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UNIMODULAR ROWS OVER LAURENT POLYNOMIAL RINGS

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Abstract. We prove that for any ring $\mathbf{R}$ of Krull dimension not greater than 1 and $n \geqslant 3$, the group $\mathrm{E}_n(\mathbf{R}[X, X^{-1}])$ acts transitively on $\mathrm{Um}_n(\mathbf{R}[X, X^{-1}])$. In particular, we obtain that for any ring $\mathbf{R}$ with Krull dimension not greater than 1, all finitely generated stably free modules over $\mathbf{R}[X, X^{-1}]$ are free. All the obtained results are proved constructively.

Keywords: Quillen-Suslin theorem; stably free module; Hermite ring conjecture; Laurent polynomial ring; constructive mathematics

MSC 2020: 13C10, 19A13, 14Q20, 03F65

1. Introduction

Let us begin by fixing some notations. Recall that for any ring $\mathbf{B}$ and $n \geqslant 1$, an $n \times n$ elementary matrix $E_{i,j}(a)$ over $\mathbf{B}$, where $i \neq j$ and $a \in \mathbf{B}$, is the matrix with 1 on the diagonal, a on position (i, j) and 0 elsewhere, that is, $E_{i,j}(a)$ is the matrix corresponding to the elementary operation $L_i \to L_i + aL_j$. The symbol $\mathrm{E}_n(\mathbf{B})$ denote the subgroup of $\mathrm{SL}_n(\mathbf{B})$ generated by elementary matrices and $\mathrm{Um}_n(\mathbf{B})$ denote the set of unimodular vectors in $\mathbf{B}$ of length n, that is, $\mathrm{Um}_n(\mathbf{B}) = \{{}^{\mathrm{t}}(x_1, \ldots, x_n) \in \mathbf{B}^n \colon \langle x_1, \ldots, x_n \rangle = \mathbf{B}\}$. Given $u, v \in \mathrm{Um}_n(\mathbf{B})$, we write $u \sim_{\mathrm{E}_n(\mathbf{B})} v$ (in short, $u \sim_{\mathrm{E}} v$) if there exists M in $\mathrm{E}_n(\mathbf{B})$ such that $v = Mu$.

In [16], the author has proven that for any ring $\mathbf{R}$ of Krull dimension not greater than 1 and $n \geqslant 3$, the group $\mathrm{E}_n(\mathbf{R}[X])$ acts transitively on $\mathrm{Um}_n(\mathbf{R}[X])$. As a consequence, he obtained that for any ring $\mathbf{R}$ with Krull dimension not greater than 1, all finitely generated stably free modules over $\mathbf{R}[X]$ are free giving a positive answer to the Hermite ring conjecture (see [8], [9]) in dimension one. Our goal in this paper is to establish analogous results over the Laurent polynomial ring $\mathbf{R}[X, X^{-1}]$.

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Some classical facts are reminded in order to give a self-contained paper. All the considered rings are unitary and commutative. The undefined terminology is standard as in [7], [9], and for constructive algebra in [10], [12], [17].

2. An analogue of the lemma of Suslin for Laurent polynomials

Definition 2.1. Let $\mathbf{A}$ be a discrete ring.

(1) If $f \in \mathbf{A}[X, X^{-1}]$, a minimal shifted version of f is $\tilde{f} = X^n f \in \mathbf{A}[X]$, where $n \in \mathbb{Z}$ is minimal possible. For example, a minimal shifted version of $X^{-3} + X + X^2$ is $1 + X^4 + X^5$, a minimal shifted version of $X^2 + X^4$ is $1 + X^2$.

(2) If $f \in \mathbf{A}[X, X^{-1}]$ is a nonzero Laurent polynomial, we denote $\deg(f) = \operatorname{hdeg}(f) - \operatorname{ldeg}(f)$, where $\operatorname{hdeg}(f)$ and $\operatorname{ldeg}(f)$ denote, respectively, the highest and lowest degrees of f.

