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ON THE HIGHER POWER MOMENTS OF CUSP FORM COEFFICIENTS OVER SUMS OF TWO SQUARES

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Abstract. Let f be a normalized primitive holomorphic cusp form of even integral weight for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Denote by $\lambda_f(n)$ the nth normalized Fourier coefficient of f. We are interested in the average behaviour of the sum

$$\sum_{a^2+b^2\leqslant x} \lambda_f^j(a^2+b^2)$$

for $x \geqslant 1$, where $a, b \in \mathbb{Z}$ and $j \geqslant 9$ is any fixed positive integer. In a similar manner, we also establish analogous results for the normalized coefficients of Dirichlet expansions of associated symmetric power L-functions and Rankin-Selberg L-functions.

Keywords: Fourier coefficient; automorphic L-function, Langlands program

MSC 2020: 11F11, 11F30, 11F66

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$ for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then the Hecke eigenform $f(z) \in H_k^*$ has the following Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} \mathrm{e}^{2\pi i n z}, \quad \Im(z) > 0,$$

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where $\lambda_f(n)$ is the nth normalized Fourier coefficient (Hecke eigenvalue) such that $\lambda_f(1) = 1$. Then $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big),$$

where $m \geqslant 1$ and $n \geqslant 1$ are positive integers. In 1974, Deligne in [4] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n), \tag{1.1}$$

where $d(n)$ is the divisor function. By (1.1), Deligne's bound is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{1.2}$$

More generally, for all positive integers $l \geqslant 1$ one has

$$\lambda_f(p^l) = \alpha_f(p)^l + \alpha_f(p)^{l-1}\beta_f(p) + \ldots + \alpha_f(p)\beta_f(p)^{l-1} + \beta_f(p)^l.$$

In 1927, Hecke in [11] proved that

$$\sum_{n\leqslant x} \lambda_f(n) \ll x^{1/2}. \tag{1.3}$$

Later, the upper bound in (1.3) has been improved by several authors, see e.g. [4], [9], [34]. The record to date is given by Wu, see [46]:

$$\sum_{n\leqslant x} \lambda_f(n) \ll x^{1/3} \log^{\varrho} x,$$

where

$$\varrho = \frac{102 + 7\sqrt{21}}{210}\Big(\frac{6 - \sqrt{21}}{5}\Big)^{1/2} + \frac{102 - 7\sqrt{21}}{210}\Big(\frac{+\sqrt{21}}{5}\Big)^{1/2} - \frac{33}{35} = -0.118\ldots$$

In 1930s, Rankin in [33] and Selberg in [42] independently proved the asymptotic formula

$$\sum_{n\leqslant x} \lambda_f^2(n) = c_f x + O(x^{3/5}), \tag{1.4}$$

where $c_f > 0$ is a positive constant depending on f and $\varepsilon > 0$ is an arbitarily small positive number. Very recently, the exponent in (2.2) has been improved to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang (see [12]), where $\delta \leqslant 1/560$. This remains the best known result in this direction.

Later, based on the works about symmetric power L-functions, Moreno and Shahidi in [30] were able to prove

$$\sum_{n\leqslant x}\tau_0^4(n)\sim c_1x\log x,\quad x\to\infty, \tag{1.5}$$

where $\tau_0(n)=\tau(n)/n^{11/2}$ is the normalized Ramanujan tau-function and $c_1>0$ is a positive constant. Moreno and Shahidi's result also holds true if we replace $\tau_0(n)$ with the normalized Fourier coefficient $\lambda_f(n)$.

Let $f\in H_k^*$ be a Hecke eigenform and denote its nth normalized Fourier coefficient by $\lambda_f(n)$. Define

$$S_j(f;x)=\sum_{n\leqslant x}\lambda_f^j(n),$$

where $j\in\mathbb{Z}^+$ and $x\geqslant 1$.

