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TRUNCATIONS OF GAUSS' SQUARE EXPONENT THEOREM

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Abstract. We establish two truncations of Gauss' square exponent theorem and a finite extension of Euler's identity. For instance, we prove that for any positive integer n,

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k+1}{2}} = \sum_{k=-n}^{n}(-1)^k q^{k^2},$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{k=1}^{m} \frac{1-q^{n-k+1}}{1-q^k} \quad \text{and} \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

Keywords: Gauss' identity; q-binomial coefficient; q-binomial theorem

MSC 2020: 11B65, 33D15

1. Introduction

One of Euler's most profound discoveries, the pentagonal number theorem (see [1], Corollary 1.7), is stated as follows:

$$\prod_{k=1}^{\infty}(1-q^k) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1.1}$$

Here and throughout this paper, we assume $|q| < 1$.

In the past decades, various generalizations of Euler's pentagonal number theorem have been widely studied. For instance, Shanks in [14] established the following

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truncation of Euler's pentagonal number theorem:

$$\sum_{k=0}^{n}(-1)^k \frac{(q;q)_n}{(q;q)_k} q^{\binom{k+1}{2}+nk} = \sum_{k=-n}^{n} (-1)^k q^{k(3k+1)/2}, \tag{1.2}$$

where the q-shifted factorials are given by $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ for $n \geqslant 1$ and $(a;q)_0 = 1$.

In 2002, Berkovich and Garvan [3] discovered another interesting finite form of Euler's pentagonal number theorem:

$$\sum_{j=-\infty}^{\infty} (-1)^j \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} q^{j(3j+1)/2} = 1, \tag{1.3}$$

where the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that both (1.2) and (1.3) reduce to (1.1) when $n \to \infty$.

The Gauss' triangular exponent identity and square exponent identity (see [1], Corollary 2.10) are stated as follows:

$$(q^2;q^2)_\infty(-q;q)_\infty = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \tag{1.4}$$

$$(q;q)_\infty(q;q^2)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}. \tag{1.5}$$

In particular, the square exponent identity (1.5) can be used to prove Lagrange's four-square theorem, see [5], page 106. Both Euler's pentagonal number theorem (1.1) and Gauss' identities (1.4)–(1.5) were historically spectacular achievements at the time of their discoveries. For various finite generalizations of (1.1), (1.4) and (1.5), we refer the interested reader to [2], [4], [7], [8], [10], [11], [12], [13].

The first aim of the paper is to establish two truncations of Gauss' identity (1.5) as follows.

Theorem 1.1. *For any positive integer n, we have*

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k+1}{2}} = \sum_{k=-n}^{n} (-1)^k q^{k^2}, \tag{1.6}$$

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q^2)_{n-k} q^{\binom{k+1}{2}} = \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{2k^2}, \tag{1.7}$$

where $\lfloor x\rfloor$ denotes the integral part of real x.

Letting $n \to \infty$ in (1.6) and (1.7) leads us to

$$(q;q^2)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}, \tag{1.8}$$

and

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} = \sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2}, \tag{1.9}$$

respectively. Letting $t = -q$ in the following identity due to Euler, see [1], Corollary 2.2:

$$\sum_{k=0}^{\infty} \frac{t^k q^{\binom{k}{2}}}{(q;q)_k} = (-t;q)_\infty, \tag{1.10}$$

we obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} = (q;q)_\infty. \tag{1.11}$$

It follows from (1.11) that both (1.8) and (1.9) become Gauss' identity (1.5).

The second aim of the paper is to prove the following identity, which has the same type as (1.3).

Theorem 1.2. *For any positive integer n, we have*

$$\sum_{k=0}^{n} \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k+1}{2}} = 1. \tag{1.12}$$

Letting $n \to \infty$ in (1.12) reduces to

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}}}{(q;q)_k} = \frac{1}{(q;q^2)_\infty} = (-q;q)_\infty,$$

which is the special case $t = q$ of Euler's identity (1.10).

The rest of the paper is organized as follows. In the next section, we first establish two preliminary results. The proofs of Theorems 1.1 and 1.2 are presented in Sections 3 and 4, respectively.

2. Preliminary results

In order to prove Theorems 1.1 and 1.2, we require the following two lemmas. The proof of the second lemma is inspired by Gu and Guo, see [6].