For example, $\deg(X^{-3} + X + X^2) = 2 - (-3) = 5$. The degree of f can be defined as the (classical) degree of a minimal shifted version of f.

Note that if $\mathbf{A}$ is a nondiscrete ring and $f = \sum\limits_{n=m}^{p} a_n X^n \in \mathbf{A}[X, X^{-1}]$ is a nonzero Laurent polynomial with $p \geqslant m \in \mathbb{Z}$, then m and p are, respectively, the formal lowest and highest degrees of f. We denote formal $\deg(f) =$ formal $\operatorname{hdeg}(f) -$ formal $\operatorname{ldeg}(f)$.

A minimal formal shifted version of f is $\tilde{f} = \sum\limits_{k=0}^{p-m} a_{m+k} X^k \in \mathbf{A}[X]$. If f is given as zero, its formal degree is -1.

In the case of a Laurent polynomial which is given as doubly monic, formal versions become useless.

Definition 2.2. If the ring $\mathbf{A}$ is nontrivial, an element $f \in \mathbf{A}[X, X^{-1}]$ is called a *doubly monic Laurent polynomial* if the coefficients of the highest degree and the lowest degree terms are units ($\in \mathbf{A}^{\times}$).

The following theorem is a generalization of a famous lemma of Suslin (see [14], Lemma 2.3) to Laurent polynomial rings following the method explained in [15]. The constructive form of Suslin lemma (by Yengui) can be found in [10], Theorem XV-6.1 and [17], Theorem 57.

Theorem 2.1. *Let $\mathbf{A}$ be a commutative ring. If $\langle v_1(X), \ldots, v_n(X) \rangle = \mathbf{A}[X, X^{-1}]$, where v_1 is doubly monic and $n \geqslant 3$, then there exist $\gamma_1, \ldots, \gamma_s \in \mathrm{E}_{n-1}(\mathbf{A}[X])$ such that:*

$$\langle \operatorname{Res}(\widetilde{v}_1, e_1.\gamma_1{}^{\mathrm{t}}(\widetilde{v}_2, \ldots, \widetilde{v}_n)), \ldots, \operatorname{Res}(\widetilde{v}_1, e_1.\gamma_s{}^{\mathrm{t}}(\widetilde{v}_2, \ldots, \widetilde{v}_n)) \rangle = \mathbf{A}.$$

In particular, $1 \in \langle \widetilde{v}_1, \ldots, \widetilde{v}_n \rangle$ in $\mathbf{A}[X]$.

Here $e_1 \cdot x$, where x is a column vector, stands for the first coordinate of x, and $\widetilde{v}_i$ is a shifted version of v_i.

Lemma 2.1 ([1], Theorem 7). *Let* $\mathbf{B}$ *be a ring. Let* $u, v \in \mathbf{B}[X]$ *with* u *doubly monic. Then*

$$\langle u, v\rangle = \langle 1\rangle \text{ in } \mathbf{B}[X, X^{-1}] \Leftrightarrow \langle \mathrm{Res}_X(u, v)\rangle = \langle 1\rangle \text{ in } \mathbf{B}.$$

In the case the base ring $\mathbf{A}$ contains an infinite field, we have the following more precise and simpler formulation of Theorem 2.1. It is worth pointing out that this result first appeared in [1] but with a nonconstructive proof. Moreover, the constructive proof we give below inspired by the proof of Theorem 1 of [11], after a shift that transforms Laurent polynomial in usual polynomials, enabled us to give more precise bounds than Proposition 9 of [1] on the degrees of the computed resultants.

