Christiaan Le Roux

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Applications of Mathematics, Vol. 68 (2023), No. 1, 1–14

Persistent URL: `http://dml.cz/dmlcz/151491`

ON THE NAVIER-STOKES EQUATIONS WITH ANISOTROPIC WALL SLIP CONDITIONS

Christiaan Le Roux, Pretoria

Received April 1, 2021. Published online November 30, 2021.

Abstract. This article deals with the solvability of the boundary-value problem for the Navier-Stokes equations with a direction-dependent Navier type slip boundary condition in a bounded domain. Such problems arise when steady flows of fluids in domains with rough boundaries are approximated as flows in domains with smooth boundaries. It is proved by means of the Galerkin method that the boundary-value problem has a unique weak solution when the body force and the variability of the surface friction are sufficiently small compared to the viscosity and the surface friction.

Keywords: Navier-Stokes equations; Stokes equations; rough boundary; slip boundary condition

MSC 2020: 76D03, 76D05, 76D07

1. Introduction

Several studies (see [5] for references, see also [4], [8]), which employ a variety of assumptions and techniques, have shown that when the flow of a liquid in a domain with a rough surface is approximated as a flow in a domain with a smooth surface, the resulting "effective boundary condition" on the smooth surface is a generalization of the Navier slip condition. We consider a boundary-value problem for the Navier-Stokes equations with the anisotropic (direction-dependent) slip boundary condition formulated in [5]:

$$(\mathbb{T}\boldsymbol{n})_\tau = -F(\cdot, |\boldsymbol{v}|^{-1}\boldsymbol{v})\boldsymbol{v} + \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\partial\Omega$ is the impermeable boundary of the flow domain $\Omega \subset \mathbb{R}^3$, $\mathbb{T}$ is the Cauchy stress tensor, $\boldsymbol{n}$ is the outward unit normal, $(\mathbb{T}\boldsymbol{n})_\tau = \mathbb{T}\boldsymbol{n} - (\boldsymbol{n}\cdot\mathbb{T}\boldsymbol{n})\boldsymbol{n}$ is the tangential

DOI: 10.21136/AM.2021.0079-21

component of the traction, $\boldsymbol{g}$ is an applied surface traction, $\boldsymbol{v}$ is the velocity, and the "friction coefficient" F is a given function defined on

$$S(\partial\Omega) = \{(x, \boldsymbol{u}) \in \partial\Omega \times \mathbb{R}^3 \colon\ \boldsymbol{n}(x) \cdot \boldsymbol{u} = 0,\ |\boldsymbol{u}| = 1\}.$$

We derive sufficient conditions for the existence and uniqueness of a weak solution of the resulting boundary-value problem (in Sections 5–6) and the existence of an associated pressure field (in Section 7). The existence proof uses the Galerkin method. For the boundary-value problem with the Stokes equations this approach yields a stronger result (in Section 8) than that of [5] (wherein the Stokes equations with a regularized version of slip condition (1.1) is considered).

2. Notation

The notation is the same as in [5]. The flow domain Ω is a bounded domain in $\mathbb{R}^3$ with a boundary $\partial\Omega$ of class $C^{1,1}$. The outward unit normal to $\partial\Omega$ is denoted by $\boldsymbol{n}$.

For $1 \leqslant q \leqslant \infty$, $L^q(\Omega)$ and $L^q(\partial\Omega)$ are the standard Lebesgue spaces with the norms $\|\cdot\|_q$ and $\|\cdot\|_{q,\partial\Omega}$, respectively. When $q = 2$, the inner products of these spaces are denoted by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\partial\Omega}$, respectively, and the norms are denoted by $\|\cdot\|$ and $\|\cdot\|_{\partial\Omega}$, respectively. The norms (and inner products, when $q = 2$) in $L^q(\Omega)^3$ and $L^q(\partial\Omega)^3$ are denoted by the same symbols as in the scalar case.

For $m \in \mathbb{N}$, $H^m(\Omega) = W^{m,2}(\Omega)$ is the standard Sobolev space, with the inner product $(\cdot,\cdot)_{m,2}$ and norm $\|\cdot\|_{m,2}$, and $H^{m-1/2}(\partial\Omega)$ is the corresponding space of traces with the norm $\|\cdot\|_{m-1/2,2,\partial\Omega}$ (see [1]). The inner products and norms in $H^m(\Omega)^3$ and $H^{m-1/2}(\partial\Omega)^3$ are denoted by the same symbols as in the scalar case. Furthermore,

$$\begin{aligned} U &:= \{\boldsymbol{v} \in H^1(\Omega)^3 \colon\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ V &:= \{\boldsymbol{v} \in U \colon\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}, \end{aligned}$$

where the boundary condition in the definition of U is understood in the sense of traces. The linear spaces U and V are closed subspaces of $H^1(\Omega)^3$ and thus Hilbert spaces with the inner product $(\cdot,\cdot)_{1,2}$. Similarly, $L^2_0(\Omega) := \{p \in L^2(\Omega) \colon\ (p, 1) = 0\}$ is a Hilbert space with the inner product $(\cdot,\cdot)$.

3. Problem formulation

Let $F\colon S(\partial\Omega) \to \mathbb{R}$ be a function with the following properties:

(a) For all $(x, \boldsymbol{u}) \in S(\partial\Omega)$, $F(\cdot, \boldsymbol{u})$ is continuous at x in the sense that there is an open ball $B(x, r(x)) = \{y \in \mathbb{R}^3\colon |y - x| < r(x)\}$ such that the function $E(\cdot, \boldsymbol{u})$ defined by

$$E(y, \boldsymbol{u}) = F(y, |(\boldsymbol{n}(y) \times \boldsymbol{u}) \times \boldsymbol{n}(y)|^{-1}(\boldsymbol{n}(y) \times \boldsymbol{u}) \times \boldsymbol{n}(y)) \tag{3.1}$$

is continuous on $\partial\Omega \cap B(x, r(x))$.

