Czechoslovak Mathematical Journal

Zhiqi Chen; Jifu Li; Ming Ding

On the classification of 3-dimensional F-manifold algebras

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 4, 1191–1204

Persistent URL: `http://dml.cz/dmlcz/151141`

Czechoslovak Mathematical Journal, 72 (147) (2022), 1191–1204

ON THE CLASSIFICATION OF 3-DIMENSIONAL F-MANIFOLD ALGEBRAS

Zhiqi Chen, Guangzhou, Jifu Li, Tianjin, Ming Ding, Guangzhou

Received January 11, 2022. Published online July 28, 2022.

Abstract. F-manifold algebras are focused on the algebraic properties of the tangent sheaf of F-manifolds. The local classification of 3-dimensional F-manifolds has been given in A. Basalaev, C. Hertling (2021). We study the classification of 3-dimensional F-manifold algebras over the complex field $\mathbb{C}$.

Keywords: F-manifold; Poisson algebra; F-manifold algebra

MSC 2020: 17A30, 17B60

1. Introduction

The concept of Frobenius manifolds was introduced by Dubrovin (see [8]) in order to give a geometrical expression of the Witten-Dijkgraaf-Verlinde-Verlinde equations. In 1999, Hertling and Manin in [11] invented the notion of F-manifolds as a relaxation of the conditions of Frobenius manifolds, i.e., F-manifolds without flat metric (see [10], [11] for more details). In [10], Hertling gave the complete classification of the germs of all 2-dimensional F-manifolds with or without Euler fields, also the partial classification of 3-dimensional cases but not pursued systematically. Basalaev and Hertling in [2] tried to obtain a systematic classification of germs of 3-dimensional F-manifolds, and succeeded in most of the cases.

Inspired by the investigation of algebraic structures of F-manifolds, the notion of an F-manifold algebra is given by Dotsenko in 2019 (see [7]) to relate the operad F-manifold algebras to the operad pre-Lie algebras. F-manifold algebra is defined

This work has been supported by the Basic Research Joint Funding Project of University and Guangzhou City under Grant 202201020103, the National Natural Science Foundation of China (11931009, 12131012, 12071228, 11771331, 11701419).

DOI: 10.21136/CMJ.2022.0017-22

as a triple $(A, \cdot, [,])$ satisfying the Hertling-Manin relation

$$P_{x\cdot y}(z,w) = x \cdot P_y(z,w) + y \cdot P_x(z,w) \quad \forall\, x,y,z,w \in A),$$

where $(A, \cdot)$ is a commutative associative algebra, $(A, [,])$ is a Lie algebra and $P_x(y,z) = [x, y \cdot z] - [x,y] \cdot z - y \cdot [x,z]$. Note that the commutative Poisson algebras are a special class of F-manifold algebras with $P_x(y,z) = 0$. Poisson algebras have been studied by many people and played an important role in various areas in mathematics and mathematical physics, such as Poisson geometry, integrable systems and noncommutative geometry (see for example [5], [12], [14], [17]). As a generalization, F-manifold algebras have played a central role in the theory of Frobenius manifolds and the theory of F-manifolds. Therefore, it is necessary to understand the structures and properties of F-manifold algebras in detail. In [4], the authors defined the notions of F-algebra-Rinehart pairs and super F-algebroids, and studied the connection between them. Recently, Liu, Sheng and Bai in [13] introduced pre-F-manifold algebras which give rise to F-manifold algebras through the subjacent associative algebras and the subjacent Lie algebras. Later, these results have been generalized to the case of F-manifold color algebras in [6]. The notion of Hom-F-manifold algebras and their proprieties were given in [3].

In this paper, we study the classification of F-manifold algebras with a multiplication with or without unit over the complex field in dimension three. Note that F-manifolds pursued in [2] are complex manifolds with a multiplication with unit. To the best of our knowledge, the corresponding F-manifold algebras should contain the unit from an algebraic viewpoint. We hope the classification given in this paper can be related to the local classification of 3-dimensional F-manifolds in [2] and regarded as a guide for further study.

The paper is organized as follows. In Section 2, we recall the classifications of 3-dimensional commutative associative algebras and Lie algebras over the complex field. In Section 3, we discuss the classification of 3-dimensional F-manifold algebras over the complex field.

2. Preliminaries

In this section, we briefly summarize the classifications of 3-dimensional commutative associative algebras and 3-dimensional Lie algebras over the complex field $\mathbb{C}$, respectively.

Let e_1, e_2, e_3 be a basis of the commutative associative algebra $(A, \cdot)$. Set $e_i \cdot e_j = \sum\limits_{k=1}^{3} a_{ij}^k e_k$, then the characteristic matrix is defined as

$$\begin{pmatrix} \sum\limits_{k=1}^{3} a_{11}^k e_k & \sum\limits_{k=1}^{3} a_{12}^k e_k & \sum\limits_{k=1}^{3} a_{13}^k e_k \\ \sum\limits_{k=1}^{3} a_{21}^k e_k & \sum\limits_{k=1}^{3} a_{22}^k e_k & \sum\limits_{k=1}^{3} a_{23}^k e_k \\ \sum\limits_{k=1}^{3} a_{31}^k e_k & \sum\limits_{k=1}^{3} a_{32}^k e_k & \sum\limits_{k=1}^{3} a_{33}^k e_k \end{pmatrix}.$$

In [1], Bai and Meng classified the Novikov algebras of dimension three over the complex field. Note that commutative associative algebras are a special class of Novikov algebras. Thus, the classification of 3-dimensional commutative associative algebras can be naturally deduced. We list this classification in Table 1:

Type	Characteristic Matrix	Type	Characteristic Matrix
A_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	A_2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$
A_3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}$	A_4	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$
B_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	B_2	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$
C_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	C_2	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$
C_{11}	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	D_1	$\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$
D_2	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	E	$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$

Table 1. The classification of 3-dimensional commutative associative algebras

Let e_1, e_2, e_3 be a basis of the Lie algebra $(A, [,])$. Listed below is the classification of 3-dimensional nonabelian Lie algebras, see [9], [15], [16] for more details.

