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INTERPOLATION SPACES AND WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO PARABOLIC EQUATIONS AND APPLICATIONS TO FLUID DYNAMICS

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Abstract. We investigate the existence, uniqueness and polynomial stability of the weighted pseudo almost automorphic solutions to a class of linear and semilinear parabolic evolution equations. The necessary tools here are interpolation spaces and interpolation theorems which help to prove the boundedness of solution operators in appropriate spaces for linear equations. Then for the semilinear equations the fixed point arguments are used to obtain the existence and stability of the weighted pseudo almost automorphic solutions. Lastly, our abstract results are applied to the Navier-Stokes equations (NSE) on some different circumstances such as the NSE on exterior domains, around rotating obstacles, and in Besov spaces.

Keywords: linear evolution equation; semilinear evolution equation; almost automorphic function; weighted pseudo almost automorphic function and solution; interpolation space

MSC 2020: 35B15, 35B35, 35Q30, 76D05

1. Introduction

The notion of almost automorphic functions was introduced by Bochner as a generalization of almost periodic functions (see [4], [5], [6]) and in an attempt to relate the function spaces with some aspects of differential geometry. Then, intensive studies of this concept and extensions to various types of solutions to differential and difference equations have been made during recent years, see [8], [17], [18], [19], [20]. Further,

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N'Guérékata et al. in [10] and Xiao et al. in [25] extended the notion of almost automorphic functions to the so-called pseudo almost automorphic functions and established the existence and uniqueness theorems of pseudo almost automorphic solutions to a certain class of semilinear abstract differential equations. Recently, Blot et al. in [3] have introduced the concept of *weighted pseudo almost automorphic functions*, which generalizes the concept of pseudo almost periodic functions (see [9]) and connects it with the pseudo almost automorphic ones in a unified approach. The authors of [3] have proved some important properties of the space of weighted pseudo almost automorphic functions such as the completeness and composition theorem.

In this paper, we prove the existence, uniqueness and stability of the weighted pseudo almost automorphic (WPAA) solutions to the linear and semilinear evolution equations on interpolation spaces. We refer the reader to the books (see [2], [24]) which provide the detailed description of the theory for interpolation spaces and the differential operators. The interpolation spaces have been used to study the solutions of the incompressible viscous fluid flow problems in many works, see [12], [13], [16], [21], [22], [23], [26].

More precisely, in the present paper we extend our recent results in the case of almost automorphic solutions obtained in [14] to the case of WPAA solutions to parabolic evolution equations in interpolation spaces. Also, we go farther to prove the polynomial stability of such WPAA solutions. The novelty in our results is lying in the fact that we allow the zero number to belong to the spectrum of the generator of the solution semigroup of the corresponding linear equation. This leads to the fact that the semigroup is no longer exponentially stable. At this point, the interpolation spaces and interpolation theorem come into play and allow us to prove the boundedness of the solution operator on WPAA. And this important fact helps us to show the existence and stability of WPAA solutions to evolution equations.

Our method is inspired by the series of works by Hishida and Shibata (see [16]), Hieber et al. (see [15]), and Nguyen and his collaborators (see [14], [21], [22], [23]). We extend such methods to a more general framework (see Assumption 5.1) so that it can be conveniently applied to obtain the solutions in the space of WPAA functions on the whole line.

For the applications to fluid flow problems, we consider the Navier-Stokes equations (NSE) in three cases. They are NSE on exterior domains, NSE around rotating obstacles, and NSE in Besov spaces. We prove the existence and polynomial stability of WPAA solutions to NSE in such situations. Our results extend the results of Yamazaki (see [26]), Borchers and Miyakawa (see [7]), Nguyen et al. (see [14]) for the case of NSE on exterior domains, results of Hieber et al. [15], Nguyen et al. (see [23]) for the case of NSE around rotating obstacles, and results of [22] for the case of NSE in the framework of Besov spaces. Precisely, we prove that WPAA solutions to NSE in such contexts exist and are polynomially stable.

This paper is organized as follows: Section 1 contains the preliminaries on the concepts of almost automorphic and weighted pseudo almost automorphic (WPAA) functions. In Section 2 we study the existence and uniqueness of WPAA solutions to linear evolution equations, and in Section 3 we investigate the semilinear equations. Section 4 devotes to the proof of stability of the solution obtained in previous Section 4, and lastly, Section 5 gives some applications to incompressible fluid flow problems.

2. Preliminaries

In this section, we recall some preliminaries on function spaces and related concepts and results for latter use. As usual, for a Banach space $(X, \|\cdot\|)$ we denote by $BC(\mathbb{R}, X)$ the Banach space of all bounded, continuous, and X-valued functions defined on $\mathbb{R}$, endowed with the sup-norm defined by $\|f\|_\infty = \|f\|_{BC(\mathbb{R},X)} := \sup_{t\in\mathbb{R}} \|f(t)\|$. We also denote by $C(\mathbb{R}, X)$ the linear space of all continuous and X-valued functions on $\mathbb{R}$.

Definition 2.1. For the Banach space X, a continuous function $f\colon \mathbb{R} \to X$ is called to be *almost automorphic* if for any sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t + s_n)$$

is well-defined for every $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n).$$

We denote the space of all almost automorphic functions $f\colon \mathbb{R} \to X$ by $AA(X)$. It is a Banach space with the sup-norm.

Let now $\mathcal{U}$ be the set of all real-valued functions defined on $\mathbb{R}$, which are positive and locally integrable over $\mathbb{R}$. Then, for a given real number $r > 0$, we put $m(r, \varrho) := \int_{-r}^{r} \varrho(x)\,\mathrm{d}x$ for every $\varrho \in \mathcal{U}$, and set

$$\mathcal{U}_\infty := \Big\{\varrho \in \mathcal{U}\colon \lim_{r\to\infty} m(r, \varrho) = \infty\Big\}$$

and

$$\mathcal{U}_b = \Big\{\varrho \in \mathcal{U}_\infty\colon \varrho \text{ is bounded and } \inf_{x\in\mathbb{R}} \varrho(x) > 0\Big\}.$$

It is clear that $\mathcal{U}_b \subset \mathcal{U}_\infty \subset \mathcal{U}$. Now for every $\varrho \in \mathcal{U}_\infty$, we put

$$PAA_0(X,\varrho) := \left\{ f \in BC(\mathbb{R},X)\colon \lim_{r\to\infty} \frac{1}{m(r,\varrho)} \int_{-r}^{r} \|f(s)\|_X \varrho(s)\,\mathrm{d}s = 0 \right\}.$$

A function belonging to $PAA_0(X,\varrho)$ is said to have "weighted property".

Definition 2.2. A continuous function $f\colon \mathbb{R} \to X$ is called *ϱ-weighted pseudo almost automorphic* if it can be decomposed as $f = g + \varphi$, where $g \in AA(X)$ and $\varphi \in PAA_0(X,\varrho)$. Denote by $WPAA(X,\varrho)$ the set of all ϱ-weighted pseudo almost automorphic (WPAA) functions. Note that $(WPAA(X,\varrho), \|\cdot\|_\infty)$ is a Banach space, where $\|\cdot\|_\infty$ is the usual sup-norm defined as above.

The concept of a weighted pseudo almost automorphic function is a generalized notion of many other classes of functions such as almost automorphic, almost periodic, pseudo almost automorphic and pseudo almost periodic functions. Clearly, an almost automorphic (or pseudo almost automorphic) function is a WPAA function. An example of a WPAA function, which is neither almost automorphic nor pseudo almost automorphic, is

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + \mathrm{e}^{-t|\alpha|}, \quad \alpha \neq 0.$$

We refer the reader to [3], [9] for more detailed properties on weighted pseudo almost automorphic functions.