(b) $F(x, \cdot)$ is uniformly Lipschitz continuous, i.e., there exists $M_0 > 0$ such that for all $(x, \boldsymbol{u}), (x, \boldsymbol{v}) \in S(\partial\Omega)$,

$$|F(x, \boldsymbol{u}) - F(x, \boldsymbol{v})| \leqslant M_0 |\boldsymbol{u} - \boldsymbol{v}|. \tag{3.2}$$

(c) There exist constants $0 < F_L < F_U$ such that for all $(x, \boldsymbol{u}) \in S(\partial\Omega)$,

$$F_L \leqslant F(x, \boldsymbol{u}) \leqslant F_U. \tag{3.3}$$

Remark 3.1. Properties (a) and (c) ensure that $H(\boldsymbol{v}) \in L^\infty(\partial\Omega)$ for all $\boldsymbol{v} \in V$, where $H(\boldsymbol{v})$ is the function defined below in (3.6). The continuity of $F(\cdot, \boldsymbol{u})$ is not necessary for this (it is sufficient that $F(\cdot, \boldsymbol{u})$ is measurable; see [2], Theorem 3.17) but is a natural property, because the effective slip boundary condition (1.1) is assumed to be the result of an averaging procedure (see [5]).

Regarding the definition of E, note that if $\boldsymbol{\tau}(y, \boldsymbol{u})$ denotes the second argument of F in (3.1), then $\boldsymbol{\tau}(y, \boldsymbol{u})$ is a unit tangential vector at y and thus $(y, \boldsymbol{\tau}(y, \boldsymbol{u})) \in S(\partial\Omega)$. Moreover, $\boldsymbol{\tau}(x, \boldsymbol{u}) = \boldsymbol{u}$ since $(x, \boldsymbol{u}) \in S(\partial\Omega)$, and $\boldsymbol{\tau}(\cdot, \boldsymbol{u})$ is continuous at x since $\boldsymbol{n} \in C^{0,1}(\partial\Omega)^3$.

For every $x \in \partial\Omega$, let $\boldsymbol{\tau}(x)$ be a unit tangential vector at x and define the average of F at x by

$$F_a(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x, \boldsymbol{\tau}(x) \cos\theta + \boldsymbol{n}(x) \times \boldsymbol{\tau}(x) \sin\theta) \, \mathrm{d}\theta.$$

Let $T(\partial\Omega) = \{(x, \boldsymbol{v}) \in \partial\Omega \times \mathbb{R}^3\colon \boldsymbol{n}(x) \cdot \boldsymbol{v} = 0\}$ and define $G\colon T(\partial\Omega) \to \mathbb{R}$ by

$$G(x, 0) = F_a(x), \ x \in \partial\Omega; \tag{3.4}$$

$$G(x, \boldsymbol{v}) = F(x, |\boldsymbol{v}|^{-1}\boldsymbol{v}), \ (x, \boldsymbol{v}) \in T(\partial\Omega), \quad \boldsymbol{v} \neq 0. \tag{3.5}$$

Remark 3.2. In the case of isotropic slip, the friction coefficient F is a function of only x. Then definition (3.4)–(3.5) reduces to $G(x, \boldsymbol{v}) = F_a(x) = F(x)$ for all $(x, \boldsymbol{v}) \in T(\partial\Omega)$. Moreover, if F is a constant function, we obtain Navier's slip condition. This special case will be considered in Section 9.

Next, for every $\boldsymbol{v} \in V$, define $H(\boldsymbol{v})$ on $\partial\Omega$ by

$$H(\boldsymbol{v})(x) = G(x, (\gamma\boldsymbol{v})(x)), \quad x \in \partial\Omega, \tag{3.6}$$

where γ denotes the trace operator. The slip boundary condition (1.1) will be formulated in terms of H.

Now consider the steady flow of an incompressible Newtonian fluid subject to slip boundary condition (1.1). The Cauchy stress tensor is $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}(\boldsymbol{v})$, where μ is the viscosity (a positive constant), $\boldsymbol{v}$ is the velocity, p is the pressure and $\mathbb{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top)$. Let $\boldsymbol{f}$ be the external body force per unit volume and let $\boldsymbol{g}$ be an applied tangential surface traction such that

$$\boldsymbol{f} \in L^2(\Omega)^3, \quad \boldsymbol{g} \in H^{1/2}(\partial\Omega)^3, \quad \boldsymbol{g}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega. \tag{3.7}$$

Then we have the following boundary-value problem.

Problem 3.1. Find $(\boldsymbol{v}, p) \in H^2(\Omega)^3 \times H^1(\Omega)$ such that

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \mu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{3.8}$$

$$\text{div } \boldsymbol{v} = 0 \quad \text{in } \Omega, \tag{3.9}$$

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3.10}$$

$$(\mathbb{T}\boldsymbol{n})_\tau + H(\boldsymbol{v})\boldsymbol{v} = \boldsymbol{g} \quad \text{on } \partial\Omega. \tag{3.11}$$

We derive and study a weak form of Problem 3.1. Suppose that $(\boldsymbol{v}, p)$ is a solution of Problem 3.1. Then, by (3.8)–(3.9), $\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \text{div } \mathbb{T} = \boldsymbol{f}$ in Ω. Thus $(\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \text{div } \mathbb{T}, \boldsymbol{\psi}) = (\boldsymbol{f}, \boldsymbol{\psi})$ for all $\boldsymbol{\psi} \in V$. By applying Green's formula, the properties of $\boldsymbol{\psi}$, the symmetry of $\mathbb{D}(\boldsymbol{v})$, and boundary conditions (3.10)–(3.11), one deduces that for all $\boldsymbol{\psi} \in V$,

$$a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} = (\boldsymbol{f}, \boldsymbol{\psi}) + (\boldsymbol{g}, \boldsymbol{\psi})_{\partial\Omega}, \tag{3.12}$$

where $a(\cdot,\cdot)$ is the bilinear form on $(H^1(\Omega)^3)^2$ defined by

$$a(\boldsymbol{v}, \boldsymbol{w}) = 2\mu(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{w})),$$

and $b(\cdot,\cdot,\cdot)$ is the trilinear form on $(H^1(\Omega)^3)^3$ defined by

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \boldsymbol{w}).$$

The right-hand side of (3.12) defines a bounded linear functional on V. Thus, by the Riesz representation theorem, there exists a unique $\boldsymbol{h} \in V$ such that $(\boldsymbol{f}, \boldsymbol{w}) + (\boldsymbol{g}, \boldsymbol{w})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{w})_{1,2}$ for all $\boldsymbol{w} \in V$. Moreover, by the Schwarz inequality and the trace theorem, $\|\boldsymbol{h}\|_{1,2} \leqslant \|\boldsymbol{f}\| + C(\Omega)\|\boldsymbol{g}\|_{\partial\Omega}$. Hence, for a given $\boldsymbol{h} \in V$, we consider the following weak form of Problem 3.1.