Type	Lie bracket multiplication
$\mathfrak{n}_{3,1}$	$[e_1, e_2] = 0, [e_2, e_3] = e_1, [e_1, e_3] = 0$
$\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$	$[e_1, e_2] = e_1, [e_2, e_3] = 0, [e_1, e_3] = 0$
$\mathfrak{s}_{3,1}$	$[e_1, e_2] = 0, [e_2, e_3] = ae_2, [e_1, e_3] = e_1$ $0 < \vert a\vert \leqslant 1$, if $\vert a\vert = 1$ then $\arg(a) \leqslant \pi$
$\mathfrak{s}_{3,2}$	$[e_1, e_2] = 0, [e_2, e_3] = e_1 + e_2, [e_1, e_3] = e_1$
$\mathfrak{sl}(2, \mathbb{C})$	$[e_1, e_2] = 2e_1, [e_2, e_3] = 2e_3, [e_1, e_3] = -e_2$

Table 2. The classification of 3-dimensional nonabelian Lie algebras.

In the following, we use Table 2 to refer the Lie algebras up to isomorphism.

3. On the classification of 3-dimensional F-manifold algebras over the complex field

Let $(A, \cdot, [,])$ be an F-manifold algebra, where the commutative associative algebra $(A, \cdot)$ with a basis e_1, e_2, e_3 belonging to one of the types in Table 1. For each case, we need to determine the Lie bracket. According to the definition of an F-manifold algebra, it is easy to see that $(A, \cdot, [,])$ is an F-manifold algebra if and only if $(A, \cdot)$ is a commutative associative algebra and $(A, [,])$ is a Lie algebra, satisfying the Hertling-Manin relation

$$P_{e_i \cdot e_j}(e_k, e_l) = e_i \cdot P_{e_j}(e_k, e_l) + e_j \cdot P_{e_i}(e_k, e_l), \quad 1 \leqslant i, j, k, l \leqslant 3,$$

where $P_x(y, z) = [x, y \cdot z] - [x, y] \cdot z - y \cdot [x, z]$ for any $x, y, z \in A$.

In what follows, we will give an explicit computation of the F-manifold algebra $(A_3, \cdot, [,])$. For other cases, as the discussion is similar, we only list the results without proofs.

3.1. The commutative associative algebra of type A_3. Note that there is a basis $\{e_1, e_2, e_3\}$ of A_3 such that $e_2 \cdot e_2 = e_1$, $e_3 \cdot e_3 = e_1$ and others are zero.

Lemma 3.1. *A triple $(A_3, \cdot, [,])$ is an F-manifold algebra if and only if for all $1 \leqslant i, j \leqslant 3$*

$$\begin{cases} P_{e_1}(e_i, e_j) = 2e_2 \cdot P_{e_2}(e_i, e_j) = 2e_3 \cdot P_{e_3}(e_i, e_j), \\ e_2 \cdot P_{e_3}(e_i, e_j) + e_3 \cdot P_{e_2}(e_i, e_j) = 0, \end{cases}$$

where $(A_3, \cdot)$ is a commutative associative algebra and $(A_3, [,])$ is a Lie algebra.

P r o o f. By the characteristic matrix of $(A_3, \cdot)$ given in Table 1, we obtain that $(A_3, \cdot, [,])$ is an F-manifold algebra if and only if

$$(3.1) \qquad \begin{cases} P_{e_1}(e_i, e_j) = 2e_k \cdot P_{e_k}(e_i, e_j), & 2 \leqslant k \leqslant 3,\ 1 \leqslant i, j \leqslant 3, \\ e_m \cdot P_{e_n}(e_i, e_j) + e_n \cdot P_{e_m}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3,\ 1 \leqslant m < n \leqslant 3, \end{cases}$$

where $(A_3, \cdot)$ is a commutative associative algebra and $(A_3, [,])$ is a Lie algebra.

Since $e_1 \cdot e_m = 0$ $(1 \leqslant m \leqslant 3)$ and $e_m^2 = e_1$ $(2 \leqslant m \leqslant 3)$, we have that (3.1) hold

$$\Leftrightarrow \begin{cases} P_{e_1}(e_i, e_j) = 2e_2 \cdot P_{e_2}(e_i, e_j) = 2e_3 \cdot P_{e_3}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \\ e_m \cdot P_{e_1}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3,\ 2 \leqslant m \leqslant 3, \\ e_2 \cdot P_{e_3}(e_i, e_j) + e_3 \cdot P_{e_2}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \end{cases}$$
$$\Leftrightarrow \begin{cases} P_{e_1}(e_i, e_j) = 2e_2 \cdot P_{e_2}(e_i, e_j) = 2e_3 \cdot P_{e_3}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \\ e_2 \cdot P_{e_3}(e_i, e_j) + e_3 \cdot P_{e_2}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3. \end{cases}$$

Hence, the proof is finished. □

Lemma 3.2. $(A_3, \cdot, [,])$ *is an* F*-manifold algebra if and only if*

$$\begin{cases} [e_1, e_2] \cdot e_2 = [e_1, e_3] \cdot e_3, \\ [e_1, e_3] \cdot e_2 + [e_1, e_2] \cdot e_3 = 0, \end{cases}$$

where $(A_3, \cdot)$ *is a commutative associative algebra and* $(A_3, [,])$ *is a Lie algebra.*

P r o o f. Through direct computations, we have

$$\begin{aligned} P_{e_i}(e_j, e_k) &= [e_i, e_j \cdot e_k] - [e_i, e_k] \cdot e_j - [e_i, e_j] \cdot e_k \\ &= \begin{cases} 0, & j = k = 1, \\ -[e_i, e_1] \cdot e_k, & j = 1,\ 2 \leqslant k \leqslant 3, \\ -[e_k, e_1] \cdot e_i, & 2 \leqslant j \leqslant 3,\ k = 1, \\ [e_i, e_1] - 2[e_i, e_j] \cdot e_j, & j = k = 2, 3, \\ -[e_i, e_2] \cdot e_3 - [e_i, e_3] \cdot e_2, & j = 2,\ k = 3, \\ -[e_i, e_2] \cdot e_3 - [e_i, e_3] \cdot e_2, & j = 3,\ k = 2. \end{cases} \end{aligned}$$

Thus, the proof can be deduced from Lemma 3.1. □

Let $[e_1, e_2] = h_1 e_1 + h_2 e_2 + h_3 e_3$, $[e_2, e_3] = k_1 e_1 + k_2 e_2 + k_3 e_3$, $[e_1, e_3] = l_1 e_1 + l_2 e_2 + l_3 e_3$. According to Lemma 3.2, we obtain the following result.