3. Weighted pseudo almost automorphic solutions to linear equations

Let X be a Banach space, (Y_1, Y_2) a couple of Banach spaces and $Y := (Y_1, Y_2)_{\theta,\infty}$ a real interpolation space for some $0 < \theta < 1$. We now consider the inhomogeneous linear evolution equation of the form

$$u'(t) + Au(t) = Bf(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $-A$ is a linear operator on $Y_1 + Y_2$ such that its restrictions on Y_1 and Y_2 are generators of C_0-semigroups (respectively), which are denoted by the same symbol e^{-tA}; f is a function from $\mathbb{R}$ to X and B is the 'connection' operator between X and Y such that $\mathrm{e}^{-tA}B \in \mathcal{L}(X, Y_i)$ for $i = 1,2$ and $t \geqslant 0$.

Definition 3.1. A function $u \in C(\mathbb{R}, Y)$ is called a *mild solution* to (3.1) if u satisfies the integral equation

$$u(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)A} Bf(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}. \tag{3.2}$$

We need the following assumptions on related operations and interpolation spaces.

Assumption 3.2. Assume that Y_i has a Banach pre-dual Z_i for $i = 1, 2$ (that means $Y_i = Z_i'$) such that $Z_1 \cap Z_2$ is dense in Z_i. Let A be a linear operator on $Y_1 + Y_2$ such that the restrictions of $-A$ on Y_1 and Y_2 are generators of bounded C_0-semigroups (respectively), which are denoted by the same symbol e^{-tA}. Furthermore, suppose that there exist constants $\alpha_1, \alpha_2 \in \mathbb{R}$ with $0 < \alpha_2 < 1 < \alpha_1$ and $L > 0$ such that

$$\begin{aligned} \|\mathrm{e}^{-tA}Bv\|_{Y_1} &\leqslant Lt^{-\alpha_1}\|v\|_X, \quad t > 0, \\ \|\mathrm{e}^{-tA}Bv\|_{Y_2} &\leqslant Lt^{-\alpha_2}\|v\|_X, \quad t > 0, \end{aligned} \tag{3.3}$$

where B is given as above.

The following lemmata are established in [15], [21].

Lemma 3.3. *Under Assumption* 3.2 *and letting* $\psi \in (Z_1, Z_2)_{\theta,1}$, *the following assertion holds: The function* $t \mapsto \|B'\mathrm{e}^{-tA'}\psi\|_{X'}$ *belongs to* $L^1(0;\infty)$ *and*

$$\int_0^\infty \|B'\mathrm{e}^{-tA'}\psi\|_{X'}\,\mathrm{d}t \leqslant \widetilde{L}\|\psi\|_{(Z_1,Z_2)_{\theta,1}} \tag{3.4}$$

for some constant $\widetilde{L}$.

Lemma 3.4. *Suppose Assumption* 3.2 *holds. Let* $f \in BC(\mathbb{R}; X)$, $\psi \in (Z_1, Z_2)_{\theta,1}$ *and* $\theta \in (0; 1)$ *such that* $1 = (1-\theta)\alpha_1 + \theta\alpha_2$. *Then the equation* (3.1) *admits a unique mild solution* u *satisfying*

$$\|u(t)\|_Y \leqslant \widetilde{L}\|f\|_{BC(\mathbb{R},X)}, \quad t \in \mathbb{R}, \tag{3.5}$$

for the constant $\widetilde{L}$ *as in* (3.4).

The existence and uniqueness of the weighted pseudo almost automorphic mild solution to the linear equation (3.1) is stated and proved in the following theorem.

Theorem 3.5. *Under Assumption* 3.2, *let* $f = g + \varphi \in WPAA(X, \varrho)$, *where* $g \in AA(X)$, $\varphi \in PAA_0(X, \varrho)$, *and* $\varrho \in \mathcal{U}_\infty$. *Then, the solution operator* $f \mapsto S(f)$ *with*

$$S(f)(t) := \int_{-\infty}^t \mathrm{e}^{-(t-\tau)A}Bf(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R},$$

maps $WPAA(X, \varrho)$ *into itself. This means that the linear equation* (3.1) *has a unique* ϱ-*weighted pseudo almost automorphic mild solution for every inhomogeneous term* $f \in WPAA(X, \varrho)$.

P r o o f. Since $f = g + \varphi$ we have

$$S(f)(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)A} Bg(\tau)\,\mathrm{d}\tau + \int_{-\infty}^{1} t\mathrm{e}^{-(t-\tau)A} B\varphi(\tau)\,\mathrm{d}\tau.$$

We prove this theorem in two steps corresponding to g and φ:

Step 1. With $g \in AA(X)$, we show that the function

$$u_1(t) := \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)A} Bg(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R},$$

belongs to $AA(X)$. Indeed, by using the change of variable $\sigma = t - \tau$, we obtain

$$u_1(t) = \int_{0}^{\infty} \mathrm{e}^{-\sigma A} Bg(t-\sigma)\,\mathrm{d}\sigma, \quad t \in \mathbb{R}.$$

Let now (σ'_n) be an arbitrary sequence of integer numbers. Since $g \in AA(X)$, there exist a subsequence (σ_n) of (σ'_n) and a function $h(t)$ such that

$$h(t) = \lim_{n\to\infty} g(t + s_n) \quad \text{and} \quad g(t) = \lim_{n\to\infty} h(t - s_n) \tag{3.6}$$

are well-defined for every $t \in \mathbb{R}$. Now fix $t \in \mathbb{R}$, we then get

$$\lim_{n\to\infty} g(t - \sigma + \sigma_n) = h(t - \sigma)$$

for every given $\sigma \in \mathbb{R}$. Clearly

$$u_1(t + \sigma_n) = \int_{0}^{\infty} \mathrm{e}^{-\sigma A} Bg(t - \sigma + \sigma_n)\,\mathrm{d}\sigma.$$

Note that by Assumption 3.2, $(Z_1, Z_2)'_{\theta,1} = (Y_1, Y_2)_{\theta,\infty}$. Denoting by $\langle \cdot, \cdot \rangle$ the dual pair between $(Y_1, Y_2)_{\theta,\infty}$ and $(Z_1, Z_2)_{\theta,1}$, we obtain for $\psi \in (Z_1, Z_2)_{\theta,1}$ that

$$\begin{aligned} |\langle \mathrm{e}^{-\sigma A} Bg(t - \sigma + \sigma_n), \psi \rangle| &= |\langle g(t - \sigma + \sigma_n), B' \mathrm{e}^{-\sigma A'} \psi \rangle| \\ &\leqslant \|g(t - \sigma + \sigma_n)\|_X \|B' \mathrm{e}^{-\sigma A'} \psi\|_{X'}. \end{aligned} \tag{3.7}$$

By using Lemma 3.3 we obtain

$$\begin{aligned} \int_0^\infty \|g(t - \sigma + \sigma_n)\|_X \|B' \mathrm{e}^{-\sigma A'} \psi\|_{X'}\,\mathrm{d}\sigma &\leqslant \|g\|_{AA(X)} \int_0^\infty \|B' \mathrm{e}^{-\sigma A'} \psi\|_{X'}\,\mathrm{d}\sigma \\ &\leqslant \widetilde{L} \|g\|_{AA(X)} \|\psi\|_{(Z_1,Z_2)_{\theta,1}}, \quad \psi \in (Z_1, Z_2)_{\theta,1}. \end{aligned}$$

Therefore, the function $\|g(t-\sigma+\sigma_n)\|_X\|B'\mathrm{e}^{-\sigma A'}\psi\|_{X'}$ is integrable on $(0,\infty)$, and so is $\langle \mathrm{e}^{-\sigma A}Bg(t-\sigma+\sigma_n),\psi\rangle$. Using now Lebesgue's dominated convergence theorem, we arrive at

$$\lim_{n\to\infty}\langle u_1(t+\sigma_n),\psi\rangle = \int_0^\infty \langle \mathrm{e}^{-\sigma A}Bg(t-\sigma),\psi\rangle\,\mathrm{d}\sigma := \langle u_1^*(t),\psi\rangle$$

for every $t\in\mathbb{R}$ and for all $\psi\in(Z_1,Z_2)_{\theta,1}$, i.e.,

$$w-\lim_{n\to\infty} u_1(t+\sigma_n) = \int_0^\infty \mathrm{e}^{-\sigma A}Bg(t-\sigma)\,\mathrm{d}\sigma.$$

In the same way, we obtain for $u_1^*(t)$ that

$$w-\lim_{n\to\infty} u_1^*(t-\sigma_n) = u_1(t)$$

for every $t\in\mathbb{R}$, which proves that $u_1(t)\in AA(X)$.