Based on the work of Moreno and Shahidi concerning the symmetric power L-functions $L(\mathrm{sym}^j f,s)$ for $j=1,2,3,4$, Fomenko in [5] established the estimates

$$S_3(f;x)\ll_{f,\varepsilon}x^{5/6+\varepsilon},\quad S_4(f;x)=c_fx\log x+d_fx+O_{f,\varepsilon}(x^{9/10+\varepsilon}),$$

where $c_f>0$ and d_f are suitable constants depending on f. Here ε is an arbitrarily small positive number. Later, Lü (see e.g. [26], [28], [23]) considered higher moments $S_j(f;x)$ for $3\leqslant l\leqslant 8$, which improved and generalized the work of Fomenko. Later Lau, Lü and Wu in [25] proved that

$$S_j(f;x)=xP_j^*(\log x)+O_{f,\varepsilon}(x^{\theta_j+\varepsilon}),\quad 3\leqslant j\leqslant 8,$$

where $P_j^*(t)\equiv 0$ are the zero functions for $j=3,5,7$, and $P_4^*(t)$, $P_6^*(t)$, $P_8^*(t)$ are polynomials of degree 1, 4, 13, respectively, and

$$\theta_3=\frac{7}{10},\quad \theta_5=\frac{40}{43},\quad \theta_7=\frac{176}{179},\quad \theta_4=\frac{151}{175},\quad \theta_6=\frac{175}{181},\quad \theta_8=\frac{2933}{2957}.$$

Lau and Lü in [24] derived general results for $S_j(f;x)$ for all $j\geqslant 2$ under the assumption that $L(\mathrm{sym}^l f,s)$ is automorphic cuspidal for a positive l. Now we know that $L(\mathrm{sym}^j f,s)$ is automorphic for all $j\geqslant 1$ due to the recent celebrated works of Newton and Thorne, see [31], [32].

In 2013, Zhai in [47] considered the average behaviour of the power sum

$$U_j(f;x):=\sum_{a^2+b^2\leqslant x}\lambda_f(a^2+b^2)^j$$

for $x\geqslant 1, 2\leqslant j\leqslant 8$ and $a,b,j\in\mathbb{Z}$. He proved that

$$U_j(f;x)=x\widetilde{P}_j(\log x)+O(x^{\alpha_j+\varepsilon}),$$

where $\widetilde{P}_j(t)$ with $j = 2, \ldots, 8$ are polynomials of t with degrees $\deg \widetilde{P}_2(t) = 0$, $\deg \widetilde{P}_4(t) = 1$, $\deg \widetilde{P}_6(t) = 4$, $\deg \widetilde{P}_8(t) = 13$, and $\deg \widetilde{P}_j(t) \equiv 0$ are the zero functions for $j = 3, 5, 7$. The powers α_j are given by

$$\alpha_2 = \frac{8}{11}, \quad \alpha_3 = \frac{17}{20}, \quad \alpha_4 = \frac{43}{46}, \quad \alpha_5 = \frac{83}{86}, \quad \alpha_6 = \frac{184}{187}, \quad \alpha_7 = \frac{355}{357}, \quad \alpha_8 = \frac{752}{755}.$$

In this paper, we firstly consider the asymptotic behavior of $U_j(f; x)$ for positive integers $j \geqslant 9$. More precisely, we will be able to establish the following results.

Theorem 1.1. *Let $f \in H_k^*$ be a Hecke eigenform. Let $j \geqslant 9$ be any fixed positive integer. Then the following hold:*

(i) *For $j = 2m$ we have*

$$U_j(f; x) = x P_{A_m - 1}(\log x) + O_{f,\varepsilon}(x^{1-2^{-j}+\varepsilon})$$

for any $\varepsilon > 0$, where $P_\omega(t)$ denotes a polynomial in t of degree ω and A_m is defined by

$$A_m = \frac{(2m)!}{m!\,(m+1)!}, \quad m \geqslant 1.$$

(ii) *For $j = 2m + 1$ we have*

$$U_j(f; x) \ll_{f,\varepsilon} x^{1-2^{-j}+\varepsilon}$$

for any $\varepsilon > 0$.

Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the nth normalized coefficient of the Dirichlet expansion of the jth symmetric power L-function. Fomenko in [6] proved that

$$\sum_{n \leqslant x} \lambda_{\mathrm{sym}^2 f}(n) \ll x^{1/2} (\log x)^2.$$

Later, this sum has been studied by many authors, see e.g. [18], [29], [41]. The analogous cases for symmetric power lifting $\mathrm{sym}^j \pi_f$ for large j were considered by Lau and Lü (see [24]), and Tang and Wu, see [45].

On the other hand, Fomenko in [7] studied the sum of $\lambda^2_{\mathrm{sym}^2 f}(n)$. Later, this result has been improved and generalized by a number of authors, see e.g. [10], [22], [40], [44]. Recently, Sankaranarayanan, Singh and Srinivas [40] proved that

$$\sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^3 f}(n) = c_1 x + O(x^{15/17+\varepsilon}), \quad \text{and} \quad \sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^4 f}(n) = c_2 x + O(x^{12/13+\varepsilon}),$$

where $c_1, c_2 > 0$ are some suitable constants. Very recently, Luo et al. in [22] established the following asymptotic formulas:

$$\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)=\tilde{c}_j x+O(x^{\tilde{\theta}_j+\varepsilon}),\quad 3\leqslant j\leqslant 6,$$
$$\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)=\tilde{c}_j x+O(x^{\tilde{\theta}_j}),\quad j=7,8,$$

where $\tilde{c}_j$ $(3 \leqslant j \leqslant 8)$ is a suitable constant, and $\tilde{\theta}_3 = \frac{551}{635}$, $\tilde{\theta}_4 = \frac{929}{1013}$, $\tilde{\theta}_5 = \frac{1391}{1475}$, $\tilde{\theta}_6 = \frac{979}{1021}$, $\tilde{\theta}_7 = \frac{63}{65}$, $\tilde{\theta}_8 = \frac{40}{41}$.

Define

$$U_j^*(f;x) := \sum_{a^2+b^2\leqslant x}\lambda^2_{\mathrm{sym}^j f}(a^2+b^2)$$

for $x \geqslant 1$, $j \geqslant 2$ and $a, b \in \mathbb{Z}$.

The second purpose of this paper is to prove the following theorem.

Theorem 1.2. *Let $f \in H_k^*$ be a Hecke eigenform. Let $j \geqslant 2$ be any fixed positive integer. Then*

$$U_j^*(f;x) = C_{f,j}x + O_{f,\varepsilon}(x^{1-1/(j+1)^2+\varepsilon}),$$

where $C_{f,j} > 0$ is a suitable constant.

Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Denote by $\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(n)$ and $\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j g}(n)$ the nth normalized coefficients of the Dirichlet expansions of the associated Rankin-Selberg L-functions $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$, respectively. Define

$$U_{i,j}(f,f;x) := \sum_{a^2+b^2\leqslant x}\lambda^2_{\mathrm{sym}^i f\times\mathrm{sym}^j f}(a^2+b^2)$$

and

$$U_{i,j}(f,g;x) := \sum_{a^2+b^2\leqslant x}\lambda^2_{\mathrm{sym}^i f\times\mathrm{sym}^j g}(a^2+b^2)$$

for $x \geqslant 1$, $1 \leqslant i \leqslant j$ and $a, b \in \mathbb{Z}$.

In a similar manner, we can also establish the following analogous results.