Proposition 3.1. *$(A_3, \cdot, [,])$ is an F-manifold algebra if and only if the structure constants satisfy*

$$\begin{cases} h_2 = l_3, \quad h_3 = -l_2, \\ h_1k_2 + l_1k_3 - 2l_3k_1 = 0, \\ l_1l_3 + l_2k_3 - l_2h_1 - l_3k_2 = 0, \\ l_2k_2 + l_1l_2 + l_3h_1 + l_3k_3 = 0. \end{cases} \tag{3.2}$$

From (3.2) we discuss the solutions (h_i, k_i, l_i) in four cases.

Case I: $(l_2, l_3) = (0, 0)$. Then (3.2) becomes

$$h_2 = 0, \quad h_3 = 0, \quad h_1k_2 + l_1k_3 = 0. \tag{3.3}$$

(1) When $(k_2, k_3) = (0, 0)$, then (3.3) holds, thus

$$[e_1, e_2] = h_1e_1, \quad [e_2, e_3] = k_1e_1, \quad [e_1, e_3] = l_1e_1.$$

Furthermore, the Lie algebra $(A_3, [,])$ is
(a) $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $h_1 \neq 0$ or $l_1 \neq 0$,
(b) $\mathfrak{n}_{3,1}$ if $h_1 = l_1 = 0$, $k_1 \neq 0$.

(2) When $(k_2, k_3) \neq (0, 0)$, we have two cases. Firstly, if $k_2 = 0$, $k_3 \neq 0$, then $l_1 = 0$, thus

$$[e_1, e_2] = h_1e_1, \quad [e_2, e_3] = k_1e_1 + k_3e_3, \quad [e_1, e_3] = 0.$$

Furthermore, the Lie algebra $(A_3, [,])$ is
(a) $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $h_1 = 0$,
(b) $\mathfrak{s}_{3,1}$ if $h_1 \neq 0, k_1 = 0$,
(c) $\mathfrak{s}_{3,2}$ if $h_1k_1 \neq 0$, $h_1 = -k_3$,
(d) $\mathfrak{s}_{3,1}$ if $h_1k_1 \neq 0$, $h_1 \neq -k_3$.

Secondly, if $k_2 \neq 0$, then

$$[e_1, e_2] = h_1e_1, \quad [e_2, e_3] = k_1e_1 + k_2e_2 + k_3e_3, [e_1, \quad e_3] = l_1e_1.$$

Thus, we have
(a) $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $k_2^2 + k_3^2 = 0$, $l_1 = 0$,
(b) $\mathfrak{s}_{3,1}$ if $k_2^2 + k_3^2 = 0$, $l_1 \neq 0$, $l_1 \neq k_2$,
(c) $\mathfrak{s}_{3,1}$ if $k_2^2 + k_3^2 = 0$, $l_1 \neq 0$, $l_1 = k_2$, $k_1 = 0$,
(d) $\mathfrak{s}_{3,2}$ if $k_2^2 + k_3^2 = 0$, $l_1 \neq 0$, $l_1 = k_2$, $k_1 \neq 0$,

(e) if $k_2^2+k_3^2 \neq 0$, let $e_1' = e_1$, $e_2' = \left(1/\sqrt{k_2^2+k_3^2}\right)(k_3e_2 - k_2e_3)$, $e_3' = \left(1/\sqrt{k_2^2+k_3^2}\right)(k_2e_2 + k_3e_3)$, then the characteristic matrix of A_3 with the basis $\{e_1', e_2', e_3'\}$ is the same as that with the basis $\{e_1, e_2, e_3\}$. Furthermore,

$$[e_2', e_3'] = k_1e_1' + \sqrt{k_2^2+k_3^2}e_3', \quad [e_1', e_2'] = h_1'e_1' + h_2'e_2' + h_3'e_3',$$
$$[e_1', e_3'] = l_1'e_1' + l_2'e_2' + l_3'e_3',$$

where

$$h_i' = \frac{k_3h_i}{\sqrt{k_2^2+k_3^2}} - \frac{k_2l_i}{\sqrt{k_2^2+k_3^2}} = 0 \quad \text{for } i = 2, 3, \quad l_1' = \frac{k_2h_1}{\sqrt{k_2^2+k_3^2}} + \frac{k_3l_1}{\sqrt{k_2^2+k_3^2}},$$
$$\text{and} \quad l_i' = \frac{k_2h_i}{\sqrt{k_2^2+k_3^2}} + \frac{k_3l_i}{\sqrt{k_2^2+k_3^2}} = 0 \quad \text{for } i = 2, 3.$$

That is the first case of (2).

Case II: $l_2 = 0$, $l_3 \neq 0$. Then (3.2) becomes

$$h_2 = l_3, \quad h_3 = k_1 = 0, \quad k_2 = l_1, \quad k_3 = -h_1.$$

Thus, we have

$$[e_1, e_2] = h_1e_1 + l_3e_2, \quad [e_2, e_3] = l_1e_2 - h_1e_3, \quad [e_1, e_3] = l_1e_1 + l_3e_3.$$

Furthermore, the Lie algebra $(A_3, [,])$ is

(1) $\mathfrak{s}_{3,1}$ if $l_1 = 0$,
(2) $\mathfrak{s}_{3,1}$ if $l_1 \neq 0$, $l_1^2 + h_1^2 = 0$,
(3) if $l_1(l_1^2 + h_1^2) \neq 0$, we can proceed as in the second case of (2) in Case I, and then change the case to the first case, i.e., $l_1 = 0$.

