$ *and* $(a,q) = 1$, *then we have*

$$\Pi_m(x,q,a) = \frac{\mathrm{e}^{\gamma} m!}{\varphi(q)} \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x\mathrm{e}^{-c(\log x)^{1/2}}) \tag{1.15}$$

for some $c > 0$ *depending on* A, *where the* O*-constant in* (1.15) *is ineffective.*

Hyperharmonic numbers were first introduced by Conway and Guy (see [13]) as an iterative generalization of harmonic numbers. Precisely, they define the nth hyperharmonic number of order r recursively as

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

for $r \geqslant 1$, where $H_n^{(0)} = n^{-1}$. In particular, $H_n = H_n^{(1)}$. A nice combinatorial approach supporting the significance of hyperharmonic numbers is due to Benjamin, Gaebler and Gaebler, see [8]. Let us say that a permutation τ of $\{1,2,\dots,n\}$ is indecomposable if

$$\tau(\{1,2,\dots,m\}) = \{1,2,\dots,m\}$$

holds for no $m < n$. Comtet in [11], [12] proved that almost all permutations of $\{1,2,\dots,n\}$ are indecomposable. Precisely, he showed that

$$\lim_{n\to\infty} \frac{a_n}{n!} = 1,$$

where a_n is the number of such permutations. He also obtained the recurrence formula

$$a_n = n! - \sum_{i=1}^{n-1} (n-i)!\, a_i, \tag{1.16}$$

and an expansion of $a_n/n!$ in terms of rising factorials $(n)_k = n(n-1)\dots(n-k+1)$. An elaboration on this is given in the proof of Theorem 2 below. Indecomposable permutations turned out to be useful in coding theory, especially in generating Gray codes. For details of this, we refer the reader to an interesting paper of King, see [23]. A few of the initial values are given below:

$$a_1 = 1 = a_2, \quad a_3 = 3, \quad a_4 = 13, \quad a_5 = 71, \quad a_6 = 461, \quad a_7 = 3447, \quad a_8 = 29093.$$

Our next result gives a family of asymptotic representations of prime counting functions in terms of hyperharmonic numbers and the number of indecomposable permutations. Representations and inequalities pertaining to $\pi(x)$ were studied extensively in the literature from a function theoretic point of view, most notably by Panaitopol, (see [28], [29], [31], also the papers of Mincu and Panaitopol [26] and Mititica and Panaitopol [27]) who was a pioneer in the elementary treatments of prime counting functions. Inspired by his work, we have the following result, which applies to the counting functions of sets of prime numbers under quite general conditions.

Theorem 2. *Let P be any set of prime numbers whose counting function satisfies the asymptotic formula*

$$\pi_P(x) = c\pi(x^\delta) + o\Big(\frac{x^\delta}{(\log x)^2}\Big)$$

for some constants $0 < c \leqslant 1$ and $0 < \delta \leqslant 1$ as $x \to \infty$. If the function $A_P(x)$ is defined by the relation

$$\pi_P(x) = \frac{cx^\delta}{\delta \log x - A_P(x)}$$

for all large enough x, then we always have

$$\lim_{x\to\infty} A_P(x) = 1. \tag{1.17}$$

If P is a set of prime numbers such that

$$\pi_P(x) = c\pi(x^\delta) + O\Big(\frac{x^\delta}{(\log x)^m}\Big) \tag{1.18}$$

holds for some integer $m \geqslant 3$, then for any given $r \geqslant 1$, we have

$$\begin{aligned}\pi_P(n) = {} & \Big(1 + O\Big(\frac{1}{(\log n)^{m-2}}\Big)\Big)cn^\delta \\ & \times \Bigg(\frac{\delta H^{(r)}_{n-r+1}}{\binom{n}{r-1}} + \delta H_{r-1} - \delta\gamma - \sum_{k=1}^{m-2} \frac{a_k}{(\delta \log n)^{k-1}}\Bigg)^{-1}\end{aligned} \tag{1.19}$$

as $n \to \infty$, where $H^{(r)}_n$ is the nth hyperharmonic number of order r and a_k is the number of indecomposable permutations of $\{1, 2, \ldots, k\}$ satisfying

$$\frac{k!}{2} \leqslant a_k \leqslant k!$$

for all $k \geqslant 1$ and

$$a_k = \Big(1 - \frac{2}{k} - \frac{1}{k^2} - \frac{5}{k^3} - \frac{32}{k^4} + O\Big(\frac{1}{k^5}\Big)\Big)k!$$

asymptotically when $k \to \infty$.

Let us remark that one can take P as the set of prime numbers belonging to any progression equivalent $a \pmod q$ with $(a, q) = 1$ in Theorem 2 by choosing $c = 1/\varphi(q)$ and $\delta = 1$. Then (1.18) holds for all $m \geqslant 2$ as a result of the prime number theorem for arithmetic progressions in the form

$$\pi(x, q, a) = \frac{\pi(x)}{\varphi(q)} + O(x\mathrm{e}^{-c\sqrt{\log x}})$$

for some $c > 0$. Therefore, (1.19) holds. Moreover, the correction function $A_P(x)$ is a generalization of the correction term historically first introduced by Legendre who suggested the empirical representation

$$\frac{x}{\log x - 1.08366}$$

as a good measure for $\pi(x)$. For measures of counting functions of sequences other than primes, we refer to [7] and the references therein. Beyond mere asymptotic representations of prime counting functions, there are various striking inequalities related to $\pi(x)$. A very explicit inequality of Schoenfeld (see [36]) given as

$$|\pi(x) - \operatorname{li} x| < \frac{1}{8\pi}\sqrt{x}\log x \quad \text{for all } x \geqslant 2657,$$

is equivalent to RH. Exploiting function theoretic properties arising from expansions of $\pi(x)$, Panaitopol in [30], [32], [33] and Hassani in [18] obtained a rich collection of inequalities and approximations involving values of $\pi(x)$ at different arguments. Panaitopol in [31] also derived the expansion

$$\frac{1}{\pi(ax)} = \frac{\log x}{ax} + \frac{\log a - 1}{ax} + \sum_{i=1}^{m} \frac{c_i(a)}{x(\log x)^i} + o\Big(\frac{1}{x(\log x)^m}\Big)$$

from which he concluded that the functions

$$\frac{x}{\pi(x)}, \quad \frac{1}{\pi(x)}$$

are not convex or concave when $x \geqslant x_1$. These results were greatly extended to prime numbers in progressions by Cobeli et al., see [10]. In light of [10], we say that a function f is asymptotically convex if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + o(1)$$

holds for any $0 \leqslant \lambda \leqslant 1$ as $x, y \to \infty$. Obviously, convex functions are asymptotically convex. Our final result gives a family of functions that are asymptotically convex but not convex. Although one can show the existence of such functions by making minuscule alterations in the values of a linear function, our examples directly relate to prime counting functions and connect us to the findings of [10]. As a byproduct of asymptotic convexity together with a classical result of Ramanujan, and its strengthening by Hassani (see [19]), we obtain a curious inequality concerning values of $\pi(x)$ at any convex combination of its arguments.

Theorem 3.

(i) *For* $1 \leqslant a \leqslant q$, $(a,q)=1$, *the function*

$$-\frac{x}{\pi(x,q,a)}$$

is not convex but asymptotically convex on any interval $x \geqslant x_1$.

(ii) *For any scalars* $0 \leqslant \lambda \leqslant 1$, $\alpha \geqslant e$ *and any positive integer* n, *we have*

$$\left(\frac{\pi(\lambda x+(1-\lambda)y)}{\lambda x+(1-\lambda)y}\right)^2 < \frac{\lambda\alpha\pi(x/\alpha)}{x\log x} + \frac{(1-\lambda)\alpha\pi(y/\alpha)}{y\log y} + o\Big(\frac{1}{(\log\min(x,y))^n}\Big) \tag{1.20}$$

when $x, y \to \infty$.

In particular, taking $\lambda = \frac{1}{2}$ in Theorem 3, we deduce the more symmetric midpoint version of (1.20) in the form

$$\frac{\pi(x+y)}{x+y} < \frac{\sqrt{\alpha}}{2}\left(\sqrt{\frac{\pi(2x/\alpha)}{x\log 2x}} + \sqrt{\frac{\pi(2y/\alpha)}{y\log 2y}} + o\Big(\frac{1}{(\log\min(x,y))^n}\Big)\right)$$

when $x, y \to \infty$ for any $\alpha \geqslant e$ and positive integer n.