Theorem 2.2. *Let* $\mathbf{A}$ *be a ring,* $v_1, \ldots, v_n, u_1, \ldots, v_n \in \mathbf{A}[X, X^{-1}]$ *such that* $\sum\limits_{i=1}^{n} u_i v_i = 1$*,* v_1 *doubly monic, and* $n \geqslant 3$*. Denote* $l = \deg v_1$*,* $s = (n-2)l + 1$*, and suppose that* $\mathbf{A}$ *contains a set* $E = \{y_1, \ldots, y_s\}$ *such that* $y_i - y_j$ *is invertible for each* $i \neq j$*. For each* $1 \leqslant r \leqslant n$ *and* $1 \leqslant i \leqslant s$*, letting* $\widetilde{v}_r$ *be a minimal shifted version of* v_r *and denoting* $r_i = \mathrm{Res}_X(\widetilde{v}_1, \widetilde{v}_2 + y_i\widetilde{v}_3 + \ldots + y_i^{n-2}\widetilde{v}_n)$*, then* $\langle r_1, \ldots, r_s\rangle = \mathbf{A}$*, that is, there exist* $\alpha_1, \ldots, \alpha_s \in \mathbf{A}$ *such that* $\alpha_1 r_1 + \ldots + \alpha_s r_s = 1$*. In particular,* $1 \in \langle \widetilde{v}_1, \ldots, \widetilde{v}_n\rangle$ *in* $\mathbf{A}[X]$*.*

Furthermore, supposing that $\mathbf{A}$ *is a polynomial ring in a finite number of variables over a basic ring* $\mathbf{B}$ *and that* $\max\limits_{1\leqslant i\leqslant n}\{\deg u_i\} \leqslant D$*,* $1 + \max\limits_{1\leqslant i\leqslant n}\{\deg v_i\} \leqslant d$ *(where* $d \geqslant 2$*), then for each* $1 \leqslant i \leqslant s$*,* $\deg(\alpha_i) \leqslant \frac{1}{16}d^4(d+D+2)^2$ *and* $\deg(\alpha_i r_i) \leqslant \frac{1}{16}d^4(d+D+3)^2$ *(here, by degree we mean the total degree).*

Proof. Let us denote $w_i = \widetilde{v}_2 + y_i\widetilde{v}_3 + \ldots + y_i^{n-2}\widetilde{v}_n$, $r_i := \mathrm{Res}_X(\widetilde{v}_1, w_i)$, $1 \leqslant i \leqslant s = (n-2)d + 1$, $l := d + 1$, where $d = \deg \widetilde{v}_1$, and suppose that $1 \in \langle \widetilde{v}_1, \ldots, \widetilde{v}_n\rangle$. Let

$$\begin{aligned}
Z_1 = \ \ldots &= Z_{n-2} = z_1,\\
Z_{n-1} = \ \ldots &= Z_{2n-4} = z_2,\\
&\vdots\\
Z_{(n-2)(k-1)+1} = \ \ldots &= Z_{(n-2)k} = z_k,\\
&\vdots\\
Z_{(n-2)(d-1)+1} = \ \ldots &= Z_{(n-2)d} = z_d,\\
Z_{(n-2)d+1} &= z_{d+1},
\end{aligned}$$

be an enumeration of l indeterminates over $\mathbf{A}$ with $n-2$ repetitions except the last one which is repeated once. Let us denote

$$I = \langle \widetilde{v}_1(Z_i), w_i(Z_i)\colon\ 1 \leqslant i \leqslant s\rangle.$$

First, we deduce that $1 = 0$ in $\mathbf{A}[Z_1^{\pm 1}, \dots, Z_s^{\pm 1}]/I$, that is, $1 \in I$. Then for $0 \leqslant k \leqslant s$, denoting $I_k = \langle \widetilde{v}_1(Z_i), w_i(Z_i) \colon 1 \leqslant i \leqslant k \rangle$, $J_k = I_k + \langle r_i \colon k < i \leqslant s \rangle$ and $\mathbf{A}_k = \mathbf{A}[Z_1^{\pm 1}, \dots, Z_k^{\pm 1}]/I_k$ and using Lemma 2.1 we get by induction on k from s to 0 that $1 \in J_k$. So $1 \in J_0 = \langle r_s, \dots, r_1 \rangle$.