Case III: $l_2 \neq 0$, $l_3 = 0$. Then (3.2) becomes

$$h_2 = 0, \quad h_3 = -l_2, \quad k_2 = -l_1, \quad k_3 = h_1.$$

Thus, we have

$$[e_1, e_2] = h_1e_1 - l_2e_3, \quad [e_2, e_3] = k_1e_1 - l_1e_2 + h_1e_3, \quad [e_1, e_3] = l_1e_1 + l_2e_2.$$

Let $e_1' = e_1$, $e_2' = e_2$, $e_3' = (l_1/l_2)e_1 + e_3$, then the characteristic matrix of A_3 with the basis $\{e_1', e_2', e_3'\}$ is the same as that with the basis $\{e_1, e_2, e_3\}$. Furthermore,

$$[e_1', e_3'] = l_2e_3',$$

which is changed into Case II.

Case IV: $l_2 \neq 0$, $l_3 \neq 0$. We have

(1) $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $l_2^2 + l_3^2 = 0$,

(2) if $l_2^2 + l_3^2 \neq 0$, let

$$e_1' = e_1, \quad e_2' = \frac{1}{\sqrt{l_2^2 + l_3^2}}(l_2 e_2 - l_3 e_3), \quad e_3' = \frac{1}{\sqrt{l_2^2 + l_3^2}}(l_3 e_2 + l_2 e_3),$$

then the characteristic matrix of A_3 with the basis $\{e_1', e_2', e_3'\}$ is the same as that with the basis $\{e_1, e_2, e_3\}$. Furthermore,

$$[e_1', e_3'] = \frac{h_1 l_3 + l_1 l_2}{\sqrt{l_2^2 + l_3^2}} e_1' + \sqrt{l_2^2 + l_3^2} e_2',$$

which is changed into Case III.

In summary, we have the following theorem.

Theorem 3.1. *Let $\{e_1, e_2, e_3\}$ be a basis of A_3 such that $e_2 \cdot e_2 = e_1$, $e_3 \cdot e_3 = e_1$ and others are zero. Then $(A_3, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on A_3 is one of the following cases:*

(1) $[e_1, e_2] = h_1 e_1$, $[e_2, e_3] = k_1 e_1$, $[e_1, e_3] = l_1 e_1$. *For this case, the nonabelian Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $(h_1, l_1) \neq (0, 0)$; $\mathfrak{n}_{3,1}$ if $(h_1, l_1) = (0, 0)$, $k_1 \neq 0$.*

(2) $[e_1, e_2] = h_1 e_1$, $[e_2, e_3] = k_1 e_1 + k_3 e_3$, $[e_1, e_3] = 0$ *for $k_3 \neq 0$. For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $h_1 = 0$; $\mathfrak{s}_{3,1}$ if $h_1 \neq 0$, $k_1 = 0$ or if $h_1 k_1 \neq 0$, $h_1 \neq -k_3$; $\mathfrak{s}_{3,2}$ if $h_1 k_1 \neq 0$, $h_1 = -k_3$.*

(3) $[e_1, e_2] = h_1 e_1$, $[e_2, e_3] = k_1 e_1 + k_2 e_2 + k_3 e_3$, $[e_1, e_3] = l_1 e_1$ *for $k_2 \neq 0$. For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $k_2^2 + k_3^2 = 0$, $l_1 = 0$ or if $k_2^2 + k_3^2 \neq 0$, $k_3 h_1 - k_2 h_1 = 0$; $\mathfrak{s}_{3,1}$ if $k_2^2 + k_3^2 = 0$, $l_1 \neq 0$, $l_1 \neq k_2$ or if $k_2^2 + k_3^2 = 0$, $l_1 = k_2 \neq 0$, $k_1 = 0$ or if $k_2^2 + k_3^2 \neq 0$, $k_3 h_1 - k_2 h_1 \neq 0$, $k_1 = 0$ or if $k_3 h_1 - k_2 h_1 \neq 0$, $k_3 h_1 - k_2 h_1 \neq -k_1 \sqrt{k_2^2 + k_3^2}$, $k_1 \neq 0$; $\mathfrak{s}_{3,2}$ if $k_2^2 + k_3^2 = 0$, $l_1 = k_2 \neq 0$, $k_1 \neq 0$ or if $k_2^2 + k_3^2 \neq 0$, $k_3 h_1 - k_2 h_1 = -k_1 \sqrt{k_2^2 + k_3^2}$, $k_1 \neq 0$.*

(4) $[e_1, e_2] = h_1 e_1 + l_3 e_2$, $[e_2, e_3] = l_1 e_2 - h_1 e_3$, $[e_1, e_3] = l_1 e_1 + l_3 e_3$ *for $l_3 \neq 0$. For this case, the Lie algebra is $\mathfrak{s}_{3,1}$.*

(5) $[e_1, e_2] = h_1 e_1 - l_2 e_3$, $[e_2, e_3] = k_1 e_1 - l_1 e_2 + h_1 e_3$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for $l_2 \neq 0$. For this case, the Lie algebra is $\mathfrak{s}_{3,1}$.*

(6) $[e_1, e_2] = h_1 e_1 + l_3 e_2 - l_2 e_3$, $[e_2, e_3] = k_1 e_1 + k_2 e_2 + k_3 e_3$, $[e_1, e_3] = l_1 e_1 + l_2 e_2 + l_3 e_3$ *for $l_2 l_3 \neq 0$. For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $l_2^2 + l_3^2 = 0$; $\mathfrak{s}_{3,1}$ if $l_2^2 + l_3^2 \neq 0$.*

3.2. The other types of commutative associative algebras. In the following, we set

$$[e_1,e_2]=h_1e_1+h_2e_2+h_3e_3,\quad [e_2,e_3]=k_1e_1+k_2e_2+k_3e_3,\quad [e_2,e_3]=l_1e_1+l_2e_2+l_3e_3,$$

where $h_i,k_i,l_i\in\mathbb{C}$ $(i=1,2,3)$.