2. Preliminaries

In this section, we collect all of the ingredients that will be necessary for the proofs of our claims. Our first result supplies the transition from a prime counting function of a progression to its Chebyshev functions and vice versa under a fairly general range of error terms.

Lemma 1. *Assume* $1 \leqslant a \leqslant q$ *and* $(a,q)=1$, *and let* $r(x)$ *be any monotonically increasing function such that*

$$\frac{x^{1/2}}{\log x} \ll r(x) \ll \frac{x}{\log x\log\log x} \tag{2.1}$$

as $x \to \infty$. *Then the formulas*

$$\pi(x,q,a) = \frac{\operatorname{li} x}{\varphi(q)} + O(r(x)) \tag{2.2}$$

and

$$\psi(x,q,a) := \sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}} \Lambda(n) = \frac{x}{\varphi(q)} + O(r(x)\log x) \tag{2.3}$$

are equivalent as $x \to \infty$.

Lemma 1 shows that for a reasonably general span of error terms in the prime counting formula, the error term of the linearization always inflates by a factor of $\log x$ as expected. The lower bound for $r(x)$ in (2.1) is optimal in some sense, since already in the case when

$$r(x) \asymp \frac{x^{1/2}}{\log x},$$

Littlewood in [24] showed that

$$\pi(x) - \mathrm{li}\, x = \Omega_{\pm}\Big(\frac{x^{1/2}\log\log\log x}{\log x}\Big) \quad \text{and} \quad \psi(x) - x = \Omega_{\pm}(x^{1/2}\log\log\log x).$$

Therefore, both of (2.2) and (2.3) (in the case $a = q = 1$) become false. However, their equivalence is a triviality in this case. On the other side, the upper bound for $r(x)$ in (2.1) is close to being optimal as well since it would be reasonable to have

$$r(x) = o\Big(\frac{x}{\log x}\Big)$$

because of $\mathrm{li}\, x \sim x/\log x$. Although q is kept fixed in the statement of Lemma 1, we should remark that, as the proof below shows, Lemma 1 is also valid when q is allowed to grow with x. Thus, we may still use this result uniformly for all q bounded by a monotonically growing function of x. Though, for reasonable applications, we have to assume that

$$\frac{x^{1/2}}{\log x} \ll r(x) = o\Big(\frac{\mathrm{li}\, x}{\varphi(q)}\Big).$$

This will be of substance in the proof of (1.15) when $q \leqslant (\log x)^A$.

Proof of Lemma 1. First assume (2.2). Then by partial summation, we have

$$\vartheta(x,q,a) := \sum_{\substack{p\leqslant x\\ p\equiv a \pmod q}} \log p = \pi(x,q,a)\log x - \int_2^x \frac{\pi(t,q,a)}{t}\,\mathrm{d}t. \tag{2.4}$$

From (2.2), we may write

$$\pi(x,q,a)\log x = \frac{\mathrm{li}\, x\log x}{\varphi(q)} + O(r(x)\log x), \tag{2.5}$$

and

$$\int_2^x \frac{\pi(t,q,a)}{t}\,\mathrm{d}t = \frac{1}{\varphi(q)}\int_2^x \frac{\mathrm{li}\, t}{t}\,\mathrm{d}t + O\Big(\int_2^x \frac{r(t)}{t}\,\mathrm{d}t\Big). \tag{2.6}$$

Next, we have

$$\frac{1}{\varphi(q)}\int_2^x \frac{\mathrm{li}\, t}{t}\,\mathrm{d}t = \frac{\mathrm{li}\, x\log x}{\varphi(q)} - \frac{x-2}{\varphi(q)}. \tag{2.7}$$

As $r(x)$ is monotonically increasing, we see that

$$\int_2^x \frac{r(t)}{t}\,\mathrm{d}t = O\biggl(r(x)\int_2^x \frac{1}{t}\,\mathrm{d}t\biggr) = O(r(x)\log x). \tag{2.8}$$

Combining (2.4)–(2.8), one infers that

$$\vartheta(x,q,a) = \frac{x}{\varphi(q)} + O(r(x)\log x). \tag{2.9}$$

We know that

$$\psi(x,q,a) = \vartheta(x,q,a) + O(x^{1/2}). \tag{2.10}$$

Since, by (2.1), we have $r(x)\log x \gg x^{1/2}$ so that (2.3) follows from (2.9) and (2.10).

Conversely, assume (2.3). This time we use the auxiliary function

$$\pi_1(x,q,a) := \sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}} \frac{\Lambda(n)}{\log n} = \pi(x,q,a) + \sum_{\substack{p^m\leqslant x\\ m\geqslant 2\\ p^m\equiv a \pmod q}} \frac{1}{m}. \tag{2.11}$$

Using Chebyshev's estimates, it is plain from (2.11) that

$$\pi(x,q,a) = \pi_1(x,q,a) + O\Bigl(\frac{x^{1/2}}{\log x}\Bigr). \tag{2.12}$$

But by partial summation, one obtains that

$$\pi_1(x,q,a) = \frac{\psi(x,q,a)}{\log x} + \int_2^x \frac{\psi(t,q,a)}{t(\log t)^2}\,\mathrm{d}t. \tag{2.13}$$

From (2.3), we have

$$\frac{\psi(x,q,a)}{\log x} = \frac{x}{\varphi(q)\log x} + O(r(x)) \tag{2.14}$$

and

$$\int_2^x \frac{\psi(t,q,a)}{t(\log t)^2}\,\mathrm{d}t = \frac{1}{\varphi(q)}\int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t + O\biggl(\int_2^x \frac{r(t)}{t\log t}\,\mathrm{d}t\biggr). \tag{2.15}$$

First, note that

$$\frac{1}{\varphi(q)}\int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t = \frac{\mathrm{li}\, x}{\varphi(q)} - \frac{x}{\varphi(q)\log x} + O(1). \tag{2.16}$$

Second, let us use the decomposition

$$\int_2^x \frac{r(t)}{t\log t}\,\mathrm{d}t = \int_2^{\sqrt{x}} \frac{r(t)}{t\log t}\,\mathrm{d}t + \int_{\sqrt{x}}^x \frac{r(t)}{t\log t}\,\mathrm{d}t. \tag{2.17}$$

Clearly, by (2.8), we have

$$\int_{\sqrt{x}}^x \frac{r(t)}{t\log t}\,\mathrm{d}t \ll \frac{1}{\log x}\int_2^x \frac{r(t)}{t}\,\mathrm{d}t \ll r(x). \tag{2.18}$$

Moreover, using (2.1), one estimates that

$$\int_2^{\sqrt{x}} \frac{r(t)}{t\log t}\,\mathrm{d}t \ll r(\sqrt{x})\log\log x \ll \frac{\sqrt{x}}{\log x} \ll r(x). \tag{2.19}$$

Patching up the estimates in (2.18) and (2.19), we deduce from (2.15)–(2.17) that

$$\int_2^x \frac{\psi(t,q,a)}{t(\log t)^2}\,\mathrm{d}t = \frac{\operatorname{li} x}{\varphi(q)} - \frac{x}{\varphi(q)\log x} + O(r(x)). \tag{2.20}$$

From (2.13), (2.14) and (2.20), we have

$$\pi_1(x,q,a) = \frac{\operatorname{li} x}{\varphi(q)} + O(r(x)). \tag{2.21}$$

Finally, (2.2) follows from (2.12) and (2.21), again by using (2.1). This completes the proof. $\square$

Our final preliminary result will be needed more than once in the sequel, and it confirms the asymptotic convexity of a family of functions satisfying fairly general conditions on their asymptotic expansions which are reminiscent of prime counting functions.

Lemma 2. *For a positive integer m, let $\{c_k\}_{k=-m}^{\infty}$ be a sequence of real numbers such that*

$$c_k \begin{cases} \leqslant 0 & \text{if } k \leqslant -1, \\ \geqslant 0 & \text{if } k \geqslant 0. \end{cases}$$

Consider a function f having an asymptotic expansion of the form

$$f(x) \sim \sum_{k=-m}^{\infty} \frac{c_k}{(\log x)^k}.$$

Then f is asymptotically convex. In particular, for any $0 \leqslant \lambda \leqslant 1$ and positive integer n,

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) + o\Big(\frac{1}{(\log\min(x,y))^n}\Big) \tag{2.22}$$

holds when $x, y \to \infty$.