For the degree bounds, by an argument similar to that of the proof of Theorem 1 of [11], we deduce that for each $1 \leqslant i \leqslant s$, $\deg(\alpha_i) \leqslant \frac{1}{4} d^4 (\frac{1}{2}(d + D) + 1)^2$. Moreover, since $\deg r_i \leqslant d^2$, we get

$$\deg(\alpha_i r_i) \leqslant \frac{d^4}{4} \Big(\frac{d + D}{2} + 1 \Big)^2 + d^2 \leqslant \Big(\frac{d^2}{4} (d + D + 3) \Big)^2.$$

□

3. The main result

We begin by extending Lemma 2 of [16] to Laurent polynomial rings.

Lemma 3.1. *Let $\mathbf{R}$ be a ring and I an ideal of $\mathbf{R}[X, X^{-1}]$ containing a doubly monic polynomial. If J is an ideal of $\mathbf{R}$ such that $I + J[X, X^{-1}] = \mathbf{R}[X, X^{-1}]$, then $(I \cap \mathbf{R}) + J = \mathbf{R}$.*

P r o o f. Let us denote by f a doubly monic polynomial in I. Without loss of generality, we assume that $f \in \mathbf{R}[X]$. As $I + J[X, X^{-1}] = \mathbf{R}[X, X^{-1}]$, there exist $g \in I$ and $h \in J[X, X^{-1}]$ such that $g + h = 1$. For some $l \in \mathbb{Z}$, we can write $X^l \widetilde{g} + h = 1$. Since we have $\langle \overline{f}, \overline{\widetilde{g}}) \rangle = (\mathbf{R}/J)[X, X^{-1}]$, we obtain by Lemma 2.1 that $\mathrm{Res}(\overline{f}, (\overline{\widetilde{g}}) \in (\mathbf{R}/J)^{\times}$. As f is monic, we have $\mathrm{Res}(\overline{f}, \overline{\widetilde{g}}) = \overline{\mathrm{Res}(f, \widetilde{g})}$ and thus $\langle \mathrm{Res}(f, \widetilde{g}) \rangle + J = \mathbf{R}$. The desired conclusion follows from the fact that $\mathrm{Res}(f, \widetilde{g}) \in I \cap \mathbf{R}$. □

Lemma 3.2 ([13], Lemma 3)**.** *Let $\mathbf{R}$ be a ring and $f(X) \in \mathbf{R}[X]$ of degree $n > 0$ such that $f(0) \in \mathbf{R}^{\times}$. Then for any $g(X) \in \mathbf{R}[X]$ and $k \geqslant \deg g - \deg f + 1$, there exists $h_k(X) \in \mathbf{R}[X]$ of degree less than n such that $g(X) \equiv X^k h_k(X) \bmod f\mathbf{R}[X]$.*

Lemma 3.3 ([13], Lemma 1)**.** *Let $\mathbf{R}$ be a ring, and*

$$^{\mathrm{t}}(x_0, \dots, x_r) \in \mathrm{Um}_{r+1}(\mathbf{R}), \quad r \geqslant 2,$$

and let t be an element of $\mathbf{R}$ which is invertible $\bmod \langle x_0, \dots, x_{r-2} \rangle$. Then there exists $E \in \mathrm{E}_{r+1}(\mathbf{R})$ such that $E\, ^{\mathrm{t}}(x_0, \dots, x_r) = {}^{\mathrm{t}}(x_0, \dots, x_{r-1}, t^2 x_r)$.

Now we are reaching a crucial stage in our objective to prove that for any ring $\mathbf{R}$ with Krull dimension not greater than 1, all finitely generated stably free modules over $\mathbf{R}[X, X^{-1}]$ are free.

