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Guodong Hua

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ON HIGHER MOMENTS OF HECKE EIGENVALUES ATTACHED TO CUSP FORMS

Guodong Hua, Weinan

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Abstract. Let f, g and h be three distinct primitive holomorphic cusp forms of even integral weights k_1, k_2 and k_3 for the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, respectively, and let $\lambda_f(n)$, $\lambda_g(n)$ and $\lambda_h(n)$ denote the nth normalized Fourier coefficients of f, g and h, respectively. We consider the cancellations of sums related to arithmetic functions $\lambda_g(n)$, $\lambda_h(n)$ twisted by $\lambda_f(n)$ and establish the following results:

$$\sum_{n\leqslant x} \lambda_f(n)\lambda_g(n)^i\lambda_h(n)^j \ll_{f,g,h,\varepsilon} x^{1-1/2^{i+j}+\varepsilon}$$

for any $\varepsilon > 0$, where $1 \leqslant i \leqslant 2$, $j \geqslant 5$ are any fixed positive integers.

Keywords: Hecke eigenform; Fourier coefficient; Rankin-Selberg L-function

MSC 2020: 11F11, 11F30, 11F66

1. Introduction

The Fourier coefficients of modular forms are important and interesting objects in number theory. Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Suppose that $\varphi(z)$ is an eigenfunction of all Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigenform $\varphi(z)$ has the following Fourier expansion at the cusp ∞:

$$\varphi(z) = \sum_{n=1}^{\infty} \lambda_\varphi(n) n^{(k-1)/2} \mathrm{e}^{2\pi i n z}, \quad \Im(z) > 0,$$

where $\lambda_\varphi(n)$ are the normalized Fourier coefficients such that $\lambda_\varphi(1) = 1$.

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Then $\lambda_\varphi(n)$ is real and satisfies the multiplicative property

$$\lambda_\varphi(m)\lambda_\varphi(n) = \sum_{d|(m,n)} \lambda_\varphi\Big(\frac{mn}{d^2}\Big), \tag{1.1}$$

where $m \geqslant 1$ and $n \geqslant 1$ are positive integers. In 1974, Deligne in [4] proved the Ramanujan-Petersson conjecture

$$|\lambda_\varphi(n)| \leqslant d(n), \tag{1.2}$$

where $d(n)$ is the divisor function. By (1.2), Deligne's bound is equivalent to the fact that there exist $\alpha_\varphi(p), \beta_\varphi(p) \in \mathbb{C}$ satisfying

$$\alpha_\varphi(p) + \beta_\varphi(p) = \lambda_\varphi(p), \quad \alpha_\varphi(p)\beta_\varphi(p) = |\alpha_\varphi(p)| = |\beta_\varphi(p)| = 1. \tag{1.3}$$

More generally, for all positive integers $l \geqslant 1$ one has

$$\lambda_\varphi(p^l) = \alpha_\varphi(p)^l + \alpha_\varphi(p)^{l-1}\beta_\varphi(p) + \ldots + \alpha_\varphi(p)\beta_\varphi(p)^{l-1} + \beta_\varphi(p)^l.$$

The average behaviour of Fourier coefficients has attracted a large number of investigations in the literature. There is a long history of the investigation of the upper estimate for

$$S(x) = \sum_{n \leqslant x} \lambda_f(n).$$

In 1927, Hecke in [7] proved that

$$\sum_{n \leqslant x} \lambda_f(n) \ll x^{1/2}.$$

Subsequently, there are a number of improvements on $S(x)$ (see e.g. [4], [25], [33], [34]), and the current best estimate is due to Wu, see [34]

$$\sum_{n \leqslant x} \lambda_f(n) \ll x^{1/3} \log^{\varrho} x,$$

where

$$\varrho = \frac{102 + 7\sqrt{21}}{210}\Big(\frac{6 - \sqrt{21}}{5}\Big)^{1/2} + \frac{102 - 7\sqrt{21}}{210}\Big(\frac{6 + \sqrt{21}}{5}\Big)^{1/2} - \frac{33}{35} = -0.118\ldots$$

In the 1930s, Rankin in [24] and Selberg in [27] inverted the powerful Rankin-Selberg method and they successfully showed that

$$\sum_{n \leqslant x} \lambda_f^2(n) = c_f x + O_f(x^{3/5}), \quad \sum_{n \leqslant x} \lambda_f(n)\lambda_g(n) = O_{f,g}(x^{3/5}) \quad (f \neq g). \tag{1.4}$$

Very recently, the exponent of the first result in (1.4) has been sharpened to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang (see [8]), where $\delta \leqslant \frac{1}{560}$. This remains the best possible result in this direction.