P r o o f of Lemma 2. To start with, because of the asymptotic expansion of f, we may write for every positive integer n that

$$f(x) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k} = \frac{c_n + o(1)}{(\log x)^n} \tag{2.23}$$

as $x \to \infty$. Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\log x)^{-k} = kx^{-2}(\log x)^{-k-1}(1 + (k+1)(\log x)^{-1}). \tag{2.24}$$

It follows from (2.24) and our assumption on c_k that

$$\frac{c_k}{(\log x)^k} \tag{2.25}$$

is a convex function when $k \geqslant 0$ and $x > 1$. Moreover, when $-m \leqslant k \leqslant -1$, we have for all large enough x (in terms of m) that

$$1 + (k+1)(\log x)^{-1} > 0. \tag{2.26}$$

It follows from (2.24), (2.26) and our assumption on c_k that (2.25) is again a convex function for all large enough x. Since convex functions are closed under finite sums, we see that

$$\sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k} \tag{2.27}$$

is convex for all large enough x in terms of m. To show (2.22), define for $0 \leqslant \lambda \leqslant 1$,

$$g(x, y, \lambda) := \lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda)y).$$

Our task is to show that $g(x, y, \lambda) \geqslant 0$ asymptotically at the expense of an $o(1)$ term which we make more explicit. To this end, we rewrite $g(x, y, \lambda)$ in the form

$$\begin{aligned} \lambda\Bigl(f(x) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k}\Bigr) &+ (1-\lambda)\Bigl(f(y) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log y)^k}\Bigr) \\ &+ \sum_{k=-m}^{n-1} \frac{c_k}{(\log(\lambda x + (1-\lambda)y))^k} - f(\lambda x + (1-\lambda)y) \\ &+ \lambda \sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k} + (1-\lambda) \sum_{k=-m}^{n-1} \frac{c_k}{(\log y)^k} \\ &- \sum_{k=-m}^{n-1} \frac{c_k}{(\log(\lambda x + (1-\lambda)y))^k}. \end{aligned} \tag{2.28}$$

First, by the convexity of the function in (2.27), we have

$$\lambda \sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k} + (1-\lambda) \sum_{k=-m}^{n-1} \frac{c_k}{(\log y)^k} - \sum_{k=-m}^{n-1} \frac{c_k}{(\log(\lambda x + (1-\lambda)y))^k} \geqslant 0 \tag{2.29}$$

for all large enough x, y. Therefore, we can dispense with (2.29), and assume that it does not affect the asymptotic nonnegativity of (2.28). Moreover, applying (2.23), we further get

$$\lambda\Big(f(x) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log x)^k}\Big) = \frac{\lambda c_n + o(1)}{(\log x)^n}, \tag{2.30}$$

$$(1-\lambda)\Big(f(y) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log y)^k}\Big) = \frac{(1-\lambda) c_n + o(1)}{(\log y)^n}, \tag{2.31}$$

$$f(\lambda x + (1-\lambda)y) - \sum_{k=-m}^{n-1} \frac{c_k}{(\log(\lambda x + (1-\lambda)y))^k} = \frac{c_n + o(1)}{(\log(\lambda x + (1-\lambda)y))^n} \tag{2.32}$$

when $x, y \to \infty$. Assembling (2.28) with (2.30)–(2.32), we are left to consider only

$$\frac{\lambda c_n + o(1)}{(\log x)^n} + \frac{(1-\lambda) c_n + o(1)}{(\log y)^n} - \frac{c_n + o(1)}{(\log(\lambda x + (1-\lambda)y))^n}. \tag{2.33}$$

However, (2.33) equals

$$\begin{aligned}&\frac{\lambda c_n}{(\log x)^n} + \frac{(1-\lambda) c_n}{(\log y)^n} - \frac{c_n}{(\log(\lambda x + (1-\lambda)y))^n} \\ &\qquad + o\Big(\frac{1}{(\log x)^n} + \frac{1}{(\log y)^n} + \frac{1}{(\log(\lambda x + (1-\lambda)y))^n}\Big).\end{aligned} \tag{2.34}$$

Again by convexity, we have

$$\frac{\lambda c_n}{(\log x)^n} + \frac{(1-\lambda) c_n}{(\log y)^n} - \frac{c_n}{(\log(\lambda x + (1-\lambda)y))^n} \geqslant 0. \tag{2.35}$$

Also note that

$$o\Big(\frac{1}{(\log x)^n} + \frac{1}{(\log y)^n} + \frac{1}{(\log(\lambda x + (1-\lambda)y))^n}\Big) = o\Big(\frac{1}{(\log \min(x,y))^n}\Big). \tag{2.36}$$

Finally, using (2.34)–(2.36), we infer that

$$g(x, y, \lambda) + o\Big(\frac{1}{(\log \min(x,y))^n}\Big) \geqslant 0$$

for every positive integer n when $x, y \to \infty$. This finishes the proof. □

3. Proof of Theorem 1

We start by making a surgery on $\operatorname{li} x$ in two main stages. Once this is accomplished, we can then obtain all parts of Theorem 1 at one stroke by relating our representation for $\operatorname{li} x$ to the theory of L-functions. Though it sounds simple in principle, this task requires us to deal with some technical details. First, by a simple elaboration on (1.3), we obtain by repeated use of partial integration that

$$\operatorname{li} x = \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} + m! \int_2^x \frac{1}{(\log t)^{m+1}}\, \mathrm{d}t + O(1) \tag{3.1}$$

for every given nonnegative integer m, where the O-term in (3.1) (depending on m) collects the total accumulation of boundary evaluations at $t = 2$ each time we apply integration by parts. Next we have

$$m! \int_2^x \frac{1}{(\log t)^{m+1}}\, \mathrm{d}t = \mathrm{e}^{\gamma} m! \int_{2/\mathrm{e}^{\gamma}}^{x/\mathrm{e}^{\gamma}} \frac{1}{(\gamma + \log t)^{m+1}}\, \mathrm{d}t. \tag{3.2}$$

A continuous generalization of harmonic numbers of the shape

$$H_t := \sum_{n \leqslant t} \frac{1}{n} \tag{3.3}$$

for any real number $t \geqslant 1$ would become handy for us. Note that (3.3) defines a monotonically increasing step function with jump discontinuities at integers not less than 2, and is subject to the formula

$$H_t = \log t + \gamma + O\Big(\frac{1}{t}\Big). \tag{3.4}$$

Using (3.4), we may write

$$\int_{2/\mathrm{e}^{\gamma}}^{x/\mathrm{e}^{\gamma}} \frac{1}{(\gamma + \log t)^{m+1}}\, \mathrm{d}t = \int_{2/\mathrm{e}^{\gamma}}^{x/\mathrm{e}^{\gamma}} \frac{1}{H_t^{m+1}(1 + O(1/(tH_t)))^{m+1}}\, \mathrm{d}t. \tag{3.5}$$

Note that

$$\frac{1}{(1 + O(1/(tH_t)))^{m+1}} = \Big(1 + O\Big(\frac{1}{tH_t}\Big)\Big)^{m+1} = 1 + O\Big(\frac{1}{tH_t}\Big), \tag{3.6}$$

where the last O-term in (3.6) would depend on m but we abuse this. Feeding (3.6) into (3.5), we arrive at the term

$$\int_{2/\mathrm{e}^{\gamma}}^{x/\mathrm{e}^{\gamma}} \frac{1}{H_t^{m+1}}\, \mathrm{d}t + O\Big(\int_{2/\mathrm{e}^{\gamma}}^{x/\mathrm{e}^{\gamma}} \frac{1}{tH_t^{m+2}}\, \mathrm{d}t\Big). \tag{3.7}$$

Clearly, we have

$$\int_{2/\mathrm{e}^\gamma}^{x/\mathrm{e}^\gamma} \frac{1}{tH_t^{m+2}}\,\mathrm{d}t \ll \int_2^x \frac{1}{t(\log t)^{m+2}}\,\mathrm{d}t = O(1). \tag{3.8}$$