Step 2. In this step, we show that the function $u_2(t) := \int_{-\infty}^t \mathrm{e}^{-(t-\tau)A}B\varphi(\tau)\,\mathrm{d}\tau$, $t\in\mathbb{R}$, belongs to $PAA_0(X,\varrho)$, i.e., we prove that

$$\lim_{r\to\infty}\frac{1}{m(r,\varrho)}\int_{-r}^r\left\|\int_{-\infty}^s \mathrm{e}^{-(s-\tau)A}B\varphi(\tau)\,\mathrm{d}\tau\right\|_Y\varrho(s)\,\mathrm{d}s = 0.$$

Indeed, for all $\psi\in(Z_1,Z_2)_{\theta,1}$ we have

$$\begin{aligned}(3.8)\qquad &\frac{1}{m(r,\varrho)}\int_{-r}^r\int_{-\infty}^s |\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle|\,\mathrm{d}\tau\,\varrho(s)\,\mathrm{d}s\\ &= \frac{1}{m(r,\varrho)}\int_{-r}^r\int_{-\infty}^{-r} |\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle|\,\mathrm{d}\tau\,\varrho(s)\,\mathrm{d}s\\ &\quad+\frac{1}{m(r,\varrho)}\int_{-r}^r\int_{-r}^s |\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle|\,\mathrm{d}\tau\,\varrho(s)\,\mathrm{d}s.\end{aligned}$$

From Lemma 3.3, we get

$$\int_{-\infty}^s \|B'\mathrm{e}^{-(s-\tau)A'}\psi(\tau)\|_{X'}\,\mathrm{d}\tau < \widetilde{L}\|\psi\|_{(Z_1,Z_2)_{\theta,1}},$$

hence for all $\varepsilon>0$, there exists $r_0\in\mathbb{R}$ such that for all $r>r_0$, we have

$$\int_{-\infty}^{-r} \|B'\mathrm{e}^{-(s-\tau)A'}\psi(\tau)\|_{X'}\,\mathrm{d}\tau < \varepsilon\|\psi\|_{(Z_1,Z_2)_{\theta,1}}.$$

Therefore for $\|\varphi\|_{BC(\mathbb{R},X)} < M$, we have

$$\begin{aligned}
(3.9)\qquad & \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-\infty}^{-r}|\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi(\tau)\rangle|\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&= \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-\infty}^{-r}|\langle \varphi(\tau),B'\mathrm{e}^{-(s-\tau)A'}\psi(\tau)\rangle|\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&\leqslant \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-\infty}^{-r}\|\varphi(\tau)\|_X\|B'\mathrm{e}^{-(s-\tau)A'}\psi(\tau)\|_{X'}\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&\leqslant \frac{\widetilde{L}M\|\psi\|_{(Z_1,Z_2)_{\theta,1}}}{\int_{-r}^{r}\varrho(s)\,\mathrm{d}s}\int_{-r}^{r}\varepsilon\varrho(s)\,\mathrm{d}s\\
&= \widetilde{L}M\varepsilon\|\psi\|_{(Z_1,Z_2)_{\theta,1}}.
\end{aligned}$$

On the other hand, using again Lemma 3.3 we obtain the estimate

$$\begin{aligned}
&\frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-r}^{s}|\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle|\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&\leqslant \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-r}^{s}|\langle \varphi(\tau),B'\mathrm{e}^{-(s-\tau)A'}\psi\rangle|\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&\leqslant \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-r}^{s}\|\varphi(\tau)\|_X\|B'\mathrm{e}^{-(s-\tau)A'}\psi\|_{X'}\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s\\
&\leqslant \frac{\widetilde{L}\|\psi\|_{(Z_1,Z_2)_{\theta,1}}}{m(r,\varrho)}\int_{-r}^{r}\|\varphi(s)\|_X\varrho(s)\,\mathrm{d}s.
\end{aligned}$$

From the fact that $\varphi \in PAA_0(X,\varrho)$, we get

$$\lim_{r\to\infty}\frac{1}{m(r,\varrho)}\int_{-r}^{r}\|\varphi(s)\|_X\varrho(s)\,\mathrm{d}s = 0.$$

Hence, there exists $r_1 \in \mathbb{R}$ such that for all $r > r_1$, then

$$(3.10)\qquad \frac{1}{m(r,\varrho)}\int_{-r}^{r}\int_{-r}^{s}|\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle|\,\mathrm{d}\tau\varrho(s)\,\mathrm{d}s \leqslant \widetilde{L}\varepsilon\|\psi\|_{(Z_1,Z_2)_{\theta,1}}.$$

By combining (3.8), (3.9) and (3.10), we get

$$\frac{1}{m(r,\varrho)}\int_{-r}^{r}\left|\int_{-\infty}^{s}\langle \mathrm{e}^{-(s-\tau)A}B\varphi(\tau),\psi\rangle\,\mathrm{d}\tau\right|\varrho(s)\,\mathrm{d}s \leqslant \widetilde{L}\varepsilon(M+1)\|\psi\|_{(Z_1,Z_2)_{\theta,1}}$$

for all $r \geqslant \max\{r_0,r_1\}$ and all $\psi \in (Z_1,Z_2)_{\theta,1}$. Therefore,

$$\lim_{r\to\infty}\frac{1}{m(r,\varrho)}\int_{-r}^{r}\left\|\int_{-\infty}^{s}\mathrm{e}^{-(s-\tau)A}B\varphi(\tau)\,\mathrm{d}\tau\right\|_Y\varrho(s)\,\mathrm{d}s = 0.$$

□

4. Weighted pseudo almost automorphic solutions to semilinear equations

In the section we consider the semilinear evolution equation

$$u'(t) + Au(t) = BG(u)(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where A and B satisfy Assumption 3.2 and G maps from $BC(\mathbb{R}; Y)$ into $BC(\mathbb{R}, X)$.

Similarly to the linear equation (3.1), by a *mild solution* to (4.1) we mean the function $u \in C(\mathbb{R}, Y)$ satisfying the integral equation

$$u(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)A} BG(u)(\tau)\,\mathrm{d}\tau. \tag{4.2}$$

Lemma 3.4 implies that the solution operator $S\colon BC(\mathbb{R}; X) \to BC(\mathbb{R}; Y)$ defined by

$$S(f)(t) := \int_{-\infty}^{t} \mathrm{e}^{-(t-s)A} Bf(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.3}$$

is a bounded operator and $\|S\| \leqslant \widetilde{L}$ for the constant $\widetilde{L}$ appearing in Lemma 3.5.

We need the following assumptions on the function G.