Now we list the sufficient and necessary conditions for other cases of $(A,\cdot,[,])$ to be an F-manifold algebra, where $(A,\cdot)$ (other than $(A_3,\cdot)$) is a commutative associative algebra listed in Table 1 and $[,]$ is the Lie bracket on A.

Type A_1: $(A_1,\cdot,[,])$ is an F-manifold algebra if and only if $(A_1,[,])$ is any 3-dimensional Lie algebra.

Type A_2: $(A_2,\cdot,[,])$ is an F-manifold algebra if and only if

$$\begin{cases} P_{e_1}(e_i,e_j)=2e_3\cdot P_{e_3}(e_i,e_j), & 1\leqslant i,j\leqslant 3,\\ e_3\cdot P_{e_2}(e_i,e_j)=0, & 1\leqslant i,j\leqslant 3,\end{cases}$$

where $(A_2,\cdot)$ is a commutative associative algebra and $(A_2,[,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_3=0,\quad k_2h_1+k_3l_1-k_1l_3-k_1h_2=0,$$
$$h_2l_1+k_3l_2-k_2l_3-h_1l_2=0,\quad h_1l_3+k_3h_2=0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.2. *Let $\{e_1,e_2,e_3\}$ be a basis of A_2 such that $e_3\cdot e_3=e_1$ and others are zero. Then $(A_2,\cdot,[,])$ is an F-manifold algebra if and only if the bracket $[,]$ on A_2 is one of the following cases:*

(1) $[e_1,e_2]=0$, $[e_2,e_3]=k_1e_1+k_2e_2$, $[e_1,e_3]=l_1e_1+l_2e_2$. *For this case, the nonabelian Lie algebra is $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$ if $k_1l_2-k_2l_1=0$; $\mathfrak{s}_{3,1}$ if $(k_1l_2-k_2l_1)\Delta\neq 0$ or if $k_1l_2-k_2l_1\neq 0$, $\Delta=0$, $l_2=\overline{k_1}$; $\mathfrak{s}_{3,2}$ if $k_1l_2-k_2l_1\neq 0$, $\Delta=0$, $l_2\neq\overline{k_1}$. Here $\Delta=(l_1+k_2)^2-4(l_1k_2-k_1l_2)$.*

(2) $[e_1,e_2]=0$, $[e_2,e_3]=k_1e_1+k_2e_2+k_3e_3$, $[e_1,e_3]=l_1e_1+l_2e_2+l_3e_3$ *for* $(k_3,l_3)\neq(0,0)$. *For this case, the Lie algebra is $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$.*

(3) $[e_1,e_2]=h_2e_2$, $[e_2,e_3]=k_2e_2$, $[e_1,e_3]=tk_2e_1+l_2e_2+l_3e_3$ *for* $h_2\neq 0$, $t=l_3/h_2$. *For this case, the Lie algebra is $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$ if $l_3=0$; $\mathfrak{s}_{3,1}$ if $l_3\neq 0$, $h_2\neq l_3$ or if $l_3\neq 0$, $l_2=0$, $h_2=l_3$; $\mathfrak{s}_{3,2}$ if $l_2l_3\neq 0$, $h_2=l_3$.*

(4) $[e_1,e_2]=h_2e_2$, $[e_2,e_3]=k_1e_1+k_2e_2$, $[e_1,e_3]=-k_2e_1+l_2e_2-h_2e_3$ *for* $h_2k_1\neq 0$. *For this case, the Lie algebra is $\mathfrak{sl}(2,\mathbb{C})$.*

(5) $[e_1,e_2]=h_1e_1$, $[e_2,e_3]=k_1e_1-l_1k_3/h_1e_2+k_3e_3$, $[e_1,e_3]=l_1e_1$ *for* $h_1\neq 0$. *For this case, the Lie algebra is $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$ if $(k_1,k_3)=(0,0)$; $\mathfrak{s}_{3,1}$ if $(k_1,l_1)=(0,0),k_3\neq 0$ or if $(k_1,k_3)=(0,0)$, $l_1\neq 0$; $\mathfrak{s}_{3,2}$ if $k_1k_3\neq 0$, $l_1=0$ or if $k_1l_1\neq 0,k_3=0$.*

(6) $[e_1,e_2]=h_1e_1$, $[e_2,e_3]=k_1e_1-l_1e_2+h_1e_3$, $[e_1,e_3]=l_1e_1+l_2e_2$ *for* $h_1l_2\neq 0$. *For this case, the Lie algebra is* $\mathfrak{sl}(2,\mathbb{C})$.

Type A_4*:* $(A_4,\cdot,[,])$ is an F-manifold algebra if and only if

$$\begin{cases} P_{e_1}(e_i,e_j)=3e_2\cdot P_{e_3}(e_i,e_j), & 1\leqslant i,j\leqslant 3,\\ P_{e_2}(e_i,e_j)=2e_3\cdot P_{e_3}(e_i,e_j), & 1\leqslant i,j\leqslant 3,\end{cases}$$

where $(A_4,\cdot)$ is a commutative associative algebra and $(A_4,[,])$ is a Lie algebra. This is equivalent to the following equations:

$$\begin{cases} h_1=2l_2-3k_3, \quad h_2=2l_3, \quad h_3=0,\\ k_2(2l_2-3k_3)+l_1k_3-3l_3k_1=0,\\ 2l_1l_3-l_3k_2+4l_2k_3-2l_2^2=0,\\ l_3k_3-2l_2l_3=0.\end{cases}$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.3. *Let* $\{e_1,e_2,e_3\}$ *be a basis of* A_4 *such that* $e_2\cdot e_3=e_1$, $e_3\cdot e_3=e_2$ *and others are zero. Then* $(A_4,\cdot,[,])$ *is an* F*-manifold algebra if and only if the bracket* $[,]$ *on* A_4 *is one of the following cases:*