Later, based on the works about symmetric power L-functions, Moreno and Shahidi in [19] proved that

$$\sum_{n \leqslant x} \tau_0^4(n) \sim c_1 x \log x, \quad x \to \infty,$$

where $c_1 > 0$ is a suitable constant and $\tau_0(n) = \tau(n)/n^{11/2}$ is the normalized Ramanujan tau-function. Obviously, Merono and Shahidi's result also holds true if we replace $\tau_0(n)$ with the normalized Fourier coefficient $\lambda_f(n)$.

Based on work of Gelbart and Jacquet (see [6]), we know that the automorphy of symmetric power lifting $\mathrm{sym}^j \pi_f$ attached to f is proved for $j = 2$, and similarly for g. In 2001, Fomenko in [5] refined and generalized the above results by showing that

$$\sum_{n \leqslant x} \lambda_f(n)^4 = c_2 x \log x + c_3 x + O_{f,\varepsilon}(x^{9/10+\varepsilon})$$

and

$$\sum_{n \leqslant x} \lambda_f^2(n) \lambda_g^2(n) = c_4 x + O_{f,g,\varepsilon}(x^{9/10+\varepsilon}) \tag{1.5}$$

for any $\varepsilon > 0$, where the result in (1.5) required the condition that $\mathrm{sym}^2 \pi_f \not\cong \mathrm{sym}^2 \pi_g$, and he also established some other results.

Let $f \in S_{k_1}(\Gamma)$, $g \in S_{k_2}(\Gamma)$ and $h \in S_{k_3}(\Gamma)$ be three distinct primitive Hecke cusp forms, and denote by $\lambda_f(n)$, $\lambda_g(n)$ and $\lambda_h(n)$ the nth normalized Fourier coefficients of f, g and h, respectively. In 2013, Lü in [17] by using Ramakrishnan's modularity theorem (see [22]) on the Rankin-Selberg L-function and some analytic properties of automorphic L-functions showed that

$$\sum_{n \leqslant x} \lambda_f(n) \lambda_g(n) \lambda_h(n) \ll_{f,g,h,\varepsilon} x^{7/9+\varepsilon}$$

and

$$\sum_{n \leqslant x} \lambda_f(n) \lambda_g(n) \lambda_h^l(n) \ll_{f,g,h,\varepsilon} x^{(2^{l+1}-1)/2^{l+1}+\varepsilon} \tag{1.6}$$

for any $\varepsilon > 0$, where $2 \leqslant l \leqslant 4$ is any fixed positive integer. In the case $l = 3$, the estimate (1.6) holds with the assumption that $\mathrm{sym}^3 \pi_h \not\cong \pi_{f \otimes g}$.

Later, Lü and Sankaranarayanan [18] in another paper generalized this to other cases by showing that

$$\sum_{n\leqslant x}\lambda_f(n)\lambda_g^2(n)\lambda_h^j(n) \ll_{f,g,h,\varepsilon} x^{1-1/2^{j+2}+\varepsilon}$$

for any $\varepsilon > 0$, where $2 \leqslant j \leqslant 4$ is any fixed positive integer.

In this paper, we consider the sums of arithmetic functions of the type

$$\sum_{n\leqslant x}\lambda_f(n)\lambda_g^i(n)\lambda_h^j(n),$$

where $1 \leqslant i \leqslant 2$, $j \geqslant 5$ are any fixed positive integers. More precisely, we are able to establish the following theorem.

Theorem 1.1. *Let $f \in S_{k_1}(\Gamma)$, $g \in S_{k_2}(\Gamma)$ and $h \in S_{k_3}(\Gamma)$ be three distinct primitive Hecke cusp forms. For any $\varepsilon > 0$, by assuming the conditions $\pi_{f\times g} \ncong \mathrm{sym}^3\pi_h$ and $\pi_{f\times \mathrm{sym}^2 g} \ncong \mathrm{sym}^5\pi_h$, we have*

$$\sum_{n\leqslant x}\lambda_f(n)\lambda_g^i(n)\lambda_h^j(n) \ll_{f,g,h,\varepsilon} x^{1-1/2^{i+j}+\varepsilon},$$

where $1 \leqslant i \leqslant 2$, $j \geqslant 5$ are any fixed positive integers.