Theorem 1.3. *Let $f \in H_k^*$ be a Hecke eigenform. Let $1 \leqslant i \leqslant j$ be any fixed positive integers. Then*

$$U_{i,j}(f,f;x) = C_{f,i,j}P_i(\log x) + O_{f,\varepsilon}(x^{1-1/((i+1)(j+1))^2+\varepsilon}),$$

where $C_{f,i,j} > 0$ is a suitable constant and $P_\omega(t)$ denotes a polynomial in t of degree ω.

Theorem 1.4. *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Let $1 \leqslant i \leqslant j$ be any fixed positive integers. Assume that* $\mathrm{sym}^l \pi_f \not\cong \mathrm{sym}^l \pi_g$ *for $1 \leqslant l \leqslant 2i$. Then we have*

$$U_{i,j}(f,g;x) = C_{f,g,i,j}x + O_{f,g,\varepsilon}(x^{1+\varepsilon-1/((i+1)(j+1))^2}),$$

where $C_{f,g,i,j} > 0$ is some suitable constant.

Remark 1.1. By applying Perron's formula (see [14], Proposition 5.54) and using better individual or average subconvexity bounds for the automorphic L-functions, we can improve the upper bounds and the remainder terms in Theorems 1.1–1.4 slightly. But here we emphasize the methods for dealing with such kinds of problems.

Throughout the paper, we always assume that $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. Denote by $\varepsilon > 0$ an arbitrarily small positive constant that may vary in different occurrences.

2. Preliminaries

In this section, we introduce some background on the analytic properties of automorphic L-functions and give some useful lemmas which play important roles in the proof of the main results of this paper.

Let $f \in H_{k_1}^*$ be a Hecke eigenform of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and let $\lambda_f(n)$ denote its nth normalized Fourier coefficient. The Hecke L-function $L(f,s)$ associated to f is defined by

$$\begin{aligned} L(f,s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \Big(1-\frac{\alpha_f(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta_f(p)}{p^s}\Big)^{-1}, \quad \Re(s) > 1, \end{aligned}$$

where $\alpha_f(p)$, $\beta_f(p)$ are the local parameters satisfying (1.2). The jth symmetric power L-function associated with f is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}, \quad \Re(s) > 1.$$

We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \Big(1+\frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s}+\ldots+\frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}+\ldots\Big), \quad \Re(s) > 1. \end{aligned} \tag{2.1}$$

Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. For $j = 1$ we have

$$L(\mathrm{sym}^1 f, s) = L(f, s).$$

Let $g \in H_{k_2}^*$ be a Hecke eigenform. The Rankin-Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined as

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \Big(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-m'}\beta_g(p)^{m'}}{p^s}\Big)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}}^i f \times \mathrm{sym}^j g(n)}{n^s}, \quad \Re(s) > 1. \end{aligned} \tag{2.2}$$

Here $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ are not necessarily different.

It is standard that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m,$$

which can be written as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \widetilde{U}_j\Big(\frac{\lambda_f(p)}{2}\Big),$$

where $\widetilde{U}_j(x)$ is the jth Chebyshev polynomial of the second kind. For any prime number p, we also have

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p) = \lambda_f(p^i)\lambda_g(p^j). \tag{2.3}$$

As is well-known, an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$ is associated to a primitive form f. It is predicted that π_f gives rise to a symmetric power lift—an automorphic representation whose L-function is the symmetric power L-function attached to f.

For $1 \leqslant j \leqslant 8$, the special Langlands functoriality conjecture which states that $\mathrm{sym}^j f$ is automorphic cuspidal has been established in a series of important works of Gelbart and Jacquet (see [8]), Kim (see [19]), Kim and Shahidi (see [20], [21]), Shahidi (see [39]), Clozel and Thorne, see [1], [2], [3]. Very recently, Newton and Thorne in [31], [32] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geqslant 1$ (with f being a holomorphic cusp form). From the works about the Rankin-Selberg theory developed by Jacquet, Piatetski-Shapiro, and Shalika (see [15]), Jacquet and Shalika (see [16], [17]), Shahidi (see [43],

[36], [37], [38]), and the reformulation of Rudnick and Sarnak (see [35]), we know that $L(\mathrm{sym}^j f, s), L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ $(1 \leqslant i \leqslant j)$ have analytic continuations to the whole complex plane except possibly for simple poles at $s = 0, 1$ (in this case $\mathrm{sym}^j \pi_f \cong \mathrm{sym}^j \pi_g$) and satisfy certain Riemann-type functional equations. We refer the interested reader to [14], Chapter 5 for a more comprehensive treatment.