Lemma 3.4 (key lemma). *Let $(v_0(X), v_1(X), \ldots, v_n(X)) \in \mathrm{Um}_{n+1}(\mathbf{R}[X, X^{-1}])$, where $\mathbf{R}$ is a reduced ring of Krull dimension not greater than 1, $n \geqslant 2$. Then*

$$(\widetilde{v}_0(X), \widetilde{v}_1(X), \ldots, \widetilde{v}_n(X)) \sim_{\mathrm{E}} (w_0(X), w_1(X), c_2, \ldots, c_n),$$

where c_i are constants for $i \geqslant 2$, $w_i \in \mathbf{R}[X]$ with $\deg w_1(X) \leqslant 1$.

Proof. By virtue of Lemma 3.3, as X is a unit in $R[X, X^{-1}]$, we can suppose that $v_i \in \mathbf{R}[X]$.

We may also assume $m_0 := \deg v_0 > 0$. Let a and b be, respectively, the highest and the lowest coefficients of v_0. If $a, b \in \mathbf{R}^\times$, then v_0 is a doubly monic polynomial, and as a consequence of Theorem 2.1 and the algorithm given in [1], we obtain

$$^{\mathrm{t}}(v_0, v_1, \ldots, v_n) \sim_E {}^{\mathrm{t}}(1, 0, \ldots, 0).$$

Now suppose that v_0 is not doubly monic. As $\dim(\mathbf{R}/\langle a\rangle) \leqslant 0$ and over any zero-dimensional ring, all unimodular vectors of length not less than 2 are elementarily completable, we can suppose that $^{\mathrm{t}}(v_0, v_1, \ldots, v_n) \equiv {}^{\mathrm{t}}(1, 0, \ldots, 0)$ $(\mathrm{mod}\ a\mathbf{R}[X, X^{-1}]^{n+1})$.

By Lemma 3.2, as $v_0(0) \in \mathbf{R}^\times$, we assume now $v_i = X^{2k} w_i$, where $\deg w_i < m_0$ for $1 \leqslant i \leqslant n$. By Lemma 3.3, we assume $\deg v_i < m_0$.

If $m_0 \leqslant 1$, our claim is established. Assume now that $m_0 \geqslant 2$.

Let $c_1, \ldots, c_{m_0(n-1)}$ be the coefficients of $1, X, \ldots, X^{m_0-1}$ in the polynomials $v_2, \ldots, v_n$, J the ideal generated by the coefficients of $v_2, \ldots, v_n$, and $I = v_0\mathbf{R}[X, X^{-1}] + v_1\mathbf{R}[X, X^{-1}]$. As

$$I + J[X, X^{-1}] = \mathbf{R}[X, X^{-1}],$$

by Lemma 3.1 the ideal generated in $\mathbf{R}_a$ (where $\mathbf{R}_a := S^{-1}\mathbf{R}$, $S := a^{\mathbb{N}}$) by $I \cap \mathbf{R}_a$ and J is $\mathbf{R}_a$. As $n \geqslant 2 > \dim \mathbf{R}_a$, by Lemma 2 of [13] (this is the stable range theorem, see [4] for a constructive proof), there exists $(c'_1, \ldots, c'_{m_0(n-1)}) \equiv (c_1, \ldots, c_{m_0(n-1)})$ $(\mathrm{mod}\ v_0\mathbf{R}[X] + v_1\mathbf{R}[X]) \cap \mathbf{R}$ such that

$$\mathbf{R}_a {c'}_1 + \ldots + \mathbf{R}_a {c'}_{m_0(n-1)} = \mathbf{R}_a.$$

Assume now that

$$\mathbf{R}_a c_1 + \ldots + \mathbf{R}_a c_{m_0(n-1)} = \mathbf{R}_a.$$

By [3], §4, 1 (b), the ideal $v_0\mathbf{R}[X] + v_2\mathbf{R}[X] + \ldots + v_n\mathbf{R}[X]$ contains a polynomial $h(X)$ of degree $m_0 - 1$ which is unitary in $\mathbf{R}_a[X]$. Let $\mathrm{LC}(h) = ua^k$ (the leading coefficient of h), where $u \in \mathbf{R}^\times$. Using Lemma 3.3, we can perform by elementary transformations