Assumption 4.1. We assume that the operator $G\colon BC(\mathbb{R}, Y) \to BC(\mathbb{R}, X)$ maps weighted pseudo almost automorphic functions to weighted pseudo almost automorphic functions and there are positive constants κ and K such that the following conditions hold.

(i) $\|G(0)\|_{BC(\mathbb{R},X)} \leqslant K/\widetilde{L} - (\kappa + K)K$.

(ii) $\|G(u) - G(v)\|_{BC(\mathbb{R},X)} \leqslant (\kappa + \|u\|_{BC(\mathbb{R},Y)} + \|v\|_{BC(\mathbb{R},Y)})\|u - v\|_{BC(\mathbb{R},Y)}$ for $u, v \in WPAA(X, \varrho)$.

Our following result shows the existence of a unique weighted pseudo almost automorphic solution to the equation (4.1) on $WPAA(X, \varrho)$.

Theorem 4.2. *Assume that Assumptions* 3.2 *and* 4.1 *hold with* $K < 1$ *and* $\widetilde{L}(K + \kappa) < 1$. *Let* $\theta \in (0; 1)$ *be given such that* $1 = (1 - \theta)\alpha_1 + \theta\alpha_2$. *There exists a unique weighted pseudo almost automorphic mild solution* $\hat{u}$ *to the equation* (4.1) *in the Banach space* $WPAA(X, \varrho)$.

P r o o f. Denote by $B(0, K)$ the ball centered at 0 with radius $K > 0$ in the space $WPAA(X, \varrho)$ and let S be the solution operator of the linearized equation (3.1) defined as in (4.3). Then, we define the mapping Φ by

$$v \mapsto S(G(v)).$$

Next, we prove that Φ maps $B(0, K)$ into itself and is a contraction mapping on $B(0, K)$. Indeed, we have

$$(\Phi u)(t) = S(Gu)(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\sigma)A} BG(u)(\sigma)\, \mathrm{d}\sigma.$$

Then, applying Lemma 3.4 and using the conditions (i) and (ii) in Assumption 4.1 we can estimate:

$$\begin{aligned}
\|\Phi(u)\|_{BC(\mathbb{R},Y)} &\leqslant \widetilde{L}\|G(u)\|_{BC(\mathbb{R},X)} = \widetilde{L}\|G(0) + G(u) - G(0)\|_{BC(\mathbb{R},X)} \\
&\leqslant \widetilde{L}(\|G(0)\|_{BC(\mathbb{R},X)} + (\kappa + \|u\|_{BC(\mathbb{R},Y)})\|u\|_{BC(\mathbb{R},Y)}) \\
&\leqslant \widetilde{L}\Big(\frac{K}{\widetilde{L}} - (\kappa + K)K + (\kappa + K)K\Big) \\
&= K.
\end{aligned}$$

Therefore, Φ maps from $B(0, K)$ into itself.

We now show that Φ is a contraction mapping. Let u and v belong to the ball $B(0, K)$. It follows from $\|S\| \leqslant \widetilde{L}$ and Assumption 4.1 that

$$\begin{aligned}
\|(\Phi u) - (\Phi v)\|_{BC(\mathbb{R},Y)} &\leqslant \widetilde{L}\|G(u) - G(v)\|_{L^\infty(\mathbb{R},X)} \\
&\leqslant \widetilde{L}(\kappa + \|u\|_{BC(\mathbb{R},Y)} + \|v\|_{BC(\mathbb{R},Y)})\|u - v\|_{BC(\mathbb{R},Y)} \\
&\leqslant \widetilde{L}(\kappa + K)\|u - v\|_{BC(\mathbb{R},Y)}
\end{aligned}$$

for all $u, v \in B(0, K)$. Since $\widetilde{L}(\kappa + K) < 1$, we obtain that Φ is a contraction mapping. Then, the contraction principle yields the existence of a unique fixed point $\hat{u} \in B(0, K) \subset WPAA(X, \varrho)$ of Φ. By the definition of Φ, $\hat{u}$ is a weighted pseudo almost automorphic mild solution of the equation (4.2) in $WPAA(X, \varrho)$.

In order to prove the uniqueness, we assume that $u, v \in WPAA(X, \varrho)$ are two weighted pseudo almost automorphic mild solutions to (4.2). Then

$$\begin{aligned}
\|u - v\|_{BC(\mathbb{R};Y)} &= \|S(G(u) - G(v))\|_{BC(\mathbb{R};Y)} \\
&\leqslant \widetilde{L}\|G(u) - G(v)\|_{BC(\mathbb{R},Y)} \\
&\leqslant \widetilde{L}(\kappa + K)\|u - v\|_{BC(\mathbb{R};Y)}.
\end{aligned}$$

By assumption $\widetilde{L}(\kappa + K) < 1$, hence the uniqueness of the WPAA solution follows. □

To obtain the stability of the solution in the previous section, we need some more conditions on $B'\mathrm{e}^{-tA'}$ and G which are stated in the following assumption.

Assumption 5.1. Assume that there exist the Banach space T with the dual T', the interpolation space $(Q_1,Q_2)_{\tilde{\theta},1}$ with the dual $(Q_1,Q_2)'_{(\tilde{\theta},1)} =: Q$, and the positive numbers $0<\beta_1<1<\beta_2$, $0<\widetilde{M}$ such that

$$\begin{aligned} \|B'\mathrm{e}^{-tA'}\psi\|_{T'} &\leqslant \widetilde{M}t^{-\beta_1}\|\psi\|_{Q_1}, \quad t>0,\\ \|B'\mathrm{e}^{-tA'}\psi\|_{T'} &\leqslant \widetilde{M}t^{-\beta_2}\|\psi\|_{Q_2}, \quad t>0,\\ \|\mathrm{e}^{-tA}\psi\|_{Q} &\leqslant Ct^{-\gamma}\|\psi\|_{Y}, \quad t>0,\ 0<\gamma<1,\\ \|G(v_1)-G(v_2)\|_{L^\infty(\mathbb{R},T)} &\leqslant (\kappa+\|v_1\|_{BC(\mathbb{R},Y)}+\|v_2\|_{BC(\mathbb{R},Y)})\|v_1-v_2\|_{BC(\mathbb{R},Q)}. \end{aligned}$$

Now, we state and prove the polynomial stability of the WPAA mild solutions to (4.1) in the following theorem.

Theorem 5.2. *Let $\hat{u}$ be the WPAA mild solution to* (4.1) *obtained in the previous section. For any other bounded solution $u\in BC(\mathbb{R},Y)$ of the semilinear equation* (4.1) *we put $u(0)=u_0$, $\hat{u}(0)=\hat{u}_0$. If $\|\hat{u}\|_{BC(\mathbb{R},Y)}$ and $\|u_0-\hat{u}_0\|_Y$ are small enough, then we have*

$$(5.1)\qquad \|u(t)-\hat{u}(t)\|_Q \leqslant Dt^{-\gamma} \quad \forall\, t>0,$$

where D is a positive constant independent of u and $\hat{u}$.