Note that $2/\mathrm{e}^\gamma = 1.12\ldots$ so that

$$\int_{2/\mathrm{e}^\gamma}^{x/\mathrm{e}^\gamma} \frac{1}{H_t^{m+1}}\,\mathrm{d}t = \left(\int_{2/\mathrm{e}^\gamma}^{2} + \int_2^{[x/\mathrm{e}^\gamma]} + \int_{[x/\mathrm{e}^\gamma]}^{x/\mathrm{e}^\gamma}\right) \frac{1}{H_t^{m+1}}\,\mathrm{d}t. \tag{3.9}$$

Moreover, we see that

$$\int_{2/\mathrm{e}^\gamma}^{2} \frac{1}{H_t^{m+1}}\,\mathrm{d}t = 2 - \frac{2}{\mathrm{e}^\gamma} = O(1) \tag{3.10}$$

and obviously that

$$\int_{[x/\mathrm{e}^\gamma]}^{x/\mathrm{e}^\gamma} \frac{1}{H_t^{m+1}}\,\mathrm{d}t = O(1). \tag{3.11}$$

Assembling (3.9)–(3.11), one infers that

$$\int_{2/\mathrm{e}^\gamma}^{x/\mathrm{e}^\gamma} \frac{1}{H_t^{m+1}}\,\mathrm{d}t = \int_2^{[x/\mathrm{e}^\gamma]} \frac{1}{H_t^{m+1}}\,\mathrm{d}t + O(1). \tag{3.12}$$

Using the fact that

$$\frac{1}{H_t^{m+1}} = \frac{1}{H_n^{m+1}}$$

when $n \leqslant t < n+1$, one obtains

$$\int_2^{[x/\mathrm{e}^\gamma]} \frac{1}{H_t^{m+1}}\,\mathrm{d}t = \sum_{n=2}^{[x/\mathrm{e}^\gamma]-1} \frac{1}{H_n^{m+1}} = \sum_{n \leqslant x/\mathrm{e}^\gamma} \frac{1}{H_n^{m+1}} + O(1). \tag{3.13}$$

As a consequence of (3.2), (3.7), (3.8), (3.12) and (3.13), we verify the formula

$$m! \int_2^x \frac{1}{(\log t)^{m+1}}\,\mathrm{d}t = \mathrm{e}^\gamma m! \sum_{n \leqslant x/\mathrm{e}^\gamma} \frac{1}{H_n^{m+1}} + O(1). \tag{3.14}$$

From (3.1) and (3.14), we record the result of the first stage of our surgery in the form

$$\mathrm{li}\, x = \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} + \mathrm{e}^\gamma m! \sum_{n \leqslant x/\mathrm{e}^\gamma} \frac{1}{H_n^{m+1}} + O(1) \tag{3.15}$$

for every given nonnegative integer m. For the second stage, let us find a representation for β_n. Since among the n independent trials of an experiment, the single failure can be on the first attempt or on the second attempt, or in general on the ith attempt, where $1 \leqslant i \leqslant n$, we see by the mutual disjointness of these events that the probability of having exactly one failure is given by

$$\beta_n = \sum_{i=1}^{n} \frac{1}{i+1} \prod_{j \neq i} \Big(1 - \frac{1}{j+1}\Big). \tag{3.16}$$

The right hand side of (3.16) can be rewritten as

$$\prod_{j=1}^{n} \Big(1 - \frac{1}{j+1}\Big) \sum_{i=1}^{n} \frac{1}{i+1} \Big(1 - \frac{1}{i+1}\Big)^{-1}. \tag{3.17}$$

As a consequence of the telescoping product, we have

$$\prod_{j=1}^{n} \Big(1 - \frac{1}{j+1}\Big) = \frac{1}{n+1}. \tag{3.18}$$

Moreover, noting that

$$\sum_{i=1}^{n} \frac{1}{i+1} \Big(1 - \frac{1}{i+1}\Big)^{-1} = \sum_{i=1}^{n} \frac{1}{i} = H_n, \tag{3.19}$$

we may gather (3.16)–(3.19) to justify the relation

$$H_n = \beta_n (n+1) \tag{3.20}$$

for every positive integer n. Bringing together the two stages of our surgery from (3.15) and (3.20), we arrive at the representation

$$\operatorname{li} x = \sum_{k=0}^{m-1} \frac{k!\, x}{(\log x)^{k+1}} + \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1} (n+1)^{m+1}} + O(1) \tag{3.21}$$

for every given nonnegative integer m. We have now completed all the preliminary work to prove parts of Theorem 1. For part (i), we recall the well-known criterion (see [14], Chapter 18) that RH is equivalent to the asymptotic formula

$$\psi(x) = x + O(x^{1/2} (\log x)^2). \tag{3.22}$$

However, by Lemma 1, (3.22) is equivalent to the formula

$$\pi(x) = \operatorname{li} x + O(x^{1/2} \log x). \tag{3.23}$$

Since the $O(1)$ term of (3.21) can always be buried into the O-term of (3.23), (3.23) is equivalent to the formula

$$\pi(x) = \sum_{k=0}^{m-1} \frac{k!\,x}{(\log x)^{k+1}} + \mathrm{e}^{\gamma} m! \sum_{n \leqslant x/\mathrm{e}^{\gamma}} \frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(x^{1/2}\log x). \tag{3.24}$$

Finally, using (3.24) and the definition of $\Pi_m(x)$, we deduce that (1.8) is an equivalent condition for RH as $x \to \infty$. This proves part (i).

Next we show (ii) by paying specific attention to why the O-term of (1.9) turns out to be better than the O-term of (1.8) in the logarithmic factor. To this end, first assume that

$$\psi(x) = x + O(x^{\theta}) \tag{3.25}$$

holds asymptotically for some $\frac{1}{2} < \theta < 1$. Then, from the generating function technology, we know that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\,\mathrm{d}x$$

for $\Re(s) > 1$. Consequently, using (3.25), one obtains

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} + s\int_1^{\infty} \frac{E(x)}{x^{s+1}}\,\mathrm{d}x, \tag{3.26}$$

where $E(x) := \psi(x) - x = O(x^{\theta})$. But then

$$s\int_1^{\infty} \frac{E(x)}{x^{s+1}}\,\mathrm{d}x$$

is analytic when $\Re(s) > \theta$, and it follows from (3.26) that $\zeta(s)$ can have no zeros with $\Re(s) > \theta$ so that RH(θ) holds. Conversely, if RH(θ) holds, then referring to a result of Grosswald (see [17]), we know that (3.25) holds. Therefore, (3.25) is an equivalent condition for RH(θ). However, by Lemma 1, we further know that (and this is the only reason why we gain in the logarithmic factor as a consequence of Grosswald's improvement when $\frac{1}{2} < \theta < 1$):

$$\pi(x) = \operatorname{li} x + O\Big(\frac{x^{\theta}}{\log x}\Big). \tag{3.27}$$

Combining (3.21) and (3.27), we see that (1.9) is equivalent to RH(θ). This completes the proof of (ii).

Although the proof of (iii) follows similar lines as in (i) and (ii), the details are heavier, so we sketch them carefully for the sake of completeness. Let q be fixed and assume that GRH(θ) holds for all Dirichlet L-functions modulo q for a given $\frac{1}{2} \leqslant \theta < 1$. For $1 \leqslant a \leqslant q$ and $(a, q) = 1$, we know that

$$\psi(x, q, a) = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \psi(x, \chi), \tag{3.28}$$

where the sum on the right hand side of (3.28) is over all characters modulo q and

$$\psi(x, \chi) := \sum_{n \leqslant x} \chi(n) \Lambda(n). \tag{3.29}$$

Let χ_1 be the principal character modulo q. Rewriting (3.28), we have

$$\psi(x, q, a) = \frac{\psi(x, \chi_1)}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_1} \overline{\chi}(a) \psi(x, \chi). \tag{3.30}$$

Using (3.29), one gets

$$|\psi(x, \chi_1) - \psi(x)| \leqslant \sum_{\substack{n \leqslant x \\ (n,q)>1}} \Lambda(n) \ll (\log q)(\log x) \ll \log x. \tag{3.31}$$