Our proof of Theorem 1.1 is based on two important progress on Langlands program, namely Ramakrishnan's modularity theorem (see [22]) and functorial products for $GL_2 \times GL_3$ (see [13]), together with the celebrated series of vital works of Gelbart and Jacquet (see [6]), Kim (see [13]), Kim and Shahidi (see [14], [15]), Shahidi (see [31]), Clozel and Thorne (see [1], [2], [3]), and Newton and Thorne (see [20], [21]). The analytic properties of the automorphic L-functions plays an important role in the proof of the main results in this paper.

Throughout the paper, we always assume that $f \in S_{k_1}(\Gamma)$, $g \in S_{k_2}(\Gamma)$ and $h \in S_{k_3}(\Gamma)$ be three distinct primitive Hecke eigenforms and denote by $\varepsilon > 0$ the arbitrarily small positive number that may vary from one occurrence to other occurrence. The symbol p always denotes a prime number.

2. Preliminaries

Let $f \in S_{k_1}(\Gamma)$ be a Hecke eigenform of even integral weight k_1 for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and let $\lambda_f(n)$ denote its nth normalized Fourier coefficient. It is natural to define the Hecke L-function $L(f,s)$ associated to f by

$$L(f,s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s} = \prod_p(1-\lambda_f(p)p^{-s}+p^{-2s})^{-1} = \prod_p\Big(1-\frac{\alpha_f(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta_f(p)}{p^s}\Big)^{-1},$$

$\Re(s) > 1$, where $\alpha_f(p)$, $\beta_f(p)$ are the local parameters satisfying (1.3). The jth symmetric power L-function associated with f is defined by
(2.1)

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1} := \prod_p L_p(\mathrm{sym}^j f, s), \quad \Re(s) > 1.$$

We may expand it into Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \Big(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \ldots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \ldots\Big), \quad \Re(s) > 1. \end{aligned}$$

Apparently $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. For $j = 1$ we have $L(\mathrm{sym}^1 f, s) = L(f, s)$. The Rankin-Selberg L-function $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined as
(2.2)

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) &= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \Big(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-m'}\beta_g(p)^{m'}}{p^s}\Big)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s}, \quad \Re(s) > 1. \end{aligned}$$

For $i = j = 1$ we have $L(\mathrm{sym}^1 f \times \mathrm{sym}^1 g, s) = L(f \times g, s)$.

Associated to a primitive cusp form f, there is an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and hence, an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$, namely

$$L(\pi_f, s) = L(f, s).$$

It is predicted by the Langlands functoriality conjecture that π_f gives rise to a symmetric power lift $\mathrm{sym}^j \pi_f$ – an automorphic representation whose L-function is the symmetric power L-function attached to f,

$$L(\mathrm{sym}^j \pi_f, s) = L(\mathrm{sym}^j f, s).$$

It is conjectured that there exists an automorphic cuspidal self-dual representation $\mathrm{sym}^j \pi_f$ of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function is the same as $L(\mathrm{sym}^j f, s)$.

For $1 \leqslant j \leqslant 8$, this special Langlands functoriality conjecture that $\mathrm{sym}^j f$ is automorphic cuspidal is shown by a series of important works by Gelbart and Jacquet (see [6]), Kim (see [13]), Kim and Shahidi (see [14], [15]), Shahidi (see [31]), Clozel and Thorne, see [1], [2], [3]. Very recently, Newton and Thorne in [20], [21] proved that $\text{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geqslant 1$ (with f being holomorphic cusp forms). Then we know that for all $j \geqslant 1$ there exists an automorphic cuspidal self-dual representation, denoted by $\text{sym}^j \pi_f = \otimes' \text{sym}^j \pi_{f,v}$, of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$, whose local L-factors $L(\text{sym}^j \pi_{f,p}, s)$ agree the local L-factors $L_p(\text{sym}^j f, s)$ in (2.1). In particular, $L(\text{sym}^j f, s)$ has an analytic continuation to the whole complex plane as an entire function and satisfies a certain Riemann-type functional equation for all $j \geqslant 1$.