Proof. We can see that both $u(t)$ and $\hat{u}(t)$ satisfy the integral equation

$$\begin{aligned} u(t) &= \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-\tau)A}B(G(u)(\tau))\,\mathrm{d}\tau,\\ \hat{u}(t) &= \mathrm{e}^{-tA}\hat{u}_0 + \int_0^t \mathrm{e}^{-(t-\tau)A}B(G(\hat{u})(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Therefore, putting $v:=u-\hat{u}$ we obtain that v satisfies the equation

$$(5.2)\qquad v(t) = \mathrm{e}^{-tA}(u_0-\hat{u}_0) + \int_0^t \mathrm{e}^{-(t-\tau)A}B(G(u)(\tau)-G(\hat{u})(\tau))\,\mathrm{d}\tau.$$

Next, we set $M:=\bigl\{v\in BC(\mathbb{R}_+,Y)\colon \sup\limits_{t\in\mathbb{R}_+} t^\gamma\|v(t)\|_Q<\infty\bigr\}$ endowed with the norm

$$\|v\|_M = \|v\|_{BC(\mathbb{R}_+,Y)} + \sup_{t\in\mathbb{R}_+} t^\gamma\|v(t)\|_Q.$$

Now we prove that if $\|\hat{u}\|_{BC(\mathbb{R}_+,Y)}$ and $\|u_0-\hat{u}_0\|_Y$ are small enough, then the equation (5.2) has a unique solution on a small ball of M. Indeed, for $v\in M$, we consider the mapping $v\mapsto \Phi(v)$ defined by

$$\Phi(v)(t) := \mathrm{e}^{-tA}(u_0-\hat{u}_0) + \int_0^t \mathrm{e}^{-(t-\tau)A}B(G(v+\hat{u})(\tau)-G(\hat{u})(\tau))\,\mathrm{d}\tau.$$

We set $B_\varrho := \{v\in M\colon \|v\|_M\leqslant \varrho\}$. We now prove that Φ acts from B_ϱ to itself and is a contraction mapping. Clearly, $\Phi(v)\in BC(\mathbb{R}_+,Y)$. Moreover,

$$\begin{aligned} t^\gamma\Phi(v)(t) &= t^\gamma \mathrm{e}^{-tA}(u_0-\hat{u}_0) + t^\gamma\int_0^t \mathrm{e}^{-(t-\tau)A}B(G(v+\hat{u})(\tau)-G(\hat{u})(\tau))\,\mathrm{d}\tau \\ &= t^\gamma \mathrm{e}^{-tA}(u_0-\hat{u}_0) + t^\gamma\int_0^t \mathrm{e}^{-\tau A}B(G(v(t-\tau)+\hat{u}(t-\tau))-G(\hat{u}(t-\tau)))\,\mathrm{d}\tau \\ &= t^\gamma \mathrm{e}^{-tA}(u_0-\hat{u}_0) + t^\gamma\int_0^t F(\tau)\,\mathrm{d}\tau \end{aligned}$$

where

$$F(\tau) := \mathrm{e}^{-\tau A}B(G(v(t-\tau)+\hat{u}(t-\tau))-G(\hat{u}(t-\tau))).$$

Using Assumption 5.1 (iii) we have

$$\|t^\gamma \mathrm{e}^{-tA}(u_0-\hat{u}_0)\|_Q = t^\gamma\|\mathrm{e}^{-tA}(u_0-\hat{u}_0)\|_Q \leqslant C\|u_0-\hat{u}_0\|_Y. \tag{5.3}$$

For $\psi\in(Q_1,Q_2)_{(\tilde{\theta},1)}$ we have

$$\Big|\Big\langle \int_0^t F(\tau)\,\mathrm{d}\tau,\psi\Big\rangle\Big| \leqslant \int_0^t |\langle F(\tau),\psi\rangle|\,\mathrm{d}\tau \leqslant \int_0^{t/2}|\langle F(\tau),\psi\rangle|\,\mathrm{d}\tau + \int_{t/2}^t |\langle F(\tau),\psi\rangle|\,\mathrm{d}\tau. \tag{5.4}$$

From Assumption 5.1 and using the interpolation theorem in the same way as in Lemma 3.3 we can establish the inequality

$$\int_0^\infty \|B'\mathrm{e}^{-\tau A'}\psi\|_{T'}\,\mathrm{d}\tau \leqslant \widetilde{N}\|\psi\|_{(Q_1,Q_2)_{(\tilde{\theta},1)}}.$$

Now, the first integral in (5.4) can be estimated as

$$\begin{aligned} \int_0^{t/2}|\langle F(\tau),\psi\rangle|\,\mathrm{d}\tau &\leqslant \int_0^{t/2}\|G(v(t-\tau)+\hat{u}(t-\tau))-G(\hat{u}(t-\tau))\|_T\|B'\mathrm{e}^{-\tau A'}\psi\|_{T'}\,\mathrm{d}\tau \\ &\leqslant \int_0^{t/2}(\kappa+\|v\|_M+2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v(t-\tau)\|_Q\|B'\mathrm{e}^{-\tau A'}\psi\|_{T'}\,\mathrm{d}\tau \\ &\leqslant \Big(\frac{t}{2}\Big)^{-\gamma}(\kappa+\|v\|_M+2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M\int_0^\infty \|B'\mathrm{e}^{-\tau A'}\psi\|_{T'}\,\mathrm{d}\tau \\ &\leqslant \widetilde{N}2^\gamma t^{-\gamma}(\kappa+\|v\|_M+2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M\|\psi\|_{(Q_1,Q_2)_{(\tilde{\theta},1)}}. \end{aligned}$$

From Assumption 3.2 we have $\|B'\mathrm{e}^{-\tau A'}\psi\|_{T'} < C'\tau^{-1}\|\psi\|_{(Q_1,Q_2)_{\theta,1}}$, $C>0$ by interpolation. Therefore, the second integral in (5.4) can be estimated as

$$
\begin{aligned}
(5.5)\quad & \int_{t/2}^{t} |\langle F(\tau),\psi\rangle| \,\mathrm{d}\tau \\
&\leqslant \int_{t/2}^{t} \|G(v(t-\tau)+\hat{u}(t-\tau)) - G(\hat{u}(t-\tau))\|_T \|B'\mathrm{e}^{-\tau A'}\psi\|_{T'} \,\mathrm{d}\tau \\
&\leqslant (\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M \int_{t/2}^{t} (t-\tau)^{-\gamma}\|B'\mathrm{e}^{-\tau A'}\psi\|_{T'} \,\mathrm{d}\tau \\
&\leqslant C'(\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M \Big(\int_{t/2}^{t} (t-\tau)^{-\gamma}\tau^{-1} \,\mathrm{d}\tau\Big) \|\psi\|_{(Q_1,Q_2)_{(\tilde{\theta},1)}} \\
&\leqslant C'(\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M \frac{2}{t}\int_{t/2}^{t} (t-\tau)^{-\gamma} \,\mathrm{d}\tau \|\psi\|_{(Q_1,Q_2)_{(\tilde{\theta},1)}} \\
&\leqslant \frac{C'2^{\gamma}}{1-\gamma} t^{-\gamma}(\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M \|\psi\|_{(Q_1,Q_2)_{(\tilde{\theta},1)}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(5.6)\quad & \Big\|\int_0^t \mathrm{e}^{-\tau A} B(G(v(t-\tau)+\hat{u}(t-\tau)) - G(\hat{u}(t-\tau)))\,\mathrm{d}\tau\Big\|_Q \\
&\qquad\qquad \leqslant \Big(\frac{C'}{1-\gamma} + \widetilde{N}\Big) 2^{\gamma} t^{-\gamma}(\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M
\end{aligned}
$$

for all $t>0$. Combining (5.3) and (5.6) we obtain that

$$
\|\Phi(v)\|_M \leqslant C\|u_0 - \hat{u}_0\|_Y + D'(\kappa + \|v\|_M + 2\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v\|_M
$$

for $D' = (C'/(1-\gamma) + \widetilde{N})2^{\gamma} > 0$. Therefore, if $\|u_0 - \hat{u}_0\|_Y$ and $\|\hat{u}\|_{BC(\mathbb{R},Y)}$ are small enough, the mapping Φ maps the ball B_{ϱ} with a small enough radius into itself.