Since, for all complex s,

$$L(s, \chi_1) = \zeta(s) \prod_{p \mid q} \Big(1 - \frac{1}{p^s}\Big)$$

holds, and the product

$$\prod_{p \mid q} \Big(1 - \frac{1}{p^s}\Big)$$

has zeros only on the imaginary axis, GRH(θ) for $L(s, \chi_1)$ implies RH(θ). Therefore, we may write from (3.31) that

$$\psi(x, \chi_1) = \psi(x) + O(\log x) = x + O(x^\theta (\log x)^2) \tag{3.32}$$

for $\frac{1}{2} \leqslant \theta < 1$, where $(\log x)^2$ can be deleted from (3.32) when $\frac{1}{2} < \theta < 1$ by Grosswald's result. From (3.30) and (3.32), one has

$$\psi(x, q, a) = \frac{x}{\varphi(q)} + \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_1} \overline{\chi}(a) \psi(x, \chi) + O(x^\theta (\log x)^2). \tag{3.33}$$

Recall the explicit formula representing $\psi(x, \chi)$ when $\chi \neq \chi_1$ (see [14], Chapter 19) in the form

$$\psi(x, \chi) = -\frac{x^{\varrho_1}}{\varrho_1} - \sum_{|t|<T} \frac{x^\omega}{\omega} + O\Big(\frac{x(\log qx)^2}{T} + x^{1/4} \log x\Big) \tag{3.34}$$

for $2 \leqslant T \leqslant x$, where the sum in (3.34) is over all critical zeros ω of $L(s,\chi)$ with ordinate t excluding the Siegel zeros at ϱ_1 and $1-\varrho_1$ (if they exist). The term $-x^{\varrho_1}/\varrho_1$ is omitted from (3.34) if the Siegel zero does not exist. Now GRH(θ) implies that $\varrho_1 \leqslant \theta$ and $\Re(\omega) \leqslant \theta$ for any critical zero ω. Consequently, we obtain from (3.34) that

$$|\psi(x,\chi)| \ll x^\theta + x^\theta \sum_{|t|<T} \frac{1}{|\omega|} + \frac{x(\log x)^2}{T} + x^{1/4}\log x \tag{3.35}$$

for any nonprincipal character χ modulo q. We also know that (see [14], Chapter 20)

$$\sum_{|t|<T} \frac{1}{|\omega|} \ll (\log qT)^2 \ll (\log qx)^2 \ll (\log x)^2. \tag{3.36}$$

Choosing $T = x^{1-\theta}$, it follows from (3.35) and (3.36) that

$$|\psi(x,\chi)| \ll x^\theta (\log x)^2 \tag{3.37}$$

for all $\chi \neq \chi_1$, and from (3.37) that

$$\frac{1}{\varphi(q)}\left|\sum_{\chi\neq\chi_1} \overline{\chi}(a)\psi(x,\chi)\right| \ll x^\theta(\log x)^2. \tag{3.38}$$

At this point, we remark that Grosswald's improvement in (3.32) for the log factor when $\frac{1}{2} < \theta < 1$ no longer helps because of (3.36). Therefore, assuming that GRH(θ) holds for all Dirichlet L-functions modulo q, we have shown from (3.33) and (3.38) that

$$\psi(x,q,a) = \frac{x}{\varphi(q)} + O(x^\theta(\log x)^2) \tag{3.39}$$

for $\frac{1}{2} \leqslant \theta < 1$. Thus by Lemma 1, (3.39) gives that

$$\pi(x,q,a) = \frac{\operatorname{li} x}{\varphi(q)} + O(x^\theta \log x). \tag{3.40}$$

Conversely, assuming (3.40) for all $1 \leqslant a \leqslant q$ with $(a,q) = 1$, we get (3.39) by Lemma 1. Let us rewrite (3.28) as a linear system of equations over the $\varphi(q)$ reduced residues modulo q. In matrix notation, we would get

$$B \begin{bmatrix} \psi(x,\chi_1) \\ \vdots \\ \psi(x,\chi_{\varphi(q)}) \end{bmatrix} = \begin{bmatrix} \varphi(q)\psi(x,q,a_1) \\ \vdots \\ \varphi(q)\psi(x,q,a_{\varphi(q)}) \end{bmatrix}, \tag{3.41}$$

where $\chi_1, \ldots, \chi_{\varphi(q)}$ and $a_1, \ldots, a_{\varphi(q)}$ are all the characters and reduced residues modulo q, respectively. Moreover, the $\varphi(q) \times \varphi(q)$ matrix B in (3.41) is defined by $B = [b_{ij}]$, where $b_{ij} = \overline{\chi_j}(a_i)$ for all $1 \leqslant i, j \leqslant \varphi(q)$. Let B^* be the conjugate transpose of B. Note that as a vector in $\mathbb{C}^{\varphi(q)}$, the ith row of B^* would be the same as the conjugate of the ith column of B which is the vector

$$[\chi_i(a_1), \ldots, \chi_i(a_{\varphi(q)})] \in \mathbb{C}^{\varphi(q)}.$$

Therefore, using the orthogonality of characters, the ordinary dot product of the ith row of B^* with the jth column of B is easily computed to be

$$\sum_a (\chi_i \overline{\chi_j})(a) = \begin{cases} \varphi(q) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{3.42}$$

where the sum in (3.42) is over all the reduced residues modulo q. Now (3.42) shows that

$$B^* B = \varphi(q) I, \tag{3.43}$$

where I is the identity matrix of the same size. In particular, this shows that the normalized matrix

$$\Big(\frac{1}{\sqrt{\varphi(q)}}\Big) B$$

is unitary. Multiplying both sides of (3.41) by B^* and using (3.43), one achieves an inversion of the form

$$\begin{bmatrix} \psi(x, \chi_1) \\ \vdots \\ \psi(x, \chi_{\varphi(q)}) \end{bmatrix} = B^* \begin{bmatrix} \psi(x, q, a_1) \\ \vdots \\ \psi(x, q, a_{\varphi(q)}) \end{bmatrix}. \tag{3.44}$$

Precisely, (3.44) means that, for the principal character χ_1, we have

$$\psi(x, \chi_1) = \sum_{j=1}^{\varphi(q)} \psi(x, q, a_j), \tag{3.45}$$

and

$$\psi(x, \chi) = \sum_{j=1}^{\varphi(q)} \chi(a_j) \psi(x, q, a_j) \tag{3.46}$$

when $\chi \neq \chi_1$. Combining (3.39) and (3.45), we see that

$$\psi(x, \chi_1) = x + O(\varphi(q) x^{\theta} (\log x)^2) = x + O(x^{\theta} (\log x)^2) \tag{3.47}$$

as q is fixed. Now from (3.31) and (3.47),

$$\psi(x) = x + O(x^{\theta} (\log x)^2) \tag{3.48}$$

follows. Thus (3.48) shows that RH(θ) holds and consequently GRH(θ) holds for $L(s,\chi_1)$. From (3.39) and (3.46), we get

$$\psi(x,\chi) = \frac{x}{\varphi(q)}\sum_{j=1}^{\varphi(q)}\chi(a_j) + O(x^\theta(\log x)^2) = O(x^\theta(\log x)^2) = O(x^{\theta+\varepsilon}) \tag{3.49}$$

for any $\varepsilon > 0$ since

$$\sum_{j=1}^{\varphi(q)}\chi(a_j) = 0$$

when $\chi \neq \chi_1$. It follows from (3.49) and the identity

$$-\frac{L'(s,\chi)}{L(s,\chi)} = s\int_1^\infty \frac{\psi(x,\chi)}{x^{s+1}}\,\mathrm{d}x \tag{3.50}$$

that both sides of (3.50) are analytic when $\Re(s) > \theta$. Thus, GRH(θ) holds for $L(s,\chi)$ when $\chi \neq \chi_1$. This completes the proof that GRH(θ) modulo q is equivalent to (3.40). Note that from (3.15), one has

$$\frac{\operatorname{li}x}{\varphi(q)} = \frac{1}{\varphi(q)}\sum_{k=0}^{m-1}\frac{k!\,x}{(\log x)^{k+1}} + \frac{\mathrm{e}^\gamma m!}{\varphi(q)}\sum_{n\leqslant x/\mathrm{e}^\gamma}\frac{1}{\beta_n^{m+1}(n+1)^{m+1}} + O(1). \tag{3.51}$$

Assembling (3.40) and (3.51) and the definition of $\Pi_m(x,q,a)$, we deduce that GRH(θ) modulo q is equivalent to (1.10). This finishes the argument of part (iii).