$$\begin{aligned} ^{\mathrm{t}}(v_0, v_1, \ldots, v_n) &\to {}^{\mathrm{t}}(v_0, (a^k)^2 v_1, \ldots, v_n) \\ &\to {}^{\mathrm{t}}(v_0, (a^k)^2 v_1 + (1 - a^k u^{-1}\mathrm{LC}(v_1))h, \ldots, v_n). \end{aligned}$$

Since $(a^k)^2v_1+(1-a^ku^{-1}\mathrm{LC}(v_1))h$ is unitary in $\mathbf{R}_a[X]$, we can suppose that v_1 is unitary in $\mathbf{R}_a[X]$, $\deg v_1 = m_0 - 1$. By Lemma 3.3, we can assume that $\deg v_i < \deg v_1$ for each $2 \leqslant i \leqslant n$. By an element of $E_{n+1}(\mathbf{R}[X])$, we can exchange v_0 and $-v_1$. So we can suppose that v_0 is unitary in $\mathbf{R}_a[X]$ and $\deg v_0 = m_0 - 1$. By Lemma 3.3, we can also assume that $\deg v_1 < \deg v_0$. Repeating the argument above, we lower the degree of v_i and we obtain finally a vector of the form ${}^{\mathrm{t}}(v_0, v_1, \ldots, v_n)$ with $\deg v_0 > \mathrm{degv}_1 = 1$ and $v_i \in \mathbf{R}$ for $2 \leqslant i \leqslant n$. □

Recall that the boundary ideal of an element a of a ring $\mathbf{R}$ is the ideal $\mathcal{I}(a)$ of $\mathbf{R}$ generated by a and all the $y \in \mathbf{R}$ such that ay is nilpotent. Moreover, $\dim \mathbf{R} \leqslant d \Leftrightarrow \dim(\mathbf{R}/\mathcal{I}(a)) \leqslant d-1$ for all $a \in \mathbf{R}$ (this defines the Krull dimension recursively initializing with "$\dim \mathbf{R} \leqslant -1 \Leftrightarrow \mathbf{R}$ being trivial"), see [10].

Recall also that for any ring $\mathbf{R}$, the ring $\mathbf{R}\langle X\rangle$ (or $\mathbf{R}(X)$) is the localization of $\mathbf{R}[X]$ at monic polynomials (or primitive polynomials). We have $\mathbf{R}[X] \subseteq \mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$, and the containment $\mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ becomes an equality if and only if $\dim\mathbf{R} \leqslant 0$, see [6] and for a constructive proof see [5].

Theorem 3.1. *Let $\mathbf{R}$ be a ring of dimension not greater than 1, $n \geqslant 2$, and $V = {}^{\mathrm{t}}(v_0, \ldots, v_n) \in \mathrm{Um}_{n+1}(\mathbf{R}[X, X^{-1}])$. Then there exists $\Gamma \in \mathrm{GL}_{n+1}(\mathbf{R}[X, X^{-1}])$ such that $\Gamma V = {}^{\mathrm{t}}(1, 0, \ldots, 0)$.*

P r o o f. Let us recall that $\mathrm{GK}_0\mathbf{A}_{\mathrm{red}} = \mathrm{GK}_0\mathbf{A}$ and $\mathrm{GK}_0\mathbf{B}_{\mathrm{red}} = \mathrm{GK}_0\mathbf{B}$, see [10], Chapter XVI. In particular, if $\mathbf{B} = \mathbf{R}[X, X^{-1}]$, we have $\mathbf{B}_{\mathrm{red}} = \mathbf{R}_{\mathrm{red}}[X, X^{-1}]$ (a Laurent polynomial f is nilpotent if and only if $\tilde{f}$ is nilpotent and by virtue of Lemma II-2.6 of [10] it is nilpotent if and only if all its coefficients are nilpotent). So we can suppose that $\mathbf{R}$ is a reduced ring.