Similarly as above we have the estimate

$$
\|\Phi(v_1) - \Phi(v_2)\|_M \leqslant D'(2\kappa + \|v_1\|_M + \|v_2\|_M + 4\|\hat{u}\|_{BC(\mathbb{R},Y)})\|v_1 - v_2\|_M,
$$

hence, Φ is a contraction mapping. As the fixed point of Φ, the function $v = u - \hat{u}$ belongs to M. Inequality (5.1) hence, follows and we obtain the stability of the small solution $\hat{u}$. The proof is completed. □

6. Applications

6.1. Navier-Stokes equations on exterior domains. We consider the Navier-Stokes equations on exterior domains $\Omega \subset \mathbb{R}^3$ with C^3-boundary $\partial\Omega$:

$$\begin{cases} u_t + (u\nabla)u - \Delta u + \nabla p = \operatorname{div} F & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{in } \partial\Omega \times \mathbb{R}, \\ \nabla u = 0 & \text{in } \Omega \times \mathbb{R}. \end{cases} \tag{6.1}$$

Applying the Helmholtz projection $\mathbb{P}$ we obtain the equation

$$u_t + Au = \mathbb{P}\operatorname{div}(-u \otimes u + F) \quad \text{in } \Omega \times \mathbb{R}, \tag{6.2}$$

where $A := -\mathbb{P}\Delta$ denotes the Stokes operator on $L^3_{\sigma,\omega}(\Omega)$. The $L^{r,q} - L^{p,q}$ smoothing properties of the Stokes semigroup on an exterior domain with C^3-boundary are well-known (see, e.g., [26], Theorem 2.2 and [7], Proposition 6.6). Therefore, we can see that Assumptions 3.2 and 5.1 are fulfilled for all $r > 3$ and

$$B = \mathbb{P}\operatorname{div}, \quad X = L^{3/2}_{\omega}(\Omega)^{3\times 3}, \quad Y = L^3_{\sigma,\omega}(\Omega), \quad Q = L^r_{\sigma,\omega}(\Omega), \quad T = L^{(r+3)/3r}_{\sigma,\omega}(\Omega).$$

Then, for a second-order tensor $F \in BC(\mathbb{R}, X)$, we consider the linear problem

$$u_t + Au = \mathbb{P}\operatorname{div} F(t) \quad \text{on } \Omega \times \mathbb{R} \tag{6.3}$$

and the semilinear problem

$$u_t + Au = \mathbb{P}\operatorname{div} G(u)(t) \quad \text{on } \Omega \times \mathbb{R}, \tag{6.4}$$

where $G(u)(t) = -u(t) \otimes u(t) + F(t)$.

By using Hölder's inequality, we have the estimate

$$\begin{aligned} \|G(u) - G(v)\|_{BC(\mathbb{R},X)} &\leqslant \|u \otimes u - v \otimes v\|_{BC(\mathbb{R},X)} \\ &\leqslant \|(u - v) \otimes u\|_{BC(\mathbb{R},X)} + \|v \otimes (u - v)\|_{BC(\mathbb{R},X)} \\ &\leqslant (\|u\|_{BC(\mathbb{R},Y)} + \|v\|_{BC(\mathbb{R},Y)})\|u - v\|_{BC(\mathbb{R},Y)}. \end{aligned}$$

Furthermore,

$$\|G(0)\|_{L^\infty(\mathbb{R},X)} = \|F\|_{L^\infty(\mathbb{R},X)}.$$

This shows that for $\|F\|_{L^\infty(\mathbb{R},X)} \leqslant K/\widetilde{L} - K^2$ we obtain that $G(\cdot)$ satisfies Assumption 4.1 with $\kappa = 0$. Next, we have

$$\begin{aligned} \|G(u) - G(v)\|_{BC(\mathbb{R},T)} &= \|u \otimes u - v \otimes v\|_{BC(\mathbb{R},T)} \\ &\leqslant \|(u - v) \otimes u\|_{BC(\mathbb{R},T)} + \|v \otimes (u - v)\|_{BC(\mathbb{R},T)} \\ &\leqslant (\|u\|_{BC(\mathbb{R},Y)} + \|v\|_{BC(\mathbb{R},Y)})\|u - v\|_{BC(\mathbb{R},Q)}. \end{aligned}$$

Hence, $G(\cdot)$ satisfies the condition for G in Assumption 5.1.

Considering the solution operator

$$S(F)(t) := \int_{-\infty}^{t} \mathrm{e}^{-(t-s)A}\mathbb{P}\,\mathrm{div}\,F(s)\,\mathrm{d}s, \quad t\in\mathbb{R},$$

we have

$$\begin{aligned}\|S(G(u))\|_{BC(\mathbb{R},Y)} &\leqslant \widetilde{L}(\|u\otimes u\|_{BC(\mathbb{R},X)} + \|F\|_{BC(\mathbb{R},X)})\\ &\leqslant \widetilde{L}\Big(K^2 + \frac{K}{\widetilde{L}} - K^2\Big) = K \quad \text{for } \|u\|_{BC(\mathbb{R},X)} < K.\end{aligned}$$

Applying Theorems 4.2 and 5.2 with the 'connection' operator

$$\begin{aligned}&B = \mathbb{P}\,\mathrm{div}, && X = L^{3/2}_{\omega}(\Omega)^{3\times 3}, && Y = L^{3}_{\sigma,\omega}(\Omega),\\ &Q = L^{r}_{\sigma,\omega}(\Omega), && T = L^{(r+3)/3r}_{\sigma,\omega}(\Omega), && \gamma = \frac{1}{2} - \frac{3}{2r},\end{aligned}$$

we obtain the following results on the existence and polynomial stability of the WPAA mild solutions of the Navier-Stokes equations on exterior domains.

Theorem 6.1. *Let $F\in BC(\mathbb{R}, L^{3/2}_{\omega}(\Omega)^{3\times 3})$.*

(a) *If F is weighted pseudo almost automorphic function, then there exist a WPAA mild solution $\hat{u}\in WPAA(L^{3}_{\sigma,\omega}(\Omega),\varrho)$ of (6.3) and a constant $\widetilde{L}>0$ such that*

$$\|\hat{u}\|_{BC(\mathbb{R},Y)} \leqslant \widetilde{L}\|F\|_{BC(\mathbb{R},X)}.$$

(b) *If $F\in WPAA(L^{3/2}_{\omega}(\Omega)^{3\times 3},\varrho)$ satisfies*

$$\|F\|_{L^{\infty}(\mathbb{R},X)} \leqslant \frac{K}{\widetilde{L}} - K^2$$

for positive constants K, $\widetilde{L}$ satisfying $\widetilde{L}K<1$, then the equation (6.4) admits a unique mild solution $\hat{u}\in WPAA(Y = L^{3}_{\sigma,\omega}(\Omega),\varrho)$ such that $\|\hat{u}\|_{BC(\mathbb{R},X)} < K$.

(c) *The small WPAA solution $\hat{u}$ of (6.4) is stable in the sense that for any other bounded mild solution $u\in BC(\mathbb{R}; L^{3}_{\sigma,\omega}(\Omega))$ of (6.4), if the $\|u(0)-\hat{u}(0)\|_{Y}$ is small enough, then we have*

$$\|u(t)-\hat{u}(t)\|_{r,\omega} \leqslant Ct^{-(1/2-3/(2r))}$$

for $r>3$ and $t>0$.

An example of a driving force F which is WPAA can be taken as follows: Let $H\in BC(\mathbb{R}, L^{3/2}_{\omega}(\Omega)^{3\times 3})$, then a (second-order) tensor function of the form

$$F(t) = \Big[\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big) + \mathrm{e}^{-t|\alpha|}\Big]H(t)$$

(with $\alpha\neq 0$) is a WPAA function.