From the works about the Rankin-Selberg theory associated with two automorphic cuspidal representations developed by Jacquet, Piatetski-Shapiro and Shalika (see [10]), Jacquet and Shalika (see [11], [12]), Shahidi (see [28], [29], [30], [32], and the reformulation of Rudnick and Sarnak (see [26]), we know the analytic properties for the Rankin-Selberg L-functions $L(\text{sym}^i f \times \text{sym}^j g, s)$ with $i, j \geqslant 1$.

Lemma 2.1. *Let $f \in S_{k_1}(\Gamma)$ and $g(z) \in S_{k_2}(\Gamma)$ be two distinct primitive Hecke cusp forms and the Rankin-Selberg L-function $L(f \times g, s)$ let be defined by* (2.2). *Then there exists a cuspidal representation $\pi_{f\times g}$ on $GL_4(\mathbb{A}_{\mathbb{Q}})$ such that*

$$L(f \times g, s) = L(\pi_{f\times g}, s).$$

In particular, $L(f \times g, s)$ has an analytic continuation to the whole complex plane as an entire function and satisfies the functional equation of Riemann-type.

Proof. This is a special case of Ramakrishnan's modularity theorem on the Rankin-Selberg L-function, see [22]. □

Lemma 2.2. *Let $f(z) \in S_{k_1}(\Gamma)$ and $g(z) \in S_{k_2}(\Gamma)$ be two distinct primitive Hecke cusp forms, and be $L(\pi_f \times \text{sym}^2 \pi_g, s)$ the Rankin-Selberg L-function associated with π_f on $GL_2(\mathbb{A}_{\mathbb{Q}})$ and $\text{sym}^2 \pi_g$ on $GL_3(\mathbb{A}_{\mathbb{Q}})$. Then there exists a cuspidal representation $\pi_f \boxtimes \text{sym}^2 \pi_g$ on $GL_6(\mathbb{A}_{\mathbb{Q}})$ such that*

$$L(\pi_f \times \text{sym}^2 \pi_g, s) = L(\pi_f \boxtimes \text{sym}^2 \pi_g, s).$$

Proof. Let π_2 and π_3 be unitary automorphic cuspidal representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and $GL_3(\mathbb{A}_{\mathbb{Q}})$, respectively. Then, by Kim and Shahidi (see [13]), $\pi_2 \boxtimes \pi_3$ is an automorphic representation of $GL_6(\mathbb{A}_{\mathbb{Q}})$. It is isobaric and cuspidal or irreducibly induced from unitary cuspidal representations. When π_2 is not dihedral, $\pi_2 \boxtimes \pi_3$ is cuspidal unless π_3 is a twist of $\text{Ad}(\pi_2)$ by a grössencharacter, see [23]. Then the lemma follows the assertions. Interested readers can also consult in Lemma 2.3 of [18]. □

Lemma 2.3. *For* $\Re(s) > 1$, *define*

$$L_{i,j}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g^i(n)\lambda_h^j(n)}{n^s},$$

where $1 \leqslant i \leqslant 2$ *and* $j \geqslant 5$ *are any fixed positive integers. Then we have*

$$L_{i,j}(s) = \Big\{\prod_{l,r} L((f \times \mathrm{sym}^l g) \times \mathrm{sym}^r h, s)\Big\} U_{i,j}(s),$$

where the product of the L-functions $L((f \times \mathrm{sym}^l g) \times \mathrm{sym}^r h, s)$ *is another automorphic L-function of degree* 2^{1+i+j} *with* $0 \leqslant l \leqslant i$ *and* $0 \leqslant r \leqslant j$, *here* $1 \leqslant i \leqslant 2$ *and* $j \geqslant 5$ *are any fixed positive integers, and the* $U_{i,j}(s)$ *is a Dirichlet series which converges uniformly and absolutely in the half-plane* $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ *for any* $\varepsilon > 0$.