In order to simplify, we will suppose that $\widetilde{v}_i = v_i$. Moreover, by virtue of Lemma 3.4, we can suppose that $v_0 = a$ is constant. Let us consider the ring $\mathbf{T} := \mathbf{R}/\mathcal{I}(a)$. Since $\dim \mathbf{T} \leqslant 0$, we have that $\mathbf{T}\langle X\rangle = \mathbf{T}\langle X, X^{-1}\rangle$ (see [2]), where $\mathbf{T}\langle X, X^{-1}\rangle$ is the localization of $\mathbf{T}[X, X^{-1}]$ at doubly monic polynomials, and $\dim \mathbf{T}\langle X, X^{-1}\rangle \leqslant 0$. By the stable range theorem (see [4] for a constructive proof), there exists $v_1' \equiv v_1 \bmod v_2$ which is invertible in $T\langle X, X^{-1}\rangle$, that is, it divides a doubly monic polynomial in $\mathbf{T}[X, X^{-1}]$. So without loss of generality, we can assume that v_1 divides a doubly monic polynomial in $\mathbf{T}[X, X^{-1}]$, that is, there exists a doubly monic polynomial $u \in \mathbf{R}[X, X^{-1}]$, $w, h_1, h_2, \in \mathbf{R}[X, X^{-1}]$ with ah_2 nilpotent such that

$$wv_1 = u + ah_1 + h_2.$$

This means that $1 \in \langle v_1, a, h_2\rangle$ in the ring $\mathbf{R}\langle X, X^{-1}\rangle$. Since ah_2 is nilpotent, by virtue of Lemma 2.3 of [4], we have $1 \in \langle v_1, a + h_2\rangle$. It follows that there exist $w_1, w_2 \in \mathbf{R}[X, X^{-1}]$ such that $v_1w_1 + (a+h_2)w_2 =: u'$ is a doubly monic polynomial.

Let $d \in \mathbb{N}$ and denote by $u_0, \ldots, u_n \in \mathbf{R}[X, X^{-1}]$ Laurent polynomials such that $u_0v_0 + \ldots + u_nv_n = 1$. Denoting

$$\begin{aligned}\gamma_1 &:= E_{1,2}(h_2u_1)\ldots E_{1,n+1}(h_2u_n),\\ \gamma_2 &:= E_{3,2}((X^d + X^{-d})w_1)E_{3,1}((X^d + X^{-d})w_2),\\ \gamma &:= \gamma_2\gamma_1,\end{aligned}$$

we have

$$\gamma_1 V = {}^{\mathrm{t}}(a + h_2, v_1, \ldots, v_n)$$

and

$$\gamma V = {}^{\mathrm{t}}(a + h_2, v_1, v_2 + (X^d + X^{-d})u', v_3, \ldots, v_n).$$

For sufficiently large d, the third entry of γV becomes a doubly monic polynomial. Thus, as stated in the proof of Lemma 3.4, by virtue of Theorem 2.1 and the algorithm given in [1], we can transform the obtained vector into ${}^{\mathrm{t}}(1, 0, \ldots, 0)$. Note that the shifted vector $\widetilde{V} = {}^{\mathrm{t}}(\widetilde{v}_0, \ldots, \widetilde{v}_n)$ can be obtained from V via multiplication by a diagonal matrix D with suitable powers of X on the diagonal. $\square$

Corollary 3.1. *For any ring* $\mathbf{R}$ *of Krull dimension not greater than* 1, *all finitely generated stably free modules over* $\mathbf{R}[X, X^{-1}]$ *are free.*

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