To prove part (iv), we recall from the works of Ingham (see [22], pages 60–65) and Turán (see [37]) that existence of a zero-free region for the Riemann zeta function as in (1.7) is equivalent to the asymptotic formula

$$\psi(x) = x + O(x\mathrm{e}^{-c(\log x)^{1/(1+\alpha)}}) \tag{3.52}$$

for some constant $c > 0$. Of course, by Lemma 1, (3.52) is equivalent to

$$\pi(x) = \operatorname{li}x + O(x\mathrm{e}^{-c(\log x)^{1/(1+\alpha)}}) \tag{3.53}$$

for some (possibly different) constant $c > 0$. Once again, (3.21) and (3.53) tell us that (1.11) is an equivalent statement to the above mentioned zero-free region for every given $m \geqslant 0$.

For the proof of part (v), first assume that (1.12) holds. Then Malliavin in [25] showed that

$$\pi_P(x) = \operatorname{li}x + O(x\mathrm{e}^{-c(\log x)^{a'}}), \tag{3.54}$$

holds with a (possibly different) constant $c > 0$ and $a' = a/10$. From (3.21) and (3.54), we infer that (1.13) holds. Conversely, if (1.13) holds for some $m \geqslant 0$, $c > 0$

and $a' > 0$, then combining this with (3.21), we would get (3.54) for some $c > 0$ and $a' > 0$. But conversely, Malliavin [25] also showed that (3.54) for some $c > 0$ and $a' > 0$ implies (1.12) for some $0 < b \leqslant 1$, $c > 0$ and $a = a'/(a'+2)$. This completes the proof of part (v).

The proof of part (vi) follows in a similar fashion. First we know by the Vinogradov-Korobov type zero-free region for the Riemann zeta function together with its correction by Walfisz (see [38]) and its perfection with explicit constants by Ford (see [16]) that

$$\pi(x) = \mathrm{li}\, x + O(x\mathrm{e}^{-(0.2098)(\log x)^{3/5}/(\log\log x)^{1/5}}). \tag{3.55}$$

Thus (1.14) is a consequence of (3.21) and (3.55) for every given $m \geqslant 0$. To obtain (1.15), note that by the Siegel-Walfisz theorem (see [14], Chapter 22), we have

$$\psi(x,q,a) = \frac{x}{\varphi(q)} + O(x\mathrm{e}^{-c(\log x)^{1/2}}) \tag{3.56}$$

uniformly for $q \leqslant (\log x)^A$, where $c > 0$ depends on A and the O-constant is ineffective in (3.56). By the remark after Lemma 1, we know that (3.56) implies

$$\pi(x,q,a) = \frac{\mathrm{li}\, x}{\varphi(q)} + O(x\mathrm{e}^{-c(\log x)^{1/2}}) \tag{3.57}$$

uniformly for $q \leqslant (\log x)^A$. Therefore, (1.15) follows from (3.51) and (3.57) for every given $m \geqslant 0$. The proof of Theorem 1 is now complete. □

4. Proof of Theorem 2

We are given that

$$c\pi(x^\delta) + o\Big(\frac{x^\delta}{(\log x)^2}\Big) = \frac{cx^\delta}{\delta\log x - A_P(x)} \tag{4.1}$$

as $x \to \infty$. We may also write

$$\pi(x^\delta) = \frac{cx^\delta}{\delta\log x} + \frac{cx^\delta}{\delta^2(\log x)^2} + o\Big(\frac{x^\delta}{(\log x)^2}\Big). \tag{4.2}$$

Feeding (4.2) into (4.1), and solving for $A_P(x)$, one obtains that

$$A_P(x) = \frac{1+o(1)}{1 + 1/(\delta\log x) + o(1/\log x)} \tag{4.3}$$

as $x \to \infty$. It is plain that (1.17) follows from (4.3). Next assume that (1.18) holds for some $m \geqslant 3$. From the asymptotic expansion of $\pi(x)$, we know that

$$\pi(x^\delta) = \mathrm{li}(x^\delta) + O\Big(\frac{x^\delta}{(\log x)^m}\Big). \tag{4.4}$$

Using (4.4), we may write

$$\pi_P(x) = c\,\mathrm{li}(x^\delta) + O\Big(\frac{x^\delta}{(\log x)^m}\Big). \tag{4.5}$$

We may bring (4.5) to the form

$$\frac{\pi_P(x)}{x^\delta} = \frac{c}{\delta \log x}\Big(\sum_{k=0}^{m-2} \frac{k!}{\delta^k (\log x)^k} + O\Big(\frac{1}{(\log x)^{m-1}}\Big)\Big). \tag{4.6}$$

Our next task is to reciprocate (4.6). For simplicity, we use the notation

$$\frac{1}{\delta \log x} = X.$$

Then essentially our task is to find

$$\Big(\sum_{k=0}^{m-2} k!\, X^k + O(X^{m-1})\Big)^{-1} = (1 + O(X^{m-1}))\Big(\sum_{k=0}^{m-2} k!\, X^k\Big)^{-1} \tag{4.7}$$

as $X \to 0$. By the theory of power series, since the constant term of $\sum_{k=0}^{m-2} k!\, X^k$ is nonzero, it follows that

$$\Big(\sum_{k=0}^{m-2} k!\, X^k\Big)^{-1} = \sum_{n=0}^{\infty} b_n X^n \tag{4.8}$$

is a power series around zero with positive radius of convergence, where the b_n's in (4.8) can be obtained from the equality of power series

$$\Big(\sum_{k=0}^{m-2} k! X^k\Big)\Big(\sum_{n=0}^{\infty} b_n X^n\Big) = 1, \tag{4.9}$$

where the left hand side of (4.9) is computed as a Cauchy product. However, we prefer to rewrite (4.9) in the form

$$\Big(\sum_{k=0}^{m-2} k!\, X^k\Big)\Big(\sum_{n=0}^{m-2} b_n X^n + O(X^{m-1})\Big) = 1, \tag{4.10}$$

and it follows from (4.7) and (4.10) that the representation

$$\Big(\sum_{k=0}^{m-2} k!\, X^k + O(X^{m-1})\Big)^{-1} = (1 + O(X^{m-1}))\Big(\sum_{n=0}^{m-2} b_n X^n\Big) \tag{4.11}$$

holds, where the coefficients in (4.11) are subject to the recurrence formula

$$\sum_{i=0}^{n} (n-i)!\, b_i = 0 \tag{4.12}$$

for $1 \leqslant n \leqslant m-2$, and $b_0 = 1$. We find that $b_1 = -1 = b_2$. For $n \geqslant 1$, put $a_n^* = -b_n$. Then (4.12) becomes

$$a_n^* = n! - \sum_{i=1}^{n-1} (n-i)!\, a_i^* \tag{4.13}$$

Comparing (1.16) and (4.13), since the initial values are also the same, we deduce that

$$a_n = a_n^* = -b_n \tag{4.14}$$

for all $n \geqslant 1$, where a_n is the number of indecomposable permutations. Therefore, the reciprocation of (4.6) is justified with a complete description of coefficients by (4.14) in terms of the number of indecomposable permutations. Precisely, we get

$$\frac{x^\delta}{\pi_P(x)} = \frac{\delta \log x}{c}\left(1 - \sum_{k=1}^{m-2} \frac{a_k}{(\delta \log x)^k} + O\Big(\frac{1}{(\log x)^{m-1}}\Big)\right) \tag{4.15}$$

for every given $m \geqslant 3$ as $x \to \infty$. Letting $x = n$ to be an integer tending to infinity in (4.15), one has

$$\frac{cn^\delta}{\pi_P(n)} = \delta \log n - \sum_{k=1}^{m-2} \frac{a_k}{(\delta \log n)^{k-1}} + O\Big(\frac{1}{(\log n)^{m-2}}\Big). \tag{4.16}$$

Using the asymptotic formula for H_n, we may rewrite (4.16) as

$$\frac{cn^\delta}{\pi_P(n)} = \delta H_n - \delta\gamma - \sum_{k=1}^{m-2} \frac{a_k}{(\delta \log n)^{k-1}} + O\Big(\frac{1}{(\log n)^{m-2}}\Big). \tag{4.17}$$