Problem 3.2. Find $\boldsymbol{v} \in V$ such that for all $\boldsymbol{\psi} \in V$,

$$a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}. \tag{3.13}$$

We call a solution $\boldsymbol{v} \in V$ of Problem 3.2 a *weak solution* of Problem 3.1.

4. Preliminaries

Let $\gamma\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ be the trace operator. Then $\gamma\colon H^1(\Omega) \to L^q(\partial\Omega)$ is continuous for all $q \in [1, 4]$ and compact for all $q \in [1, 4)$. In particular, there is a constant $C_t = C_t(\Omega)$ such that

$$\|\boldsymbol{v}\|_{\partial\Omega} \leqslant \sqrt{C_t}\|\boldsymbol{v}\|_{1,2} \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^3. \tag{4.1}$$

The bilinear form $a(\cdot, \cdot)$ is continuous on $(H^1(\Omega)^3)^2$, since

$$|a(\boldsymbol{v}, \boldsymbol{w})| \leqslant 2\mu\|\nabla\boldsymbol{v}\|\cdot\|\nabla\boldsymbol{w}\|. \tag{4.2}$$

The imbedding $H^1(\Omega) \to L^q(\Omega)$ is continuous for all $q \in [1, 6]$ and compact for all $q \in [1, 6)$. The trilinear form $b(\cdot, \cdot, \cdot)$ is continuous on $(H^1(\Omega)^3)^3$, since it follows from the generalized Hölder inequality and the imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ that

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leqslant \|\boldsymbol{u}\|_4\|\nabla\boldsymbol{v}\|\cdot\|\boldsymbol{w}\|_4 \leqslant C_b\|\boldsymbol{u}\|_{1,2}\|\nabla\boldsymbol{v}\|\cdot\|\boldsymbol{w}\|_{1,2} \tag{4.3}$$

for some constant $C_b = C_b(\Omega)$. Furthermore, it follows from Green's formula that if $\boldsymbol{u} \in V$, then $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$ for all $\boldsymbol{v}, \boldsymbol{w} \in H^1(\Omega)^3$. Thus

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{u} \in V,\ \boldsymbol{v} \in H^1(\Omega)^3. \tag{4.4}$$

The following Korn type inequality (see [5], Lemma 1) is fundamental to the analysis of Problem 3.2.

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$ and let $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$. Then there exist positive constants K_1 and K_2, which depend at most on Ω and Γ, such that for all $A, B > 0$ and all $\boldsymbol{v} \in H^1(\Omega)^3$,*

$$A\|\mathbb{D}(\boldsymbol{v})\|^2 + B\|\boldsymbol{v}\|_\Gamma^2 \geqslant K_1 \min\{K_2 A, B\}\|\boldsymbol{v}\|_{1,2}^2. \tag{4.5}$$

The next lemma shows that although $G(x,\cdot)$ is discontinuous at 0, the mapping $\boldsymbol{v} \mapsto G(x,\boldsymbol{v})\boldsymbol{v}$ is uniformly Lipschitz continuous.

Lemma 4.2. *For all* $(x,\boldsymbol{v})$ *and* $(x,\boldsymbol{w})$ *in* $T(\partial\Omega)$,

$$|G(x,\boldsymbol{v})\boldsymbol{v} - G(x,\boldsymbol{w})\boldsymbol{w}| \leqslant (F_U + 2M_0)|\boldsymbol{v}-\boldsymbol{w}|, \tag{4.6}$$

$$|(G(x,\boldsymbol{v}) - G(x,\boldsymbol{w}))\boldsymbol{v}| \leqslant \max\{F_U - F_L, 2M_0\}|\boldsymbol{v}-\boldsymbol{w}|. \tag{4.7}$$

P r o o f. Let $(x,\boldsymbol{v}),(x,\boldsymbol{w}) \in T(\partial\Omega)$.

(a) If $\boldsymbol{v}=0$ or $\boldsymbol{w}=0$, then

$$|G(x,\boldsymbol{v})\boldsymbol{v} - G(x,\boldsymbol{w})\boldsymbol{w}| \leqslant F_U|\boldsymbol{v}-\boldsymbol{w}|$$

by $(3.3)_2$. Assume that $\boldsymbol{v}\neq 0$ and $\boldsymbol{w}\neq 0$. Then, by (3.2),

$$\begin{aligned} |F(x,|\boldsymbol{v}|^{-1}\boldsymbol{v}) - F(x,|\boldsymbol{w}|^{-1}\boldsymbol{w})| &\leqslant M_0\big||\boldsymbol{v}|^{-1}\boldsymbol{v} - |\boldsymbol{w}|^{-1}\boldsymbol{w}\big| \\ &= M_0|\boldsymbol{v}|^{-1}|\boldsymbol{w}|^{-1}\big||\boldsymbol{w}|(\boldsymbol{v}-\boldsymbol{w}) + (|\boldsymbol{w}|-|\boldsymbol{v}|)\boldsymbol{w}\big| \\ &\leqslant 2M_0|\boldsymbol{v}|^{-1}|\boldsymbol{v}-\boldsymbol{w}|. \end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned} |G(x,\boldsymbol{v})\boldsymbol{v} - G(x,\boldsymbol{w})\boldsymbol{w}| &= |F(x,|\boldsymbol{v}|^{-1}\boldsymbol{v})\boldsymbol{v} - F(x,|\boldsymbol{w}|^{-1}\boldsymbol{w})\boldsymbol{w}| \\ &\leqslant |(F(x,|\boldsymbol{v}|^{-1}\boldsymbol{v}) - F(x,|\boldsymbol{w}|^{-1}\boldsymbol{w}))\boldsymbol{v}| + |F(x,|\boldsymbol{w}|^{-1}\boldsymbol{w})(\boldsymbol{v}-\boldsymbol{w})| \\ &\leqslant 2M_0|\boldsymbol{v}-\boldsymbol{w}| + F_U|\boldsymbol{v}-\boldsymbol{w}| \end{aligned}$$

by (3.3) and (4.8). This proves (4.6).