We firstly state some basic definitions and analytic properties of general L-functions. Let $L(\varphi, s)$ be a Dirichlet series (associated with the object φ) that admits an Euler product of degree $m \geqslant 1$, namely

$$L(\varphi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\varphi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \Bigl(1 - \frac{\alpha_\varphi(p, j)}{p^s}\Bigr)^{-1},$$

where $\alpha_\varphi(p, j)$, $j = 1, 2, \ldots, m$ are the local parameters of $L(\varphi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\varphi, s) = \prod_{j=1}^{m} \pi^{-s+\mu_\varphi(j)/2} \Gamma\Bigl(\frac{s + \mu_\varphi(j)}{2}\Bigr)$$

with local parameters $\mu_\varphi(j)$, $j = 1, 2, \ldots, m$ of $L(\varphi, s)$ at ∞. The complete L-function $\Lambda(\varphi, s)$ is defined by

$$\Lambda(\varphi, s) = q(\varphi)^{s/2} L_\infty(\varphi, s) L(\varphi, s),$$

where $q(\varphi)$ is the conductor of $L(\varphi, s)$. We assume that $\Lambda(\varphi, s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, we assume that it satisfies a functional equation of the Riemann-type

$$\Lambda(\varphi, s) = \varepsilon_\varphi \Lambda(\widetilde{\varphi}, 1 - s),$$

where ε_φ is the root number with $|\varepsilon_\varphi| = 1$ and $\widetilde{\varphi}$ is the dual of φ such that $\lambda_{\widetilde{\varphi}}(n) = \overline{\lambda_\varphi(n)}$, $L_\infty(\widetilde{\varphi}, s) = L_\infty(\varphi, s)$ and $q(\widetilde{\varphi}) = q(\varphi)$. We write $\varphi \in S_e^{\#}$ if it is satisfies with the above conditions. We say the L-function $L(\varphi, s)$ satisfies the Ramanujan conjecture if $\lambda_\varphi(n) \ll n^\varepsilon$ for any ε.

Here we state a very general theorem due to Lau and Lü, see [24].

Lemma 2.1 ([24], Lemma 2.4). *Let $L(f, s)$ be a product of two L-functions $L_1, L_2 \in S_e^{\#}$ with both $\deg L_i \geqslant 2$, $i = 1, 2$ and suppose that $L(f, s)$ satisfies the Ramanujan conjecture. Then for any $\varepsilon > 0$ we have*

$$\sum_{n \leqslant x} \lambda_f(n) = M(x) + O(x^{1-2/m+\varepsilon}),$$

where $M(x) = \mathrm{Res}_{s=1}\{L(f, s)x^s/s\}$ and $m = \deg L$.

Define $r_2(n)$ by

$$r_2(n) = \sharp\{(a,b) \in \mathbb{Z}^2 \colon\ n = a^2 + b^2\},$$

then it is well-known that

$$r_2(n) = 4 \sum_{d|n} \chi_4(d), \tag{2.4}$$

where χ_4 is the nontrivial character to modulus 4. In fact, we have

$$\sum_{n=0}^{\infty} r_2(n) e(nz) = \theta^2(z),$$

here $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$, where $\theta(z)$ is the classical theta function defined by

$$\theta(z) = 1 + 2 \sum_{n=1}^{\infty} e(n^2 z).$$

It is well-known that $\theta^2(z)$ is a modular form of weight 1 for $\Gamma_0(4)$ with character χ_4. We set