(1) $[e_1,e_2]=0$, $[e_2,e_3]=k_1e_1+k_2e_2$, $[e_1,e_3]=l_1e_1$. *For this case, the non-abelian Lie algebra is* $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$ *if* $k_2=0$, $l_1\neq 0$ *or if* $k_2\neq 0$, $l_1=0$*;* $\mathfrak{n}_{3,1}$ *if* $(k_2,l_1)=(0,0)$, $k_1\neq 0$*;* $\mathfrak{s}_{3,1}$ *if* $k_2l_1\neq 0$, $k_2\neq l_1$ *or if* $k_2=l_1\neq 0$, $k_1=0$*;* $\mathfrak{s}_{3,2}$ *if* $k_2=l_1\neq 0$, $k_1\neq 0$.

(2) $[e_1,e_2]=-3k_3e_1$, $[e_2,e_3]=k_1e_1+k_2e_2+k_3e_3$, $[e_1,e_3]=3k_2e_1$ *for* $k_3\neq 0$. *For this case, the Lie algebra is* $\mathfrak{s}_{3,1}$.

(3) $[e_1,e_2]=k_3e_1$, $[e_2,e_3]=k_1e_1-l_1e_2+k_3e_3$, $[e_1,e_3]=l_1e_1+2k_3e_2$ *for* $k_3\neq 0$. *For this case, the Lie algebra is* $\mathfrak{sl}(2,\mathbb{C})$.

(4) $[e_1,e_2]=-4l_2e_1+2e_2$, $[e_2,e_3]=-8l_2^3e_1+6l_2^2e_2+2l_2e_3$, $[e_1,e_3]=l_2e_2+e_3$. *For this case, the Lie algebra is* $\mathfrak{s}_{3,1}$.

Type B_1*:* $(B_1,\cdot,[,])$ is an F-manifold algebra if and only if

$$\begin{cases} e_2\cdot P_{e_1}(e_i,e_j)=e_3\cdot P_{e_1}(e_i,e_j)=0, & 1\leqslant i,j\leqslant 3,\\ P_{e_2}(e_i,e_j)=P_{e_3}(e_i,e_j)=0, & 1\leqslant i,j\leqslant 3,\end{cases}$$

where $(B_1,\cdot)$ is a commutative associative algebra and $(B_1,[,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_2=h_3=k_1=k_2=k_3=l_2=l_3=0.$$

Thus, we have $[e_1,e_2]=h_1e_1$, $[e_2,e_3]=0$, $[e_1,e_3]=l_1e_1$. Furthermore, the non-abelian Lie algebra $(B_1,[,])$ is $\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$.

Type B_2: $(B_2, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} e_1 \cdot P_{e_1}(e_i, e_j) = P_{e_2}(e_i, e_j) = P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ P_{e_1}(e_i, e_j) = e_3 \cdot P_{e_1}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(B_2, \cdot)$ is a commutative associative algebra and $(B_2, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_2 = h_3 = k_1 = k_2 = k_3 = l_2 = l_3 = 0.$$

Thus, we have $[e_1, e_2] = h_1 e_1$, $[e_2, e_3] = 0$, $[e_1, e_3] = l_1 e_1$. Furthermore, the nonabelian Lie algebra $(B_2, [,])$ is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.

Type C_1: $(C_1, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} e_3 \cdot P_{e_1}(e_i, e_j) = e_3 \cdot P_{e_2}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(C_1, \cdot)$ is a commutative associative algebra and $(C_1, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_3 = k_3 = l_3 = 0, \quad h_1 k_2 - h_2 k_1 = 0, \quad h_1 l_2 - h_2 l_1 = 0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.4. *Let $\{e_1, e_2, e_3\}$ be a basis of C_1 such that $e_3 \cdot e_3 = e_3$ and others are zero. Then $(C_1, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on C_1 is one of the following cases:*

(1) *$[e_1, e_2] = 0$, $[e_2, e_3] = k_1 e_1 + k_2 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$. For this case, the nonabelian Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ if $k_1 l_2 - k_2 l_1 = 0$; $\mathfrak{s}_{3,1}$ if $(k_1 l_2 - k_2 l_1)\Delta \neq 0$ or if $k_1 l_2 - k_2 l_1 \neq 0$, $\Delta = 0$, $l_2 = \overline{k_1}$; $\mathfrak{s}_{3,2}$ if $k_1 l_2 - k_2 l_1 \neq 0$, $\Delta = 0$, $l_2 \neq \overline{k_1}$. Here $\Delta = (l_1 + k_2)^2 - 4(l_1 k_2 - k_1 l_2)$.*

(2) *$[e_1, e_2] = h_1 e_1 + h_2 e_2$, $[e_2, e_3] = k_1 e_1 + k_2 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ for $(h_1, h_2) \neq (0, 0)$. For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.*

Type C_2: $(C_2, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} P_{e_1}(e_i, e_j) = e_3 \cdot P_{e_1}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \\ e_1 \cdot P_{e_1}(e_i, e_j) = e_1 \cdot P_{e_2}(e_i, e_j) = e_3 \cdot P_{e_2}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(C_2, \cdot)$ is a commutative associative algebra and $(C_2, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_3 = k_1 = k_3 = l_2 = l_3 = h_1 k_2 = h_2 l_1 = 0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.5. *Let $\{e_1, e_2, e_3\}$ be a basis of C_2 such that $e_1 \cdot e_3 = e_1$, $e_3 \cdot e_3 = e_3$ and others are zero. Then $(C_2, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on C_2 is one of the following cases:*

(1) $[e_1, e_2] = 0$, $[e_2, e_3] = k_2 e_2$, $[e_1, e_3] = l_1 e_1$. *For this case, the nonabelian Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ *if* $k_2 \neq 0$, $l_1 = 0$ *or if* $k_2 = 0$, $l_1 \neq 0$*;* $\mathfrak{s}_{3,1}$ *if* $k_2 l_1 \neq 0$.