(b) If $\boldsymbol{v}=0$, (4.7) holds trivially. If $\boldsymbol{v}\neq 0$ and $\boldsymbol{w}=0$, then

$$|(G(x,\boldsymbol{v}) - G(x,\boldsymbol{w}))\boldsymbol{v}| = |(F(x,|\boldsymbol{v}|^{-1}\boldsymbol{v}) - F_a(x))\boldsymbol{v}| \leqslant (F_U - F_L)|\boldsymbol{v}-\boldsymbol{w}|$$

by (3.3). If $\boldsymbol{v}\neq 0$ and $\boldsymbol{w}\neq 0$, then

$$|(G(x,\boldsymbol{v}) - G(x,\boldsymbol{w}))\boldsymbol{v}| = |(F(x,|\boldsymbol{v}|^{-1}\boldsymbol{v}) - F(x,|\boldsymbol{w}|^{-1}\boldsymbol{w}))\boldsymbol{v}| \leqslant 2M_0|\boldsymbol{v}-\boldsymbol{w}|$$

by (4.8). This proves (4.7). □

Lemma 4.3. *Let* $m \geqslant 2$. *Then there exists an orthonormal sequence* $(\boldsymbol{\psi}_n)$ *in* V *such that* $\{\boldsymbol{\psi}_n\colon n\in\mathbb{N}\}$ *is a basis for* V *and* $\boldsymbol{\psi}_n \in H^m(\Omega)^3$ *for all* $n\in\mathbb{N}$.

P r o o f. Let $\boldsymbol{u} \in V$. Then $(\boldsymbol{u}, \cdot)_{1,2}$ defines a bounded linear functional on $V_m := H^m(\Omega)^3 \cap V$, which is a closed subspace of $H^m(\Omega)^3$ and thus a Hilbert space. Thus, by the Riesz representation theorem, there exists a unique $\boldsymbol{v} \in V_m$ such that $(\boldsymbol{u}, \boldsymbol{w})_{1,2} = (\boldsymbol{v}, \boldsymbol{w})_{m,2}$ for all $\boldsymbol{w} \in V_m$, and $\|\boldsymbol{v}\|_{m,2} \leqslant \|\boldsymbol{u}\|_{1,2}$. Define $T\colon V \to V$ by $T(\boldsymbol{u}) = \boldsymbol{v}$. Then T is linear, self-adjoint, injective (since V_m is dense in V) and bounded. Moreover, T is compact since $H^m(\Omega) \hookrightarrow\hookrightarrow H^1(\Omega)$. Hence, by a classical result (see, e.g., [7], Theorem I.7.C), T has an orthonormal sequence $(\boldsymbol{\psi}_n)$ of eigenvectors, which is a basis for V, and the corresponding sequence of eigenvalues (λ_n) converges to zero. For each n, $(\boldsymbol{\psi}_n, \boldsymbol{w})_{1,2} = \lambda_n(\boldsymbol{\psi}_n, \boldsymbol{w})_{m,2}$ for all $\boldsymbol{w} \in V_m$. Thus $\lambda_n \neq 0$ (since V_m is dense in V) and $\boldsymbol{\psi}_n = \lambda_n^{-1} T\boldsymbol{\psi}_n \in V_m$. □

The following lemma, due to Miranda [6] (see also [3], Lemma IX.3.1), is equivalent to Brouwer's fixed point theorem.

Lemma 4.4. *Let $n \in \mathbb{N}$ and let $\boldsymbol{P}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that for some $R > 0$,*

$$\boldsymbol{P}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant 0 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ with } |\boldsymbol{\xi}| = R. \tag{4.9}$$

Then there exists $\boldsymbol{\xi}^\star \in \mathbb{R}^n$ such that $\boldsymbol{P}(\boldsymbol{\xi}^\star) = 0$ and $|\boldsymbol{\xi}^\star| \leqslant R$.

5. Uniqueness of a solution

For brevity, let

$$K := K_1 \min\{2K_2\mu, F_L\}, \quad D := C_t \max\{F_U - F_L, 2M_0\}. \tag{5.1}$$

In essence, K represents the viscosity (internal friction) and surface friction of the liquid and D represents the extent to which the surface friction varies with position and direction.

Lemma 5.1. *If $\boldsymbol{v}$ is a solution of Problem 3.2, then*

$$\|\boldsymbol{v}\|_{1,2} \leqslant R := K^{-1}\|\boldsymbol{h}\|_{1,2}. \tag{5.2}$$

P r o o f. If $\boldsymbol{v}$ is a solution of Problem 3.2, set $\boldsymbol{\psi} = \boldsymbol{v}$ in (3.13). Then

$$\begin{aligned} K_1 \min\{2K_2\mu, F_L\}\|\boldsymbol{v}\|_{1,2}^2 &\leqslant a(\boldsymbol{v}, \boldsymbol{v}) + F_L\|\boldsymbol{v}\|_{\partial\Omega}^2 \\ &\leqslant a(\boldsymbol{v}, \boldsymbol{v}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{v})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{v})_{1,2} \\ &\leqslant \|\boldsymbol{h}\|_{1,2}\|\boldsymbol{v}\|_{1,2} \end{aligned}$$

by (4.5) and $(3.3)_1$ and (4.4). □

Theorem 5.1. *Let K and D be as in* (5.1) *and suppose that $D < K$.*

(a) *If $\boldsymbol{v}$ is a solution of Problem* 3.2 *and*

$$\|\nabla \boldsymbol{v}\|_2 < C_b^{-1}(K - D), \tag{5.3}$$

then $\boldsymbol{v}$ is the only solution of Problem 3.2.