Lemma 2.1. *For any positive integer n, we have*

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{2k^2}\begin{bmatrix}2n\\ n+2k\end{bmatrix}=(-q;q^2)_n, \tag{2.1}$$

$$\sum_{k=-n}^{n}(-1)^k q^{k^2}\begin{bmatrix}2n\\ n+k\end{bmatrix}=(q;q^2)_n. \tag{2.2}$$

Proof. One can refer to [9], Proposition 2 for the proof of (2.1). Next, we give a proof of (2.2). By [1] equation (3.3.8), we have

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\ k\end{bmatrix}=(q;q^2)_n,$$

which can be rewritten as

$$\sum_{k=-n}^{n}(-1)^k\begin{bmatrix}2n\\ n+k\end{bmatrix}=(-1)^n(q;q^2)_n. \tag{2.3}$$

Letting $q\to q^{-1}$ on both sides of (2.3) and noting that

$$\begin{bmatrix}n\\ m\end{bmatrix}_{q^{-1}}=\begin{bmatrix}n\\ m\end{bmatrix}_q q^{m(m-n)}, \tag{2.4}$$

and

$$(q^{-1};q^{-2})_n=(-1)^n q^{-n^2}(q;q^2)_n, \tag{2.5}$$

we obtain

$$\sum_{k=-n}^{n}(-1)^k q^{k^2}\begin{bmatrix}2n\\ n+k\end{bmatrix}=(q;q^2)_n.$$

This proves (2.2). □

Lemma 2.2. *Let n be a nonnegative integer. For any integer $-n\leqslant j\leqslant n$, we have*

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}2n-k\\ k\end{bmatrix}\begin{bmatrix}2n-2k\\ n-k+j\end{bmatrix}=1. \tag{2.6}$$

Proof. By the q-binomial theorem (see [1], Theorem 3.3), we have

$$(xq;q)_{n+j} = \sum_{k=0}^{n+j} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n+j \\ k \end{bmatrix} x^k, \quad \text{and} \quad \frac{1}{(x;q)_{n+j+1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+j+k \\ k \end{bmatrix} x^k.$$

It follows that

$$\left(\sum_{k=0}^{n+j} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n+j \\ k \end{bmatrix} x^k\right) \left(\sum_{k=0}^{\infty} \begin{bmatrix} n+j+k \\ k \end{bmatrix} x^k\right) = \frac{(xq;q)_{n+j}}{(x;q)_{n+j+1}} = \frac{1}{1-x}. \tag{2.7}$$

Equating the coefficients of x^{n-j} on both sides of (2.7), we obtain

$$\sum_{k=0}^{n-j} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n+j \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n-j-k \end{bmatrix} = 1,$$

which can be rewritten as

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix} = 1.$$

This completes the proof of (2.6). □

3. Proof of Theorem 1.1

Proof of (1.6). By using (2.2) and (2.6), we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k+1}{2}} \\
&\quad = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \sum_{j=-(n-k)}^{(n-k)} (-1)^j q^{j^2} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix} \\
&\quad = \sum_{j=-n}^{n} (-1)^j q^{j^2} \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix} \\
&\quad = \sum_{j=-n}^{n} (-1)^j q^{j^2},
\end{aligned}$$

as desired. □

Proof of (1.7). By using (2.1) and (2.6), we have

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q^2)_{n-k} q^{\binom{k+1}{2}}$$
$$= \sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \sum_{j=-\lfloor (n-k)/2 \rfloor}^{\lfloor (n-k)/2 \rfloor} (-1)^j q^{2j^2} \begin{bmatrix} 2n-2k \\ n-k+2j \end{bmatrix}$$
$$= \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^j q^{2j^2} \sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k+2j \end{bmatrix} = \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^j q^{2j^2},$$

which proves (1.7). □

4. Proof of Theorem 1.2

Letting $q \to q^{-1}$ in (1.12) and using (2.4) and (2.5), we find that (1.12) is equivalent to

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k}{2}} = (-1)^n q^{n^2}. \tag{4.1}$$

By using (2.2), we obtain

$$\begin{aligned} &\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} \\ &\quad = \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \sum_{j=-(n-k)}^{(n-k)} (-1)^j q^{j^2} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix} \\ &\quad = \sum_{j=-n}^{n} (-1)^j q^{j^2} \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix}. \end{aligned} \tag{4.2}$$

Note that, see [6], equation (2.3)

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k+j \end{bmatrix} = q^{n^2-j^2}. \tag{4.3}$$

Finally, combining (4.2) and (4.3) gives

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (q;q^2)_{n-k} q^{\binom{k}{2}} = q^{n^2} \sum_{j=-n}^{n} (-1)^j = (-1)^n q^{n^2},$$

which proves (4.1). □

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