We would like to remark that, since almost periodic functions or (pseudo) almost automorphic functions are also WPAA functions, our results cover previous results obtained for almost periodic or (pseudo) almost automorphic functions in [13], [14], [15], [22], [23], [26]. Moreover, our results pave the way for complicated driving forces to come into the play.

6.2. Navier-Stokes flows around rotating obstacles. In this subsection we consider the Navier-Stokes flows around a rotating obstacle in $\mathbb{R}^3$ described by the equations

$$(6.5)\qquad \begin{cases} u_t + (u\nabla)u - \Delta u - (\zeta \times x)\nabla u + \zeta \times u + \nabla p = \operatorname{div} F & \text{in } \Omega \times (0,\infty), \\ \nabla u = 0 & \text{in } \Omega \times (0,\infty), \\ u = \zeta \times x & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $\zeta = (0, 0, a)$ is a constant vector representing the angular velocity of the rotation around the x_3-axis of an obstacle $D \subset \mathbb{R}^3$ with the complement $\Omega = \mathbb{R}^3 \setminus D$. As in [21], to handle (6.5) we consider the stationary problem

$$(6.6)\qquad \begin{cases} (\nu\nabla)\nu - \Delta\nu - (\zeta \times x)\nabla\nu + \zeta \times \nu + \nabla p = 0 & \text{in } \Omega \times (0,\infty), \\ \nabla\nu = 0 & \text{in } \Omega, \\ \nu = \zeta \times x & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} \nu(x) = 0. \end{cases}$$

Next, by putting $u = z + \nu$ where ν is the solution of (6.6) (the existence of ν was proved in [11]), we obtain that u fulfills (6.5) if and only if z satisfies

$$(6.7)\qquad \begin{cases} z_t + (z\nabla)z - \Delta z - (\zeta \times x)\nabla z + \zeta \times z & \\ \qquad +(\nu\nabla)z + (z\nabla)\nu + \nabla p = \operatorname{div} F & \text{in } \Omega \times (0,\infty), \\ \nabla z = 0 & \text{in } \Omega \times (0,\infty), \\ z = 0 & \text{on } \partial\Omega \times (0,\infty), \\ z|_{t=0} = u_0 - \nu & \text{in } \Omega, \\ \lim_{|x|\to\infty} z = 0. \end{cases}$$

We define the operator A by

$$\begin{aligned} D(A) &:= \{u \in L^r_\sigma(\Omega) \cap W^{2,r}(\Omega) \colon\ u|_{\partial\Omega} = 0 \text{ and } (\zeta \times x)\nabla u \in L^r(\Omega)\}, \\ Au &:= -\mathbb{P}[\Delta u + (w \times x)\nabla u - \zeta \times u] \quad \text{for } u \in D(A). \end{aligned}$$

Applying the Helmholtz projection to this equation we obtain

$$(6.8)\qquad \begin{cases} z_t + Az = \mathbb{P}\operatorname{div}(-z\otimes z - \nu\otimes z - z\otimes \nu + F), \\ u|_{t=0} = z_0 \in L^3_{\sigma,\omega}(\Omega). \end{cases}$$

We set $G(z) := -z\otimes z - \nu\otimes z - z\otimes\nu + F$. Putting $X := L^{3/2}_\omega(\Omega)^{3\times 3}$ and $Y = L^3_{\sigma,\omega}(\Omega)$, by Hölder's inequality we have

$$\|G(z_1) - G(z_2)\|_{BC(\mathbb{R},X)} \leqslant (\|z_1\|_{\infty,d,\omega} + \|z_2\|_{\infty,d,\omega} + 2C|\zeta|)\|z_1 - z_2\|_{\infty,d,\omega}.$$

Furthermore,

$$\|G(0)\|_{L^\infty(\mathbb{R},X)} = \|F\|_{L^\infty(\mathbb{R},X)}.$$

Therefore, for $\|F\|_{L^\infty(\mathbb{R},X)} \leqslant K/\widetilde{L} - (2C|\zeta| + K)K$ we have that G satisfies Assumption 4.1 with $\kappa = 2C|\zeta|$. Similarly, G satisfies the corresponding condition in Assumption 5.1. Next, the solution operator

$$S(F)(t) := \int_{-\infty}^{t} \mathrm{e}^{-(t-s)A}\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s, \quad t\in\mathbb{R},$$

satisfies

$$\begin{aligned} &\|S(G(z))\|_{BC(\mathbb{R},Y)} \\ &\qquad \leqslant \widetilde{L}(\|z\otimes z\|_{BC(\mathbb{R},X)} + \|\nu\otimes z\|_{BC(\mathbb{R},X)} + \|z\otimes\nu\|_{BC(\mathbb{R},X)} + \|F\|_{BC(\mathbb{R},X)}) \\ &\qquad \leqslant \widetilde{L}(K^2 + 2KC|\zeta| + \|F\|_{BC(\mathbb{R},X)}) < K \end{aligned}$$

for $K\widetilde{L} < 1$ and $\|F\|_{BC(\mathbb{R},X)} < K/\widetilde{L} - K^2 - 2KC|\zeta|$.

The $L^{r,q} - L^{p,q}$ estimates are valid for $(\mathrm{e}^{-tA})_{t\in\mathbb{R}_+}$, see [16]. Then the conditions in Assumptions 3.2 and 5.1 are valid with

$$\begin{aligned} &B = \mathbb{P}\operatorname{div}, && X = L^{3/2}_\omega(\Omega)^{3\times 3}, && Y = L^3_{\sigma,\omega}(\Omega), \\ &Q = L^r_{\sigma,\omega}(\Omega), && T = L^{(r+3)/3r}_{\sigma,\omega}(\Omega), && \gamma = \frac{1}{2} - \frac{3}{2r}. \end{aligned}$$

Applying Theorems 4.2 and 5.2, we obtain the following results on the existence and polynomial stability of the WPAA mild solutions to the Navier-Stokes equations around rotating obstacles in $\mathbb{R}^3$.

Theorem 6.2. *Let $F \in BC(\mathbb{R}, L^{3/2}_\omega(\Omega)^{3\times 3})$.*

(a) *If F is a WPAA function, then there exist a WPAA mild solution $\hat{u} \in WPAA(L^3_{\sigma,\omega}(\Omega), \varrho)$ of (6.3) and a constant $\widetilde{L} > 0$ such that*

$$\|\hat{u}\|_{BC(\mathbb{R},Y)} \leqslant \widetilde{L}\|F\|_{BC(\mathbb{R},X)}.$$

(b) *If* $F \in WPAA(L^{3/2}_\omega(\Omega)^{3\times 3}, \varrho)$ *satisfies*

$$\|F\|_{L^\infty(\mathbb{R},X)} < \frac{K}{\widetilde{L}} - K^2 - 2KC|\zeta|$$

for the constants $K, \widetilde{L}$ *and* $|\zeta|$ *fulfilling* $\widetilde{L}(K+2C|\zeta|) < 1$, *then the equation* (6.4) *admits a unique mild solution* $\hat{u} \in WPAA(L^3_{\sigma,\omega}(\Omega), \varrho)$ *such that* $\|\hat{u}\| < K$. *Moreover, this small WPAA solution* $\hat{u}$ *is stable in the sense that for any other bounded solution* $u \in BC(\mathbb{R}; L^3_{\sigma,\omega}(\Omega))$ *to* (6.4) *if* $\|u(0)-\hat{u}(0)\|_Y$ *is small enough, then we have*

$$\|u(t) - \hat{u}(t)\|_{r,\omega} \leqslant Ct^{-(1/2-3/(2r))}$$

for $r > 3$ *and* $t > 0$.