$$r(n) = \frac{1}{4} r_2(n) = \sum_{d|n} \chi_4(d).$$

Then for each prime p we have

$$r(p) = 1 + \chi_4(p), \quad r(p^2) = 1 + \chi_4(p) + \chi_4(p^2). \tag{2.5}$$

For simplicity, we write $\chi := \chi_4$. In fact, we have

$$U_j(f;x) = \sum_{n \leqslant x} \lambda_f^j(n) r_2(n) = 4 \sum_{n \leqslant x} \lambda_f^j(n) r(n).$$

We define the generating function $\mathfrak{L}_j(f,s)$ by

$$\mathfrak{L}_j(f,s) := \sum_{n=1}^{\infty} \frac{\lambda_f^j(n) r(n)}{n^s} \tag{2.6}$$

for $\Re(s) > 1$ and $j \geqslant 1$.

Lemma 2.2. *Let $f \in H_k^*$ be a Hecke eigenform. For $j \geqslant 9$ being an integer we have*

$$\mathfrak{L}_j(f,s) = F_j(s) H_j(s),$$

where

$$\begin{aligned} F_{2m}(s) = {} & \zeta(s)^{A_m} L(\mathrm{sym}^{2m} f, s) L(s,\chi)^{A_m} L(\mathrm{sym}^{2m} f \times \chi, s) \\ & \times \prod_{1 \leqslant r \leqslant m-1} L(\mathrm{sym}^{2r} f, s)^{C_m(r)} L(\mathrm{sym}^{2r} f \times \chi, s)^{C_m(r)} \quad (j = 2m), \end{aligned} \tag{2.7}$$

and

$$F_{2m+1}(s) = L(f,s)^{B_m} L(\mathrm{sym}^{2m+1} f, s) L(f\times\chi, s)^{B_m} L(\mathrm{sym}^{2m+1} f\times\chi, s) \times \prod_{1\leqslant r\leqslant m-1} L(\mathrm{sym}^{2r+1} f, s)^{D_m(r)} L(\mathrm{sym}^{2r+1} f\times\chi, s)^{D_m(r)}, \quad (j = 2m+1), \tag{2.8}$$

where A_m, B_m, $C_m(r)$, $D_m(r)$ are suitable constants, and

$$A_m = \frac{(2m)!}{m!\,(m+1)!}, \quad m \geqslant 1, \tag{2.9}$$

where the function $H_j(s)$ is a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ and $H_j(s) \neq 0$ if $\Re(s) = 1$.

Proof. Since $\lambda_f^j(n) r(n)$ is a multiplicative function and satisfies the trivial bound $O(n^{\varepsilon})$, then for $\Re(s) > 1$ we have the Euler product

$$\mathfrak{L}_j(f,s) = \prod_p \Bigl(1 + \sum_{k\geqslant 1} \frac{\lambda_f^j(p^k) r(p^k)}{p^{ks}}\Bigr). \tag{2.10}$$

We only give the proof of the cases $j = 2m$, since other cases follow similar approach. For $j = 2m$ and $\Re(s) > 1$, the L-function

$$F_j(s) = \zeta(s)^{A_m} L(\mathrm{sym}^{2m} f, s) L(s,\chi)^{A_m} L(\mathrm{sym}^{2m} f\times\chi, s) \times \prod_{1\leqslant r\leqslant m-1} L(\mathrm{sym}^{2r} f, s)^{C_m(r)} L(\mathrm{sym}^{2r} f\times\chi, s)^{C_m(r)} \tag{2.11}$$

can be represented as

$$F_j(s) := \prod_p \Bigl(1 + \sum_{k\geqslant 1} \frac{b(p^k)}{p^{ks}}\Bigr). \tag{2.12}$$