(2) $[e_1, e_2] = h_1 e_1 + h_2 e_2$, $[e_2, e_3] = k_2 e_2$,$[e_1, e_3] = l_1 e_1$ *for* $(h_1, h_2) \neq (0, 0)$. *For this case, the Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.

Type C_{11}: $(C_{11}, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i < j \leqslant 3, \\ e_1 \cdot P_{e_1}(e_i, e_j) = e_2 \cdot P_{e_2}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ e_1 \cdot P_{e_2}(e_i, e_j) + e_2 \cdot P_{e_1}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(C_{11}, \cdot)$ is a commutative associative algebra and $(C_{11}, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$k_3 = l_3 = 0, \quad h_1 k_2 = h_2 k_1, \quad h_2 l_1 = h_1 l_2, \quad (k_2 + l_1) h_3 = 0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.6. *Let $\{e_1, e_2, e_3\}$ be a basis of C_{11} such that $e_1 \cdot e_3 = e_1$, $e_2 \cdot e_3 = e_2$, $e_3 \cdot e_3 = e_3$ and others are zero. Then $(C_{11}, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on C_{11} is one of the following cases:*

(1) $[e_1, e_2] = 0$, $[e_2, e_3] = k_1 e_1 + k_2 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$. *For this case, the nonabelian Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ *if* $k_1 l_2 - k_2 l_1 = 0$*;* $\mathfrak{s}_{3,1}$ *if* $(k_1 l_2 - k_2 l_1)\Delta \neq 0$ *or if* $k_1 l_2 - k_2 l_1 \neq 0$, $\Delta = 0$, $l_2 = \overline{k_1}$*;* $\mathfrak{s}_{3,2}$ *if* $k_1 l_2 - k_2 l_1 \neq 0$, $\Delta = 0$, $l_2 \neq \overline{k_1}$. *Here* $\Delta = (l_1 + k_2)^2 - 4(l_1 k_2 - k_1 l_2)$.

(2) $[e_1, e_2] = h_1 e_1 + h_2 e_2$, $[e_2, e_3] = k_1 e_1 + k_2 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for* $(h_1, h_2) \neq (0, 0)$. *For this case, the Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.

(3) $[e_1, e_2] = h_3 e_3$, $[e_2, e_3] = k_1 e_1$, $[e_1, e_3] = l_2 e_2$ *for* $h_3 \neq 0$. *For this case, the Lie algebra is* $\mathfrak{s}_{3,1}$ *if* $(k_1, l_2) = (0, 0)$ *or if* $(k_1, l_2) \neq (0, 0)$, $k_1 l_2 = 0$*;* $\mathfrak{sl}(2, \mathbb{C})$ *if* $k_1 l_2 \neq 0$.

(4) $[e_1, e_2] = h_3 e_3$, $[e_2, e_3] = k_1 e_1 - l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for* $h_3 l_1 \neq 0$. *For this case, the Lie algebra is* $\mathfrak{s}_{3,1}$ *if* $k_1 l_2 + l_1^2 = 0$*;* $\mathfrak{sl}(2, \mathbb{C})$ *if* $k_1 l_2 + l_1^2 \neq 0$.

(5) $[e_1, e_2] = h_1 e_1 + h_3 e_3$, $[e_2, e_3] = k_1 e_1$, $[e_1, e_3] = 0$ *for* $h_1 h_3 \neq 0$. *For this case, the Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ *if* $k_1 = 0$*;* $\mathfrak{s}_{3,1}$ *if* $k_1 \Delta \neq 0$ *or if* $k_1 \neq 0$, $\Delta = 0$, $h_3 = -\overline{k_1}$*;* $\mathfrak{s}_{3,2}$ *if* $k_1 \neq 0$, $\Delta = 0$, $h_3 \neq -\overline{k_1}$. *Here* $\Delta = h_1^2 - 4k_1 h_3$.

(6) $[e_1, e_2] = h_1 e_1 + h_2 e_2 + h_3 e_3$, $[e_2, e_3] = k_1 e_1 - l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for* $h_2 \neq 0$. *For this case, the Lie algebra is* $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$ *if* $l_2 = 0$*;* $\mathfrak{s}_{3,1}$ *if* $l_2 \Delta \neq 0$ *or if* $l_2 \neq 0$, $\Delta = 0$, $h_3 = \overline{l_2}$*;* $\mathfrak{s}_{3,2}$ *if* $l_2 \neq 0$, $\Delta = 0$, $h_3 \neq \overline{l_2}$. *Here* $\Delta = h_2^2 + 4l_2 h_3$.

Type D_1: $(D_1, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} P_{e_2}(e_i, e_j) = 2e_1 \cdot P_{e_1}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \\ P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ e_3 \cdot P_{e_1}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(D_1, \cdot)$ is a commutative associative algebra and $(D_1, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_3 = k_1 = k_3 = l_3 = 0, \quad k_2 = 2l_1, \quad l_1 h_1 = 0, \quad l_1 h_2 - h_1 l_2 = 0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.7. *Let $\{e_1, e_2, e_3\}$ be a basis of D_1 such that $e_1 \cdot e_1 = e_2$, $e_3 \cdot e_3 = e_3$ and others are zero. Then $(D_1, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on D_1 is one of the following cases:*

(1) $[e_1, e_2] = 0$, $[e_2, e_3] = 2l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$. *For this case, the nonabelian Lie algebra is $\mathfrak{n}_{3,1}$ if $l_1 = 0$, $l_2 \neq 0$; $\mathfrak{s}_{3,1}$ if $l_1 \neq 0$.*

(2) $[e_1, e_2] = h_1 e_1 + h_2 e_2$, $[e_2, e_3] = 2l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for* $(h_1, h_2) \neq (0, 0)$. *For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.*

Type D_2: $(D_2, \cdot, [,])$ is an F-manifold algebra if and only if

$$\begin{cases} e_2 \cdot P_{e_1}(e_i, e_j) = P_{e_3}(e_i, e_j) = 0, & 1 \leqslant i, j \leqslant 3, \\ P_{e_2}(e_i, e_j) = 2e_1 \cdot P_{e_1}(e_i, e_j), & 1 \leqslant i, j \leqslant 3, \end{cases}$$

where $(D_2, \cdot)$ is a commutative associative algebra and $(D_2, [,])$ is a Lie algebra. This is equivalent to the following equations:

$$h_3 = k_1 = k_3 = l_3 = 0, \quad k_2 = 2l_1, \quad l_1 h_1 = 0, \quad l_1 h_2 - h_1 l_2 = 0.$$

Similarly to the discussion in Subsection 3.1, we have the following result.