Next, let us recall the following important formula from [13]

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) \tag{4.18}$$

which is useful in connecting hyperharmonic numbers of order $r \geqslant 1$ to harmonic numbers. Replacing n by $n-r+1$ in (4.18) and feeding this into (4.17), we finally arrive at the relation

$$\frac{cn^\delta}{\pi_P(n)} = \frac{\delta H_{n-r+1}^{(r)}}{\binom{n}{r-1}} + \delta H_{r-1} - \delta\gamma - \sum_{k=1}^{m-2} \frac{a_k}{(\delta \log n)^{k-1}} + O\Big(\frac{1}{(\log n)^{m-2}}\Big) \tag{4.19}$$

for a given $r \geqslant 1$ when $n \to \infty$. Now (1.19) is an immediate consequence of (4.19). To complete the proof of Theorem 2, we make some observations and comment on the size of coefficients. It is plain that $b_1, \ldots, b_{n-1}$ are all negative integers and they all satisfy $b_k \geqslant -k!$ (since $a_k \leqslant k!$). If we only use (4.12), then note that starting with the obvious inequality

$$\sum_{i=1}^{n-1}(n-i)!\,b_i \geqslant (n-1)\sum_{i=1}^{n-1}(n-i-1)!\,b_i, \tag{4.20}$$

and

$$\sum_{i=1}^{n-1}(n-i)!\,b_i = -n!-b_n, \tag{4.21}$$

$$\sum_{i=1}^{n-1}(n-i-1)!\,b_i = -(n-1)!, \tag{4.22}$$

we infer from (4.20)–(4.22) that

$$-n! \leqslant b_n \leqslant -(n-1)! \tag{4.23}$$

for all $n \geqslant 1$. Thus, (4.23) implies that

$$(n-1)! \leqslant a_n \leqslant n! \tag{4.24}$$

for all $n \geqslant 1$. This is what we may come by analytically from (4.12) at a first glance. However, exploiting the combinatorial significance of a_n's as the number of indecomposable permutations, King in [23] obtained a significant improvement of the lower bound in (4.24). Precisely, he showed that

$$\frac{n!}{2} \leqslant a_n \leqslant n! \tag{4.25}$$

for all $n \geqslant 1$. Finally, we analyze the asymptotic behavior of a_n using only (1.16). Let q be a given positive integer. From (1.16), we may consider the decomposition

$$a_n = n! - \sum_{j=1}^{q}(n-j)!\,a_j - \sum_{j=1}^{q} j!\,a_{n-j} - \sum_{j=q+1}^{n-q-1}(n-j)!\,a_j. \tag{4.26}$$

Since $a_j \leqslant j!$ for any j, we have

$$\sum_{j=q+1}^{n-q-1}(n-j)!\,a_j \leqslant \sum_{j=q+1}^{n-q-1}(n-j)!\,j!. \tag{4.27}$$

Let us estimate the right hand side of (4.27) keeping in mind the unimodality of binomial coefficients in a row of Pascal's triangle. First, by symmetry, we obtain for all large n that

(4.28)

$$\sum_{j=q+1}^{n-q-1}(n-j)!\,j! \leqslant 2\sum_{j=q+1}^{[n/2]}(n-j)!\,j! = 2\sum_{q+1\leqslant j\leqslant n/4}(n-j)!\,j! + 2\sum_{n/4<j\leqslant[n/2]}(n-j)!\,j!.$$

Precisely, the unimodality says that the ratio of two consecutive summands in (4.28) is

$$\frac{(n-j-1)!\,(j+1)!}{(n-j)!\,j!} = \frac{j+1}{n-j}. \tag{4.29}$$

Note that the ratio in (4.29) is always not greater than 1 when $j \leqslant [n/2]-1$. Therefore, one obtains that

$$2 \sum_{n/4<j\leqslant[n/2]} (n-j)!\,j! = O\Big(n\Big(n-\Big[\frac{n}{4}\Big]-1\Big)!\,\Big(\Big[\frac{n}{4}\Big]+1\Big)!\Big). \tag{4.30}$$

Moreover, when $q+1 \leqslant j \leqslant n/4$, the ratio in (4.29) is

$$\frac{j+1}{n-j} \leqslant \frac{n/4+1}{3n/4} \leqslant \frac{1}{2},$$

provided $n \geqslant 8$. Consequently, one gets

$$2 \sum_{q+1\leqslant j\leqslant n/4} (n-j)!\,j! \leqslant 2(q+1)!\,(n-q-1)!\sum_{r=0}^{\infty}\frac{1}{2^r} = O_q((n-q-1)!). \tag{4.31}$$

Next we show that the estimate in (4.30) can be absorbed into the estimate in (4.31). We make a digression to show a more general result that suffices for this. We claim that for any given $q \geqslant 1$

$$u!\,v! \leqslant (u+v-q)! \tag{4.32}$$

holds when $u, v \to \infty$ and $u \sim \delta v$ for some $\delta > 0$. Note that (4.32) is trivial when $q \leqslant 0$. Clearly, (4.32) is equivalent to

$$\log u! + \log v! \leqslant \log(u+v-q)!. \tag{4.33}$$

Using the well-known asymptotic formula for the logarithm of a factorial, we may write

$$\log u! + \log v! = u\log u + v\log v - u - v + O(\log u) + O(\log v) \tag{4.34}$$

and

$$\log(u+v-q)! = (u+v-q)\log(u+v-q) - u - v + q + O(\log(u+v)). \tag{4.35}$$

Furthermore, we have

$$\begin{aligned}(u+v-q)\log(u+v-q) &= (u+v)\log(u+v-q) + O_q(\log(u+v)) \\ &= (u+v)\log(u+v) + (u+v)\log\Big(1-\frac{q}{u+v}\Big) \\ &\quad + O_q(\log(u+v)).\end{aligned} \tag{4.36}$$

Note that

$$\log\Big(1-\frac{q}{u+v}\Big)=-\frac{q}{u+v}+O\Big(\frac{q^2}{(u+v)^2}\Big). \tag{4.37}$$

Combining (4.35)–(4.37), one infers that

$$\log(u+v-q)!=(u+v)\log(u+v)-u-v+O_q(\log(u+v)). \tag{4.38}$$

Using now (4.34) and (4.38), we see that showing (4.33) boils down to the verification of the asymptotic inequality

$$u\log u+v\log v+O_q(\log(u+v))\leqslant(u+v)\log(u+v). \tag{4.39}$$

Since $\log(u+v)=o(u+v)$ as $u,v\to\infty$, (4.39) reduces to

$$\Big(\frac{u}{u+v}\Big)\log u+\Big(\frac{v}{u+v}\Big)\log v+o(1)\leqslant\log(u+v). \tag{4.40}$$

By concavity of the logarithm, we have

$$\Big(\frac{u}{u+v}\Big)\log u+\Big(\frac{v}{u+v}\Big)\log v\leqslant\log\Big(\frac{u^2+v^2}{u+v}\Big). \tag{4.41}$$

Finally, since $u/v\to\delta$,

$$\log(u+v)-\log\Big(\frac{u^2+v^2}{u+v}\Big)=\log\Big(1+\frac{2uv}{u^2+v^2}\Big)\to\log\Big(1+\frac{2\delta}{1+\delta^2}\Big)>0. \tag{4.42}$$

Thus, (4.40) easily follows from (4.41) and (4.42). This proves (4.33) and our claim in (4.32). Taking $u=[\frac{1}{4}n]+1$ and $v=n-[\frac{1}{4}n]-1$, and applying our claim with $\delta=\frac{1}{3}$, we deduce that

$$\Big(n-\Big[\frac{n}{4}\Big]-1\Big)!\Big(\Big[\frac{n}{4}\Big]+1\Big)!\leqslant(n-q-2)! \tag{4.43}$$

asymptotically as $n\to\infty$. As q is fixed, it follows from (4.43) that

$$n\Big(n-\Big[\frac{n}{4}\Big]-1\Big)!\Big(\Big[\frac{n}{4}\Big]+1\Big)!=O((n-q-1)!). \tag{4.44}$$