By the relations (2.5), (2.11), (2.12) and Lau-Lü (see [24], Lemma 7.1), we know that

$$\lambda_f^j(p) r(p) = \Bigl(A_m + \sum_{1\leqslant r\leqslant m-1} C_m(r) \lambda_{\mathrm{sym}^{2r} f}(p) + \lambda_{\mathrm{sym}^{2m} f}(p)\Bigr)(1 + \chi(p)) = b(p), \tag{2.13}$$

where A_m is determined by (2.9), and $C_m(r)$ are some suitable coefficients. Combining (2.10)–(2.13) for $\Re(s) > 1$ we obtain

$$\begin{aligned}\mathfrak{L}_j(f,s) = F_j(s) \times \prod_p \Big(1 + \frac{\lambda_f^j(p^2)r(p^2) - b(p^2)}{p^{2s}} + \ldots\Big) \\ := \zeta(s)^{A_m} L(\mathrm{sym}^{2m} f, s) L(s,\chi)^{A_m} L(\mathrm{sym}^{2m} f \times \chi, s) \\ \times \prod_{1 \leqslant r \leqslant m-1} L(\mathrm{sym}^{2r} f, s)^{C_m(r)} L(\mathrm{sym}^{2r} f \times \chi, s)^{C_m(r)} H_j(s).\end{aligned}$$

It is not hard to find that

$$|\lambda_f^j(p^2)r(p^2) - b(p^2)| \leqslant c_1$$

for a suitable constant $c_1 > 0$. Hence, $H_j(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. $\square$

We also define

$$\mathfrak{L}(\mathrm{sym}^j f, s) := \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^j f}(n) r(n)}{n^s}, \quad \Re(s) > 1. \tag{2.14}$$

We have the following lemma concerning the decomposition of $\mathfrak{L}(\mathrm{sym}^j f, s)$.

Lemma 2.3. *Let $f \in H_k^*$ be a Hecke eigenform. For $j \geqslant 2$ we have*

$$\mathfrak{L}(\mathrm{sym}^j f, s) = L_{f,j}(s) G_j(s),$$

where

$$L_{f,j}(s) = \zeta(s) L(s,\chi) \prod_{r=1}^{j} L(\mathrm{sym}^{2r} f, s) L(\mathrm{sym}^{2r} f \times \chi, s),$$

where the function $G_j(s)$ is a Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ and $G_j(s) \neq 0$ with $\Re(s) = 1$.

P r o o f. This follows the same argument as in the proof of Lemma 2.2 by noting the relation

$$\lambda^2_{\mathrm{sym}^j f}(p) = \lambda_f^2(p^j) = 1 + \lambda_{\mathrm{sym}^2 f}(p) + \ldots + \lambda_{\mathrm{sym}^{2j} f}(p).$$

$\square$

Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. In a similar manner, we define

$$(2.15)\qquad \mathfrak{L}(f,f,i,j,s) := \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n) r(n)}{n^s}$$

and

$$(2.16)\qquad \mathfrak{L}(f,g,i,j,s) := \sum_{n=1}^{\infty} \frac{\lambda^2_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n) r(n)}{n^s}$$

for $\Re(s) > 1$, where $1 \leqslant i \leqslant j$ and $i, j \in \mathbb{Z}^+$.

Lemma 2.4. *Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. For $1 \leqslant i \leqslant j$ being any fixed positive integers, we have*