(b) *If*

$$\|\boldsymbol{h}\|_{1,2} < C_b^{-1}K(K - D), \tag{5.4}$$

then Problem 3.2 *has at most one solution.*

P r o o f. If $\boldsymbol{h} \equiv 0$, then $\boldsymbol{v} \equiv 0$ is a solution of Problem 3.2. Moreover, by estimate (5.2), it is the only solution. If $\boldsymbol{h}$ is not identically zero, suppose that $\boldsymbol{v} \in V$ and $v \in V$ are solutions of Problem 3.2. Thus

$$a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}, \tag{5.5}$$

$$a(v, \boldsymbol{\psi}) + b(v, v, \boldsymbol{\psi}) + (H(v)v, \boldsymbol{\psi})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi})_{1,2} \tag{5.6}$$

for all $\boldsymbol{\psi} \in V$. Let $\boldsymbol{\psi} = \boldsymbol{v} - v$ and subtract (5.6) from (5.5). This yields

$$a(\boldsymbol{v} - v, \boldsymbol{v} - v) + I + J = 0, \tag{5.7}$$

where $I = (\boldsymbol{v} \cdot \nabla \boldsymbol{v} - v \cdot \nabla v, \boldsymbol{v} - v)$ and $J = (H(\boldsymbol{v})\boldsymbol{v} - H(v)v, \boldsymbol{v} - v)_{\partial\Omega}$. Now,

$$I = ((\boldsymbol{v} - v) \cdot \nabla \boldsymbol{v}, \boldsymbol{v} - v) + (v \cdot \nabla(\boldsymbol{v} - v), \boldsymbol{v} - v) = ((\boldsymbol{v} - v) \cdot \nabla \boldsymbol{v}, \boldsymbol{v} - v)$$

by (4.4) and thus

$$|I| \leqslant C_b \|\nabla \boldsymbol{v}\|_2 \|\boldsymbol{v} - v\|_{1,2}^2 \tag{5.8}$$

by (4.3). Furthermore,

$$J = ((H(\boldsymbol{v}) - H(v))\boldsymbol{v}, \boldsymbol{v} - v)_{\partial\Omega} + (H(v)(\boldsymbol{v} - v), \boldsymbol{v} - v)_{\partial\Omega}$$

and by (4.7) and (4.1),

$$\begin{aligned} |((H(\boldsymbol{v}) - H(v))\boldsymbol{v}, \boldsymbol{v} - v)_{\partial\Omega}| &\leqslant \max\{F_U - F_L, 2M_0\}\|\boldsymbol{v} - v\|_{\partial\Omega}^2 \\ &\leqslant C_t \max\{F_U - F_L, 2M_0\}\|\boldsymbol{v} - v\|_{1,2}^2. \end{aligned} \tag{5.9}$$

By applying (4.5) and $(3.3)_1$, (5.7) and estimates (5.8)–(5.9) we obtain

$$\begin{aligned}K_1 \min\{2K_2\mu, F_L\}\|\boldsymbol{v}-v\|_{1,2}^2 &\leqslant a(\boldsymbol{v}-v, \boldsymbol{v}-v) + F_L\|\boldsymbol{v}-v\|_{\partial\Omega}^2 \\ &\leqslant a(\boldsymbol{v}-v, \boldsymbol{v}-v) + (H(v)(\boldsymbol{v}-v), \boldsymbol{v}-v)_{\partial\Omega} \\ &= -I - ((H(\boldsymbol{v})-H(v))\boldsymbol{v}, \boldsymbol{v}-v)_{\partial\Omega} \\ &\leqslant (C_b\|\nabla\boldsymbol{v}\|_2 + C_t\max\{F_U-F_L, 2M_0\})\|\boldsymbol{v}-v\|_{1,2}^2.\end{aligned}$$

Hence, if $D < K$ and (5.3) holds, then $\boldsymbol{v} = v$. By Lemma 5.1, a sufficient condition for (5.3) is $K^{-1}\|\boldsymbol{h}\|_{1,2} < C_b^{-1}(K-D)$. □

6. Existence of a solution

Theorem 6.1. *Problem* 3.2 *has a solution* $\boldsymbol{v} \in V$, *which satisfies estimate* (5.2).

Proof. By Lemma 4.3 there exists a sequence $(\boldsymbol{\psi}_n)$ in $V \cap H^3(\Omega)^3$ that is an orthonormal basis for V. For each $n \in \mathbb{N}$, let $S_n := \operatorname{span}\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \dots, \boldsymbol{\psi}_n\}$ and consider the following problem:

Find $\boldsymbol{v}_n = \sum\limits_{i=1}^{n} \xi_{ni}\boldsymbol{\psi}_i \in S_n$ such that for $k = 1, 2, \dots, n$,

$$a(\boldsymbol{v}_n, \boldsymbol{\psi}_k) + b(\boldsymbol{v}_n, \boldsymbol{v}_n, \boldsymbol{\psi}_k) + (H(\boldsymbol{v}_n)\boldsymbol{v}_n, \boldsymbol{\psi}_k)_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi}_k)_{1,2}. \tag{6.1}$$

This is an algebraic system in the unknown $\boldsymbol{\xi}_n = (\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}) \in \mathbb{R}^n$.