Here $L((f \times \mathrm{sym}^l g) \times \mathrm{sym}^r h, s)$ *with* $1 \leqslant l \leqslant 2$ *and* $r \geqslant 1$ *are well-defined Rankin-Selberg L-functions associated with corresponding automorphic cuspidal representations.*

Proof. Since $\lambda_f(n)\lambda_g^i(n)\lambda_h^j(n)$ are multiplicative functions and satisfy the trivial bound $O(n^{\varepsilon})$, then for $\Re(s) > 1$ we have the Euler product

$$L_{i,j}(s) = \prod_p \Big(1 + \sum_{k \geqslant 1} \frac{\lambda_f(p^k)\lambda_g^i(p^k)\lambda_h^j(p^k)}{p^{ks}}\Big).$$

In the half-plane $\Re(s) > \frac{1}{2}$, the corresponding coefficients of p^{-s} determine analytic properties of $L_{i,j}(s)$.

By the Hecke relation (1.1) and Lau-Lü (see [16], Lemma 7.1), we know that $\lambda_f(p)\lambda_g^i(p)\lambda_h^j(p)$ can be decomposed into the sums of types $\lambda_f(p)\lambda_{\mathrm{sym}^l g}(p)\lambda_{\mathrm{sym}^r h}(p)$ with $0 \leqslant l \leqslant 2$, $r \geqslant 0$. Then the assertions follow from Lemmas 2.1 and 2.2 and these identities. □

3. Proof of Theorem 1.1

From the Rankin-Selberg theory mentioned in Section 2, by assuming the conditions $\pi_{f\times g} \ncong \mathrm{sym}^3\pi_h$ and $\pi_{f\times \mathrm{sym}^2 g} \ncong \mathrm{sym}^5\pi_h$, the Rankin-Selberg L-functions $L((f \times g) \times \mathrm{sym}^j h, s)$ and $L((f \times \mathrm{sym}^2 g) \times \mathrm{sym}^j h, s)$ with $j \geqslant 1$ can be analytically continued to the whole complex plane as entire functions and satisfy certain Riemann-type functional equations.

By Lemma 2.3, we define the general L-function

$$L^*_{i,j}(s) = \prod_{l,r} L((f \times \mathrm{sym}^l g) \times \mathrm{sym}^r h, s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \quad \text{and} \quad U_{i,j}(s) := \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

for $\Re(s) > 1$, where $U_{i,j}(s)$ is the Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geqslant \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$, and $0 \leqslant l \leqslant 2$ and $j \geqslant 5$. We learn from the Rankin-Selberg theory that $L^*_{i,j}(s)$ is an automorphic L-function (see [9], Chapter 5), which can be analytically continued to the whole complex plane as an entire function and satisfies Riemann-type functional equation. In particular, these general L-functions $L^*_{i,j}(s)$ satisfy the conditions in Lau and Lü (see [16], Lemma 2.4), which states that if we suppose that $L(f, s)$ is a product of two general L-functions L_1, L_2 with both degree $\deg L_i \geqslant 2$, and $L(s, f)$ satisfies the Ramanujan conjecture, then for $\varepsilon > 0$ we have

$$\sum_{n \leqslant x} \lambda_f(n) = M(x) + O(x^{1-2/m+\varepsilon}),$$

where $M(x) = \mathrm{Res}_{s=1}\{L(f, s)x^s/s\}$ and $m = \deg L$. Then we know from Lemma 2.3 that

$$\sum_{n \leqslant x} b(n) \ll x^{1-1/2^{i+j}+\varepsilon}.$$

By Lemma 2.3 we know that

$$\lambda_f(n)\lambda_g^i(n)\lambda_h^j(n) = \sum_{n=uv} c(v)b(u)$$

which satisfies the relation

$$\sum_{v \geqslant 1} |c(v)|v^{-\sigma} \ll_\sigma 1 \quad \text{for any } \sigma > \frac{1}{2}. \tag{3.1}$$

Hence, we can obtain

$$\begin{aligned}\sum_{n \leqslant x} \lambda_f(n)\lambda_g^i(n)\lambda_h^j(n) &= \sum_{v \leqslant x} c(v) \sum_{u \leqslant x/v} b(u) \\ &\ll x^{1-1/2^{i+j}+\varepsilon} \sum_{v \leqslant x} \frac{c(v)}{v^{1-1/2^{i+j}+\varepsilon}} \ll x^{1-1/2^{i+j}+\varepsilon}\end{aligned}$$

by noting relation (3.1). This completes the proof of Theorem 1.1. □

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Author's address: Guodong Hua, School of Mathematics and Statistics, Weinan Normal University, Chaoyang Street, Shaanxi, Weinan 714099, P. R. China, e-mail: `gdhua@mail.sdu.edu.cn`.