Theorem 3.8. *Let $\{e_1, e_2, e_3\}$ be a basis of D_2 such that $e_1 \cdot e_1 = e_2$, $e_1 \cdot e_3 = e_1$, $e_2 \cdot e_3 = e_2$, $e_3 \cdot e_3 = e_3$ and others are zero. Then $(D_2, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on D_2 is one of the following cases:*

(1) $[e_1, e_2] = 0$, $[e_2, e_3] = 2l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$. *For this case, the nonabelian Lie algebra is $\mathfrak{n}_{3,1}$ if $l_1 = 0$, $l_2 \neq 0$; $\mathfrak{s}_{3,1}$ if $l_1 \neq 0$.*

(2) $[e_1, e_2] = h_1 e_1 + h_2 e_2$, $[e_2, e_3] = 2l_1 e_2$, $[e_1, e_3] = l_1 e_1 + l_2 e_2$ *for* $(h_1, h_2) \neq (0, 0)$. *For this case, the Lie algebra is $\mathfrak{s}_{2,1} \oplus \mathfrak{n}_{1,1}$.*

Type E: $(E, \cdot, [,])$ is an F-manifold algebra if and only if the bracket $[,]$ on E is $[e_i, e_j] = 0$, $1 \leqslant i, j \leqslant 3$.

Acknowledgment. The authors are very grateful to the referee for careful reading and valuable comments which have helped to improve this paper.

References

[1] *C. Bai, D. Meng*: The classification of Novikov algebras in low dimensions. J. Phys. A, Math. Gen. *34* (2001), 1581–1594. zbl MR doi

[2] *A. Basalaev, C. Hertling*: 3-dimensional F-manifolds. Lett. Math. Phys. *111* (2021), Article ID 90, 50 pages. zbl MR doi

[3] *A. Ben Hassine, T. Chtioui, M. A. Maalaoui, S. Mabrouk*: On Hom-F-manifold algebras and quantization. Available at https://arxiv.org/abs/2102.05595 (2021), 23 pages.

[4] *J. A. Cruz Morales, J. A. Gutierrez, A. Torres-Gomez*: F-algebra-Rinehart pairs and super F-algebroids. Available at https://arxiv.org/abs/1904.04724v2 (2019), 14 pages.

[5] *V. Chari, A. Pressley*: A Guide to Quantum Groups. Cambridge University Press, Cambridge, 1994. zbl MR

[6] *M. Ding, Z. Chen, J. Li*: F-manifold color algebras. Available at https://arxiv.org/abs/2101.00959v2 (2021), 13 pages.

[7] *V. Dotsenko*: Algebraic structures of F-manifolds via pre-Lie algebras. Ann. Mat. Pura Appl. (4) *198* (2019), 517–527. zbl MR doi

[8] *B. Dubrovin*: Geometry of 2D topological field theories. Integrable Systems and Quantum Groups. Lecture Notes in Mathematics 1620. Springer, Berlin, 1996, pp. 120–348. zbl MR doi

[9] *W. Fulton, J. Harris*: Representation Theory: A First Course. Graduate Texts in Mathematics 129. Springer, New York, 1991. zbl MR doi

[10] *C. Hertling*: Frobenius Manifolds and Moduli Spaces for Singularities. Cambridge Tracts in Mathematics 151. Cambridge University Press, Cambridge, 2002. zbl MR doi

[11] *C. Hertling, Y. Manin*: Weak Frobenius manifolds. Int. Math. Res. Not. *1999* (1999), 277–286. zbl MR doi

[12] *J. Liu, C. Bai, Y. Sheng*: Noncommutative Poisson bialgebras. J. Algebra *556* (2020), 35–66. zbl MR doi

[13] *J. Liu, Y. Sheng, C. Bai*: F-manifold algebras and deformation quantization via pre-Lie algebras. J. Algebra *559* (2020), 467–495. zbl MR doi

[14] *X. Ni, C. Bai*: Poisson bialgebras. J. Math. Phys. *54* (2013), Article ID 023515, 14 pages. zbl MR doi

[15] *J. Patera, R. T. Sharp, P. Winternitz, H. Zassenhaus*: Invariants of real low dimension Lie algebras. J. Math. Phys. *17* (1976), 986–994. zbl MR doi

[16] *L. Šnobl, P. Winternitz*: Classification and Identification of Lie Algebras. CRM Monograph Series 33. AMS, Providence, 2014. zbl MR doi

[17] *K. Uchino*: Quantum analogy of Poisson geometry, related dendriform algebras and Rota-Baxter operators. Lett. Math. Phys. *85* (2008), 91–109. zbl MR doi

Authors' addresses: Zhiqi Chen, School of Mathematics and Statistics, Guangdong University of Technology, 161 Yinglong Road, Tianhe District, Guangzhou 510520, P. R. China, e-mail: chenzhiqi@nankai.edu.cn; Jifu Li, School of Science, Tianjin University of Technology, 1038 Dagu Nanlu, Hexi District, Tianjin 300384, P. R. China, e-mail: ljfanhan@126.com; Ming Ding (corresponding author), School of Mathematics and Information Science, Guangzhou University, 230 Wai Huan Xi Road, Guangzhou 510006, P. R. China, e-mail: dingming@nankai.edu.cn.