Define $\boldsymbol{P}\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{P}(\boldsymbol{\xi}) = (P_1(\boldsymbol{\xi}), P_2(\boldsymbol{\xi}), \dots, P_n(\boldsymbol{\xi}))$ with

$$P_k(\boldsymbol{\xi}) = a(\boldsymbol{v}, \boldsymbol{\psi}_k) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}_k) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega} - (\boldsymbol{h}, \boldsymbol{\psi}_k)_{1,2}$$

for $k = 1, 2, \dots, n$, where $\boldsymbol{v} := \sum\limits_{i=1}^{n} \xi_i\boldsymbol{\psi}_i$ if $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$. Then, by the linearity of $P_k(\boldsymbol{\xi})$ in $\boldsymbol{\psi}_k$, (4.4) and $(3.3)_1$ and (4.5),

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} = \sum_{k=1}^{n} P_k(\boldsymbol{\xi})\xi_k &= a(\boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{v})_{\partial\Omega} - (\boldsymbol{h}, \boldsymbol{v})_{1,2} \\ &\geqslant 2\mu\|D(\boldsymbol{v})\|^2 + F_L\|\boldsymbol{v}\|_{\partial\Omega}^2 - \|\boldsymbol{h}\|_{1,2}\|\boldsymbol{v}\|_{1,2} \\ &\geqslant K\|\boldsymbol{v}\|_{1,2}^2 - \|\boldsymbol{h}\|_{1,2}\|\boldsymbol{v}\|_{1,2}\end{aligned}$$

with K as in (5.1). Moreover, $\|\boldsymbol{v}\|_{1,2} = |\boldsymbol{\xi}|$ because $(\boldsymbol{\psi}_n)$ is orthonormal in $H^1(\Omega)^3$. Thus

$$\boldsymbol{P}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geqslant (K|\boldsymbol{\xi}| - \|\boldsymbol{h}\|_{1,2})|\boldsymbol{\xi}| = 0 \tag{6.2}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$ with $|\boldsymbol{\xi}| = R = K^{-1}\|\boldsymbol{h}\|_{1,2}$.

Next, let $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^n$, let $\boldsymbol{u} := \sum_{i=1}^{n} \eta_i \boldsymbol{\psi}_i$ and let $\boldsymbol{v} := \sum_{i=1}^{n} \xi_i \boldsymbol{\psi}_i$. Then, for $k = 1, 2, \ldots, n$,

$$P_k(\boldsymbol{\eta}) - P_k(\boldsymbol{\xi}) = a(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{\psi}_k) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_k) - b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}_k) \\ + (H(\boldsymbol{u})\boldsymbol{u} - H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega}.$$

By (4.2), $|a(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{\psi}_k)| \leqslant 2\mu \|\boldsymbol{u} - \boldsymbol{v}\|_{1,2} \|\boldsymbol{\psi}_k\|_{1,2} = 2\mu |\boldsymbol{\eta} - \boldsymbol{\xi}|$. By (4.3),

$$\begin{aligned} |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_k) - b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}_k)| &\leqslant |b(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\psi}_k)| + |b(\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{\psi}_k)| \\ &\leqslant C_b \|\boldsymbol{u} - \boldsymbol{v}\|_{1,2} (\|\boldsymbol{u}\|_{1,2} + \|\boldsymbol{v}\|_{1,2}) \|\boldsymbol{\psi}_k\|_{1,2} \\ &= C_b (|\boldsymbol{\eta}| + |\boldsymbol{\xi}|) |\boldsymbol{\eta} - \boldsymbol{\xi}|. \end{aligned}$$

By the Schwarz inequality, (4.6) and (4.1),

$$\begin{aligned} |(H(\boldsymbol{u})\boldsymbol{u} - H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega}| &\leqslant \|H(\boldsymbol{u})\boldsymbol{u} - H(\boldsymbol{v})\boldsymbol{v}\|_{\partial\Omega} \|\boldsymbol{\psi}_k\|_{\partial\Omega} \\ &\leqslant (F_U + 2M_0) \|\boldsymbol{u} - \boldsymbol{v}\|_{\partial\Omega} \|\boldsymbol{\psi}_k\|_{\partial\Omega} \\ &\leqslant C_t (F_U + 2M_0) \|\boldsymbol{u} - \boldsymbol{v}\|_{1,2} \|\boldsymbol{\psi}_k\|_{1,2} \\ &= C_t (F_U + 2M_0) |\boldsymbol{\eta} - \boldsymbol{\xi}|. \end{aligned} \tag{6.3}$$

These estimates imply that P_k is (locally Lipschitz) continuous. Thus by Lemma 4.4, there exists $\boldsymbol{\xi}_n \in \mathbb{R}^n$ such that $\boldsymbol{P}(\boldsymbol{\xi}_n) = 0$ and $|\boldsymbol{\xi}_n| \leqslant R$. Therefore $\boldsymbol{v}_n := \sum_{i=1}^{n} \xi_{ni} \boldsymbol{\psi}_i$ is a solution of problem (6.1) and $\|\boldsymbol{v}_n\|_{1,2} = |\boldsymbol{\xi}_n| \leqslant R$.

Now, since $(\boldsymbol{v}_n)$ is bounded in the Hilbert space V, there exists $\boldsymbol{v} \in V$ and a subsequence of $(\boldsymbol{v}_n)$, again denoted by $(\boldsymbol{v}_n)$, that converges weakly to $\boldsymbol{v}$ in V. Let $k \in \mathbb{N}$. Then, by (4.2), $a(\cdot, \boldsymbol{\psi}_k)$ is a bounded linear functional on V and thus $a(\boldsymbol{v}_n, \boldsymbol{\psi}_k) \to a(\boldsymbol{v}, \boldsymbol{\psi}_k)$ as $n \to \infty$. Furthermore, $b(\boldsymbol{v}_n, \boldsymbol{v}_n, \boldsymbol{\psi}_k) = b(\boldsymbol{v}_n - \boldsymbol{v}, \boldsymbol{v}_n, \boldsymbol{\psi}_k) + b(\boldsymbol{v}, \boldsymbol{v}_n, \boldsymbol{\psi}_k)$. It follows from (4.3) and the imbedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ that

$$|b(\boldsymbol{v}_n - \boldsymbol{v}, \boldsymbol{v}_n, \boldsymbol{\psi}_k)| \leqslant R \|\boldsymbol{\psi}_k\|_4 \|\boldsymbol{v}_n - \boldsymbol{v}\|_4 \to 0 \text{ as } n \to \infty.$$

Moreover, by (4.3), $b(\boldsymbol{v}, \cdot, \boldsymbol{\psi}_k)$ is a bounded linear functional on V. Hence,

$$\lim_{n\to\infty} b(\boldsymbol{v}_n, \boldsymbol{v}_n, \boldsymbol{\psi}_k) = \lim_{n\to\infty} b(\boldsymbol{v}, \boldsymbol{v}_n, \boldsymbol{\psi}_k) = b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}_k).$$