$$\begin{aligned}\mathfrak{L}(f,f,i,j,s) = \zeta(s)L(s,\chi)\prod_{l_1=1}^{i}\prod_{l_2=1}^{j} & L(\mathrm{sym}^{2l_1} f, s)\\ &\times L(\mathrm{sym}^{2l_2} f, s) L(\mathrm{sym}^{2l_1} f \times \mathrm{sym}^{2l_2} f, s)\\ &\times L(\mathrm{sym}^{2l_1} f \times \chi, s) L(\mathrm{sym}^{2l_2} f \times \chi, s)\\ &\times L(\mathrm{sym}^{2l_1} f \times \mathrm{sym}^{2l_2} f \times \chi, s) U^*_{f,f,i,j}(s)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}(f,g,i,j,s) = \zeta(s)L(s,\chi)\prod_{l_1=1}^{i}\prod_{l_2=1}^{j} & L(\mathrm{sym}^{2l_1} f, s)\\ &\times L(\mathrm{sym}^{2l_2} g, s) L(\mathrm{sym}^{2l_1} f \times \mathrm{sym}^{2l_2} g, s)\\ &\times L(\mathrm{sym}^{2l_1} f \times \chi, s) L(\mathrm{sym}^{2l_2} g \times \chi, s)\\ &\times L(\mathrm{sym}^{2l_1} f \times \mathrm{sym}^{2l_2} g \times \chi, s) U^*_{f,g,i,j}(s),\end{aligned}$$

where the functions $U^*_{f,f,i,j}(s)$, $U^*_{f,g,i,j}(s)$ are Dirichlet series which converge uniformly and absolutely in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ and $U^*_{f,f,i,j}(s), U^*_{f,g,i,j}(s) \neq 0$ if $\Re(s) = 1$.

Proof. This follows the arguments of Lao and Luo (see [27], Proposition 3.1) with some modifications. □

3. Proofs of Theorems 1.1–1.4

Now we are ready to establish the main results of this paper. We only give the proof of Theorem 1.1, and the other theorems follow essentially the same arguments by applying Lemmas 2.1–2.4.

Let $j \geqslant 9$ be any fixed integer. For $j = 2m$, by (2.7) in Lemma 2.2, define

$$F_{2m}(s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

then the L-function $F_{2m}(s)$ is of degree 2^{2m+1} and can be extended to a meromorphic function on the whole complex plane except for a pole of order A_m at $s = 1$. Then by Lemma 2.1 we can derive that

$$\sum_{n \leqslant x} b(n) = xP''_{A_m-1}(\log x) + O_{f,\varepsilon}(x^{1-2^{-2m}+\varepsilon}),$$

where the main term $xP''_{A_m-1}(\log x)$ is given by

$$xP''_{A_m-1}(\log x) = \mathrm{Res}_{s=1}\Big\{F_{2m}(s)\frac{x^s}{s}\Big\}.$$

Here $P''_\omega(t)$ denotes a polynomial in t of degree ω, and A_m is defined by (2.9).

By Lemma 2.2 we know that

$$\lambda_f^j(n)r(n) = \sum_{n=uv} c(v)b(u)$$

satisfies the relations

$$\sum_{v \geqslant 1} |c(v)|v^{-\sigma} \ll_\sigma 1 \quad \text{for any } \sigma > \frac{1}{2}. \tag{3.1}$$

Hence, we can obtain

$$\begin{aligned}
\sum_{n \leqslant x} \lambda_f^j(n)r(n) &= \sum_{v \leqslant x} c(v) \sum_{u \leqslant x/v} b(u) \\
&= \sum_{v \leqslant x} c(v)\Big(\frac{x}{v}P''_{A_m-1}\Big(\log\frac{x}{y}\Big) + O\Big(\Big(\frac{x}{y}\Big)^{1-2^{-2m}+\varepsilon}\Big)\Big) \\
&= xP_{A_m-1}(\log x) + O(x^{1-2^{-2m}+\varepsilon})
\end{aligned}$$

by noting relation (3.1). Here $P_\omega(t)$ denotes another polynomial in t of degree ω, and A_m is defined by (2.9).

For $j = 2m+1$ we know from (2.8) that the L-function $F_{2m+1}(s)$ can be extended to an entire function. Then again by applying Lemma 2.1 and arguing as above, we obtain

$$U_j(f;x) \ll_{f,\varepsilon} x^{1-2^{-(2m+1)}+\varepsilon}.$$

This completes the proof of Theorem 1.1. □

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