Assembling (4.30), (4.31) and (4.44), we know that (4.26) becomes

$$a_n=n!-\sum_{j=1}^{q}(n-j)!\,a_j-\sum_{j=1}^{q}j!\,a_{n-j}+O((n-q-1)!), \tag{4.45}$$

where the dependence of the O-constant on q in (4.45) is minor. We remark that (4.45) can be used in an inductive manner to arrive at an asymptotic expansion of a_n of the form

$$a_n = \Big(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots + \frac{c_q}{n^q} + O\Big(\frac{1}{n^{q+1}}\Big)\Big) n! \tag{4.46}$$

for some scalars $c_1, c_2, \ldots, c_q$. Let us demonstrate this when $q = 4$. From (4.45), we would get by taking $q = 4$ that

$$\begin{aligned}\frac{a_n}{n!} = 1 - \frac{1}{n} - \frac{1}{n(n-1)} - \frac{3}{n(n-1)(n-2)} - \frac{13}{n(n-1)(n-2)(n-3)} \\ - \frac{a_{n-1}}{n!} - \frac{2a_{n-2}}{n!} - \frac{6a_{n-3}}{n!} - \frac{24a_{n-4}}{n!} + O\Big(\frac{1}{n^5}\Big).\end{aligned} \tag{4.47}$$

As a result of the postulated asymptotic formula in (4.46), we need to show that

$$\frac{a_n}{n!} = 1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \frac{c_4}{n^4} + O\Big(\frac{1}{n^5}\Big), \tag{4.48}$$

where c_1, c_2, c_3, c_4 are scalars to be determined. As our inductive hypothesis, we are allowed to assume that

$$\frac{a_{n-j}}{(n-j)!} = 1 + \frac{c_1}{n-j} + \frac{c_2}{(n-j)^2} + \frac{c_3}{(n-j)^3} + \frac{c_4}{(n-j)^4} + O\Big(\frac{1}{n^5}\Big) \tag{4.49}$$

for $j = 1, 2, 3, 4$. To carry out the induction step, we combine (4.47)–(4.49) and make sure to keep only the significant terms that are superior to $O(1/n^5)$. In this way, one obtains that

$$\begin{aligned}1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \frac{c_4}{n^4} = 1 - \frac{2}{n} - \frac{c_1+3}{n(n-1)} - \frac{2c_1+9}{n(n-1)(n-2)} \\ - \frac{6c_1+37}{n(n-1)(n-2)(n-3)} - \frac{c_2}{n(n-1)^2} \\ - \frac{c_3}{n(n-1)^3} - \frac{2c_2}{n(n-1)(n-2)^2} + O\Big(\frac{1}{n^5}\Big).\end{aligned} \tag{4.50}$$

It is clear that for (4.50) to hold, we must have $c_1 = -2$. To find c_2 note that the only contributor from the right hand side of (4.50) is the term containing

$$\frac{1}{n(n-1)} = \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \ldots$$

Thus, we must have $c_2 = -c_1 - 3 = -1$. To find c_3, note that

$$\frac{1}{n(n-1)(n-2)} = \frac{1}{n^3(1-1/n)(1-2/n)} = \frac{1}{n^3}\Big(1 + \frac{3}{n} + \ldots\Big)$$

and

$$\frac{1}{n(n-1)^2} = \frac{1}{n^3} + \frac{2}{n^4} + \ldots$$

Therefore, $c_3 = -3c_1 - c_2 - 12 = -5$. Finally, the representations

$$\frac{1}{n(n-1)(n-2)(n-3)} = \frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big), \quad \frac{1}{n(n-1)^3} = \frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big),$$
$$\frac{1}{n(n-1)(n-2)^2} = \frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big)$$

hold, and note that they all start with the most significant term $1/n^4$. Taking into account all of these contributions, we find

$$c_4 = -13c_1 - 4c_2 - c_3 - 67 = -32.$$

This completes the induction step. Of course the base case of the induction is automatic since we may arrange the O-constants so that (4.48) trivially holds for small values of n. Lastly, (4.25) and (4.48) complete our comments on the behavior of the number of indecomposable permutations. This finishes the proof of Theorem 2. $\square$

5. Proof of Theorem 3

For the proof of part (i), by taking $\delta = 1$ and $c = 1/\varphi(q)$ in (4.15), we obtain as a special case of the reciprocation that

$$-\frac{x}{\pi(x,q,a)} = -\varphi(q)\log x + \sum_{k=1}^{m-2} \frac{\varphi(q)a_k}{(\log x)^{k-1}} + O\Big(\frac{1}{(\log x)^{m-2}}\Big) \tag{5.1}$$

for all $m \geqslant 3$, $1 \leqslant a \leqslant q$ and $(a,q) = 1$ when $x \to \infty$. Let us remark that $m \geqslant 3$ was assumed only for convenience in Theorem 2 just to make the O-term in (1.19) tend to zero. Therefore, (5.1) still holds for all $m \geqslant 2$, and when $m = 2$, it becomes

$$-\frac{x}{\pi(x,q,a)} = -\varphi(q)\log x + O(1). \tag{5.2}$$

Clearly, (5.1) and (5.2) signify the asymptotic expansion

$$-\frac{x}{\pi(x,q,a)} \sim -\varphi(q)\log x + \sum_{k=1}^{\infty} \frac{\varphi(q)a_k}{(\log x)^{k-1}}. \tag{5.3}$$

Note that (5.3) falls into the scope of Lemma 2, and we deduce that

$$-\frac{x}{\pi(x,q,a)}$$

is asymptotically convex. But, as mentioned in the introduction, we know from [10] that this function is never convex on any interval $x \geqslant x_1$. This proves part (i).

For part (ii), let us start with the expansion

$$\frac{\pi(x)}{x} \sim \sum_{k=0}^{\infty} \frac{k!}{(\log x)^{k+1}}. \tag{5.4}$$

Note that if we let $X = 1/\log x$, then

$$\sum_{k=0}^{\infty} \frac{k!}{(\log x)^{k+1}} = \sum_{k=0}^{\infty} k!\, X^{k+1}$$

is formally a power series around zero (whose radius of convergence is also zero, but this is not harmful to us as we only use it formally). It is known that asymptotic expansions can be multiplied in the same way that their formal power series are multiplied as a Cauchy product. Therefore, from (5.4), we may conclude that the asymptotic expansion of the function

$$f(x) := \Big(\frac{\pi(x)}{x}\Big)^2$$

is given by

$$\sum_{k=2}^{\infty} \frac{c_k}{(\log x)^k}, \tag{5.5}$$

where each c_k is a sum of terms each of which are products of two factorials. It is plain that $c_k > 0$ for $k \geqslant 2$. Because of this, (5.5) lies in the scope of Lemma 2, and we see that $f(x)$ is an asymptotically convex function satisfying (2.22). Precisely, when $0 \leqslant \lambda \leqslant 1$ and $x, y \to \infty$, we have

$$\Big(\frac{\pi(\lambda x + (1-\lambda)y)}{\lambda x + (1-\lambda)y}\Big)^2 \leqslant \lambda\Big(\frac{\pi(x)}{x}\Big)^2 + (1-\lambda)\Big(\frac{\pi(y)}{y}\Big)^2 + o\Big(\frac{1}{(\log \min(x,y))^n}\Big) \tag{5.6}$$

for any given positive integer n. At this point, we recall a pretty generalization of a curious inequality of Ramanujan

$$\Big(\frac{\pi(x)}{x}\Big)^2 < \frac{\mathrm{e}\pi(x/\mathrm{e})}{x\log x}$$

due to Hassani (see [19] and also [21] for many variations over Ramanujan's inequality) of the form

$$\Big(\frac{\pi(x)}{x}\Big)^2 < \frac{\alpha\pi(x/\alpha)}{x\log x} \tag{5.7}$$

for any $\alpha \geqslant \mathrm{e}$ and all large x (for quantifications on how large x should be so that (5.7) holds, we refer to [19] and [20]). Furthermore, Hassani's work shed full light on Ramanujan's special preference of taking $\alpha = \mathrm{e}$ in his inequality and showed that the inequality is reversed in (5.7) when $0 < \alpha < \mathrm{e}$ (so that Ramanujan's choice for the parameter α is minimal). Finally, (1.20) follows from (5.6) and (5.7). This ends part (ii) and the proof of Theorem 3. $\square$

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