Lastly, as in (6.3),

$$|(H(\boldsymbol{v}_n)\boldsymbol{v}_n, \boldsymbol{\psi}_k)_{\partial\Omega} - (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega}| \leqslant (F_U + 2M_0) \|\boldsymbol{v}_n - \boldsymbol{v}\|_{\partial\Omega} \|\boldsymbol{\psi}_k\|_{\partial\Omega}.$$

Thus $\lim_{n\to\infty} (H(\boldsymbol{v}_n)\boldsymbol{v}_n, \boldsymbol{\psi}_k)_{\partial\Omega} = (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega}$ since $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\partial\Omega)$. It follows from these limits and (6.1) that for every $k \in \mathbb{N}$,

$$a(\boldsymbol{v}, \boldsymbol{\psi}_k) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}_k) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}_k)_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi}_k)_{1,2}.$$

Each term in this equation is a bounded linear functional on V in ψ_k. Thus, since $\operatorname{span}\{\psi_n \colon n \in \mathbb{N}\}$ is dense in V, $\boldsymbol{v}$ satisfies (3.13) for all $\psi \in V$. Estimate (5.2) holds because

$$\|\boldsymbol{v}\|_{1,2} \leqslant \liminf_{n\to\infty} \|\boldsymbol{v}_n\|_{1,2} \leqslant R$$

or by virtue of Lemma 5.1. □

Remark 6.1. (a) Theorem 6.1 does not require a restriction on D or $\|\boldsymbol{h}\|_{1,2}$. It is sufficient that $K > 0$.

(b) Some of the estimates in the proof of Theorem 6.1 can be derived differently, for example by using the fact that $\psi_k \in H^3(\Omega)^3 \hookrightarrow C^1(\overline{\Omega})^3$.

(c) Theorems 5.1 and 6.1 hold for $\boldsymbol{h} \in H^1(\Omega)^3$ as well. Then $\|\boldsymbol{h}\|_{1,2}$ in estimate (5.2) can be replaced by $\|\boldsymbol{h}_V\|_{1,2}$, where $\boldsymbol{h}_V$ is the projection of $\boldsymbol{h}$ onto V.

The following corollary follows from Theorems 5.1 and 6.1.

Corollary 6.1. *Let K and D be as in* (5.1). *If $D < K$ and* (5.4) *holds, then Problem* 3.2 *has a unique solution $\boldsymbol{v} \in V$ and the solution satisfies estimate* (5.2).

7. Pressure

In the derivation of (3.13) from (3.8)–(3.11), the pressure gradient ∇p is eliminated by the application of Green's formula:

$$(\nabla p, \psi) = (p, \psi \cdot \boldsymbol{n})_{\partial\Omega} - (p, \operatorname{div} \psi) = 0 \quad \forall\, \psi \in V.$$

The next theorem shows that each solution $\boldsymbol{v}$ of Problem 3.2 determines a corresponding pressure field p. The proof is adapted from [3], Theorem III.5.3 and [3], Lemma IX.1.2, which deal with the Navier-Stokes equations with the no-slip boundary condition ($\boldsymbol{v} = 0$ on $\partial\Omega$).

Theorem 7.1. *Suppose that $\boldsymbol{v} \in V$ is a solution of Problem* 3.2. *Then there exists $p \in L^2_0(\Omega)$, uniquely determined by $\boldsymbol{v}$, such that for all $\psi \in U$,*

$$a(\boldsymbol{v}, \psi) + b(\boldsymbol{v}, \boldsymbol{v}, \psi) + (H(\boldsymbol{v})\boldsymbol{v}, \psi)_{\partial\Omega} - (p, \operatorname{div} \psi) = (\boldsymbol{h}, \psi)_{1,2}. \tag{7.1}$$

Moreover,

$$\|p\| \leqslant C(\Omega)((C_b\|\boldsymbol{v}\|_{1,2} + 2\mu)\|\boldsymbol{v}\|_{1,2} + \|\boldsymbol{h}\|_{1,2}). \tag{7.2}$$

Proof. By Green's formula, div $\boldsymbol{u} \in L_0^2(\Omega)$ for all $\boldsymbol{u} \in U$. Thus we can define an operator $A\colon U \to L_0^2(\Omega)$ by $A\boldsymbol{u} = \operatorname{div} \boldsymbol{u}$. Then A is linear and bounded, hence closed, and $\ker A = V$. Moreover, for every $f \in L_0^2(\Omega)$ there exists $\boldsymbol{u} \in H^1(\Omega)^3$ such that

$$\operatorname{div} \boldsymbol{u} = f \text{ in } \Omega, \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega, \quad \|\boldsymbol{u}\|_{1,2} \leqslant C_d\|f\| \tag{7.3}$$

for some constant $C_d = C_d(\Omega)$ (see [3], Theorem III.3.1; Ω is bounded and has the cone property). Thus $R(A) = L_0^2(\Omega)$ is closed in $L_0^2(\Omega)$.

Let $A^*\colon L_0^2(\Omega)' \to U'$ be the adjoint of A, where the prime means dual space. Then, by the Banach closed range theorem, $V^\perp = [\ker A]^\perp = R(A^*)$, where $\perp$ means annihilator. Thus, if $\mathcal{F} \in U'$ and $\mathcal{F}\boldsymbol{\psi} = 0$ for all $\boldsymbol{\psi} \in V$, there exists $\mathcal{L} \in L_0^2(\Omega)'$ such that $\mathcal{F} = \mathcal{L} \circ A$. Thus, by the Riesz representation theorem, there exists $p \in L_0^2(\Omega)$ such that

$$\mathcal{F}\boldsymbol{\psi} = \mathcal{L}(A\boldsymbol{\psi}) = (p, A\boldsymbol{\psi}) = (p, \operatorname{div} \boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in U. \tag{7.4}$$

Now, if $\boldsymbol{v} \in V$ is a solution of Problem 3.2, consider the linear functional $\mathcal{F}$ on U defined by

$$\mathcal{F}\boldsymbol{\psi} = a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} - (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}, \quad \boldsymbol{\psi} \in U.$$

Here $\mathcal{F}$ is bounded on U (by $(3.3)_2$ and (4.1)–(4.3)) and $\mathcal{F}$ vanishes on V (by (3.13)). Hence, there exists $p \in L_0^2(\Omega)$ that satisfies (7.4), which gives (7.1). If $\hat{p} \in L_0^2(\Omega)$ also satisfies (7.4), then $(p-\hat{p}, \operatorname{div} \boldsymbol{\psi}) = 0$ for all $\boldsymbol{\psi} \in U$. Since A is surjective, there exists $\boldsymbol{\psi} \in U$ with $\operatorname{div} \boldsymbol{\psi} = p - \hat{p}$. Thus $p - \hat{p} = 0$. Hence, p is uniquely determined by $\boldsymbol{v}$.