6.3. Navier-Stokes equations in Besov spaces. We now consider the Navier-Stokes equations on $\mathbb{R}^d$ in the framework of Besov spaces:

$$\begin{cases} u_t + (u\nabla)u - \Delta u + \nabla p = \operatorname{div} F & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ \nabla u = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}. \end{cases} \tag{6.9}$$

From [1], Lemmata 2.3, 2.4 we have the smoothing estimates.

Lemma 6.3. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ *and* $\sigma, \tau \in \mathbb{R}$ *such that* $\sigma \geqslant \tau$. *Then there is a* $C > 0$ *such that*

$$\begin{aligned} \|\mathrm{e}^{tA} f\|_{\dot{B}^\tau_{p,q}(\mathbb{R}^d)} &\leqslant Ct^{-(\tau-\sigma)/2}\|f\|_{\dot{B}^\sigma_{p,q}}, \\ \|\nabla \mathrm{e}^{tA} f\|_{\dot{B}^\tau_{p,q}}(\mathbb{R}^d) &\leqslant Ct^{-(1+\tau-\sigma)/2}\|f\|_{\dot{B}^\sigma_{p,q}}(\mathbb{R}^d). \end{aligned} \tag{6.10}$$

By [1], Proposition 2.20 we have the following embedding result: Let $s \in R$, $q \in [1,\infty]$ and p_1, $p_2 \in [1,\infty]$ such that $p_1 \leqslant p_2$. Then there is a constant $C > 0$ such that

$$\|u\|_{\dot{B}^{s-d(1/p_1-1/p_2)}_{p_2,q}} \leqslant C\|u\|_{\dot{B}^s_{p_1,q}}.$$

Moreover, from [15] we have the following lemma.

Lemma 6.4. *Let* $p \in [2, d)$ *and* $s = d/p - 1$. *Then there is a constant* $C > 0$ *such that*

$$\|u \otimes v\|_{\dot{B}^{s-1}_{p,\infty}} \leqslant C\|u\|_{\dot{B}^s_{p,\infty}}\|v\|_{\dot{B}^s_{p,\infty}}. \tag{6.11}$$

More generally, for $\sigma \in (1,\infty)$, $u \in \dot{B}^s_{p,\infty}(\mathbb{R}^d)$, $v \in \dot{B}^\sigma_{p,\infty}(\mathbb{R}^d)$ *it holds*

$$\|u \otimes v\|_{\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d)} \leqslant C'\|u\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}\|v\|_{\dot{B}^\sigma_{p,\infty}(\mathbb{R}^d)}. \tag{6.12}$$

Using the Helmholtz projection, the equation (6.9) may be rewritten as

$$\begin{cases} u'(t) + Au(t) = \mathbb{P}G(u), & t \in \mathbb{R}_+, \\ u(0) = u_0, \end{cases} \tag{6.13}$$

where $A := -\mathbb{P}\Delta$ and $G(u) := \operatorname{div}(u \otimes u) + F$. As previously, by a *mild solution* of (6.13) we mean a function u satisfying the integral equation

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-\tau)A}\mathbb{P}\operatorname{div} G(u)(\tau)\,\mathrm{d}\tau, \quad t > 0.$$

Put $X = \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d)^{d\times d}$, $Y = \dot{B}^{s}_{p,\infty}(\mathbb{R}^d)$. Then, for $s = d/p - 1$ from Lemma 6.4 and the fact that the operator div maps $\dot{B}^{s}_{p,q}$ continuously to $\dot{B}^{s-1}_{p,q}$ it follows that

$$\|G(u_1) - G(u_2)\|_{BC(\mathbb{R},X)} \leqslant C(d,p)(\|u_1\|_{BC(\mathbb{R},Y)} + \|u_2\|_{BC(\mathbb{R},Y)})\|u_1 - u_2\|_{BC(\mathbb{R},Y)}.$$

Moreover,

$$\|G(0)\|_{BC(\mathbb{R},X)} = \|F\|_{BC(\mathbb{R},X)}.$$

Thus, for $\|F\|_{BC(\mathbb{R},X)} \leqslant K/\widetilde{L} - C(d,p)K^2$ we obtain that G satisfies Assumption 4.1 with $\kappa = 0$. Similarly, putting $T = \dot{B}^{r-2}_{p,\infty}(\mathbb{R}^d)$, $Q = \dot{B}^{r}_{p,\infty(\mathbb{R}^d)}$ and using again Lemma 6.4 we have

$$\|G(u_1) - G(u_2)\|_{BC(\mathbb{R},T)} \leqslant C'(d,p)(\|u_1\|_{BC(\mathbb{R},Y)} + \|u_2\|_{BC(\mathbb{R},Y)})\|u_1 - u_2\|_{BC(\mathbb{R},Q)}.$$

Hence, the corresponding condition on G in Assumption 5.1 is satisfied.

As previously, the solution operator

$$S(F)(t) := \int_{-\infty}^t \mathrm{e}^{-(t-s)A}\mathbb{P}\operatorname{div} F(s)\,\mathrm{d}s, \quad t \in \mathbb{R},$$

fulfills

$$\begin{aligned} \|S(G(u))\|_{BC(\mathbb{R},Y)} &\leqslant \widetilde{L}(\|u \otimes u\|_{BC(\mathbb{R},X)} + \|F\|_{BC(\mathbb{R},X)}) \\ &\leqslant \widetilde{L}(C(d,p)K^2 + \|F\|_{BC(\mathbb{R},X)}) < K \end{aligned}$$

for $K\widetilde{L}C(d,p) < 1$ and $\|F\|_{BC(\mathbb{R},X)} < K/\widetilde{L} - C(d,p)K^2$.

Applying Theorems 4.2 and 5.2 for

$$B = \mathbb{P}, \quad X = \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d)^{d\times d}, \quad Y = \dot{B}^{s}_{p,\infty}(\mathbb{R}^d), \quad Q = \dot{B}^{r}_{p,\infty}(\mathbb{R}^d), \quad T = \dot{B}^{r-2}_{p,\infty}(\mathbb{R}^d),$$

we obtain the following result on the boundedness, stability, and weighted pseudo almost automorphy of mild solutions to the equation (6.13).

Theorem 6.5. *Let $F \in BC(\mathbb{R}, \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d)^{d\times d})$.*

(a) *If F is a WPAA function, then there exists a WPAA mild solution $\hat{u} \in WPAA(\dot{B}^{s}_{p,\infty}(\mathbb{R}^d, \varrho)$ to the equation (6.13) such that*

$$\|\hat{u}\|_{BC(\mathbb{R},Y)} \leqslant \widetilde{L}\|F\|_{BC(\mathbb{R},X)} \quad \text{for some constant } \widetilde{L} > 0.$$

(b) *If $F \in WPAA(\dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d)^{d\times d}, \varrho)$ satisfies*

$$\|F\|_{BC(\mathbb{R},X)} < \frac{K}{\widetilde{L}} - C(d,p)K^2$$

for the constants K, $\widetilde{L}$ fulfilling $\widetilde{L}KC(d,p) < 1$, then the equation (6.13) admits a unique mild solution $\hat{u} \in WPAA(\dot{B}^{s}_{p,\infty}(\mathbb{R}^d), \varrho)$ such that $\|\hat{u}\| < K$. Furthermore, this WPAA solution $\hat{u}$ is stable in the sense that for any other bounded solution $u \in BC(\mathbb{R}; \dot{B}^{s}_{p,\infty}(\mathbb{R}^d))$ of (6.13) if $\|u(0) - \hat{u}(0)\|_Y$ is small enough, then

$$\|u(t) - \hat{u}(t)\|_Q \leqslant Ct^{-(1/2-s/(2r))}$$

for $r > s$ and $t > 0$.

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