Lastly, let $\boldsymbol{\psi}$ be a solution of problem (7.3) with $f = p$. Then, by (7.1) and the fact that $\boldsymbol{\psi} = 0$ on $\partial\Omega$,

$$\|p\|^2 = a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) - (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}.$$

Thus it follows from $(3.3)_2$ and (4.1)–(4.3) and estimate $(7.3)_3$ that

$$\|p\|^2 \leqslant C_d(2\mu\|\nabla \boldsymbol{v}\| + C_b\|\boldsymbol{v}\|_{1,2}\|\nabla \boldsymbol{v}\| + \|\boldsymbol{h}\|_{1,2})\|p\|,$$

which implies estimate (7.2). □

8. Stokes problem

If (3.8) is replaced by the Stokes equations, the weak form of the boundary-value problem is as follows:

Problem 8.1. Find $\boldsymbol{v} \in V$ such that for all $\boldsymbol{\psi} \in V$,

$$a(\boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} = (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}. \tag{8.1}$$

The arguments in Sections 5–7 then yield the following theorems.

Theorem 8.1. *Let K and D be as in* (5.1).

(a) *Every solution $\boldsymbol{v}$ of Problem* 8.1 *satisfies estimate* (5.2).
(b) *If $D < K$, Problem* 8.1 *has at most one solution $\boldsymbol{v}$.*

Theorem 8.2. *Problem* 8.1 *has a solution $\boldsymbol{v} \in V$. Moreover, there exists $p \in L_0^2(\Omega)$, uniquely determined by $\boldsymbol{v}$, such that for all $\boldsymbol{\psi} \in U$,*

$$a(\boldsymbol{v}, \boldsymbol{\psi}) + (H(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi})_{\partial\Omega} - (p, \operatorname{div} \boldsymbol{\psi}) = (\boldsymbol{h}, \boldsymbol{\psi})_{1,2}. \tag{8.2}$$

In addition,

$$\|p\| \leqslant C(\Omega)(2\mu\|\boldsymbol{v}\|_{1,2} + \|\boldsymbol{h}\|_{1,2}). \tag{8.3}$$

Remark 8.1. Problem 8.1 was considered in [5] by a different approach. In [5] the function G is approximated by a smooth function to ensure that the mapping H is Lipschitz continuous. The existence and uniqueness of solution is then proved by the contraction mapping theorem. The present approach improves on this in two respects. Lemma 4.2 shows that it is not necessary to regularize G and that H does not have to be Lipschitz continuous. Furthermore, the existence proof by the Galerkin method avoids the smallness condition on $\boldsymbol{h}$ required by the contraction mapping theorem.

9. Navier slip

Consider the case of Navier slip, where F is a positive constant. Here

$$M_0 = 0, \quad F_L = F_U = F, \quad K = K_1 \min\{2K_2\mu, F\}, \quad D = 0.$$

Hence, Theorems 5.1–7.1 for the Navier-Stokes problem can be stated as follows.

Theorem 9.1. *Problem* 3.2 *has at least one solution $\boldsymbol{v} \in V$. If*

$$\|\nabla \boldsymbol{v}\|_{1,2} < C_b^{-1} K \quad \text{or} \quad \|\boldsymbol{h}\|_{1,2} < C_b^{-1} K^2, \tag{9.1}$$

the solution is unique. Every solution satisfies estimate (5.2) *and for every solution there exists a unique $p \in L_0^2(\Omega)$ that satisfies* (7.1). *Moreover, p satisfies estimate* (7.2).

Theorems 8.1–8.2 for the Stokes problem reduce to the following result.

Theorem 9.2. *Problem* 8.1 *has a unique solution* $\boldsymbol{v} \in V$ *and there exists a unique* $p \in L^2_0(\Omega)$ *that satisfies* (8.2). *Moreover,* $\boldsymbol{v}$ *satisfies estimate* (5.2) *and* p *satisfies estimate* (8.3).

R e m a r k 9.1. The slip boundary condition (1.1) can be written as

$$\boldsymbol{v} = \frac{\boldsymbol{g} - (\mathbb{T}\boldsymbol{n})_\tau}{F} \quad \text{on } \partial\Omega.$$

Formally, this becomes the no-slip condition in the limit $F \to \infty$. Since $K = 2K_1K_2\mu$ for all $F \geqslant 2K_2\mu$, estimate (5.2) and inequalities (9.1) are independent of F if $F \geqslant 2K_2\mu$. Moreover, the constants in estimate (7.2) are independent of F. These four inequalities are the same as those for the Navier-Stokes no-slip problem; see, e.g., (IX.2.4), estimates (IX.3.5)–(IX.3.6) and (IX.3.22) in [3]. (There are slight differences because the weak form of the no-slip problem is formulated in terms of the bilinear form $(\nabla\boldsymbol{v}, \nabla\boldsymbol{w})$ instead of $b(\boldsymbol{v}, \boldsymbol{w})$ and analyzed by means of Poincaré's inequality instead of Korn's inequality.) Hence, the Navier slip problem has the same solvability properties as the corresponding no-slip problem if $F \geqslant 2K_2\mu$. This also applies to the corresponding Stokes problems.

A c k n o w l e d g e m e n t. The author thanks the reviewers for their comments which helped to improve the article.

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Author's address: *Christiaan Le Roux*, Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa, e-mail: ianleroux@yahoo.com.