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Commentationes Mathematicae Universitatis Carolinae, Vol. 63 (2022), No. 2, 169–187

Persistent URL: http://dml.cz/dmlcz/151084

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Abstract. We show how the measure theory of regular compacted-Borel measures defined on the δ-ring of compacted-Borel subsets of a weighted locally compact group (G,ω) provides a compatible framework for defining the corresponding Beurling measure algebra $\mathcal{M}(G,\omega)$, thus filling a gap in the literature.

Keywords: weighted locally compact group; group algebra; measure algebra; Beurling algebra

Classification: 43A10, 22D15, 43A05, 43A20, 43A60, 28C10

Throughout this article, G denotes a locally compact group and $\omega\colon G\to(0,\infty)$ is a continuous *weight function* satisfying

$$\omega(st)\le\omega(s)\omega(t),\quad s,t\in G,\qquad\text{and}\qquad \omega(e_G)=1;$$

the pair (G,ω) is called a *weighted locally compact group.* Let λ denote a fixed Haar measure on G, with respect to which the group algebra $L^1(G)$ and $L^\infty(G)=L^1(G)^*$ are defined in the usual way. The Beurling group algebra, $L^1(G,\omega)$, is composed of all functions f such that ωf belongs to $L^1(G)$, with $\|f\|_{1,\omega}:=\|\omega f\|_1$ and convolution product. If $\mathcal{S}(G)$ is a closed subspace of $L^\infty(G)$, $\psi\in\mathcal{S}(G,\omega^{-1})$ exactly when $\psi/\omega\in\mathcal{S}(G)$; putting $\|\psi\|_{\infty,\omega^{-1}}=\|\psi/\omega\|_\infty$, $\mathcal{S}(G,\omega^{-1})$ is a Banach space and $S\colon\mathcal{S}(G,\omega^{-1})\to\mathcal{S}(G)\colon\psi\mapsto\psi/\omega$ is an isometric linear isomorphism. The Beurling group algebra $L^1(G,\omega)$ has become a classical object of study that has received significant research attention over the years, see the monographs [3], [11], [15] and the references therein; a sample of relevant articles include [6], [5], [7], [8], [17], [18], [20]. When ω is the trivial weight $\omega\equiv 1$ — the "non-weighted case" — $L^1(G,\omega)=L^1(G)$, the study of which is intimately linked with the measure algebra $M(G)$ of complex, regular, Borel measures on G, which contains $L^1(G)$ as a closed ideal.

The above definition of $L^1(G,\omega)$ is valid for any weight ω. As in the non-weighted case, it is desirable to have a Beurling measure algebra $M(G,\omega)$ that shares the same relationship with $L^1(G,\omega)$ that $M(G)$ shares with $L^1(G)$. In

DOI 10.14712/1213-7243.2022.016

This research was partially supported by an NSERC grant.

the literature, $M(G,\omega)$ is usually defined as the collection of all complex regular measures ν defined on $\mathfrak{B}(G)$, the σ-algebra of Borel subsets of G, such that $\int \omega(t)\,\mathrm{d}|\nu|(t) < \infty$, and the identification $M(G,\omega) = C_0(G,\omega^{-1})^*$ through $\langle \nu, \psi\rangle_\omega = \int \psi\,\mathrm{d}\nu$ is required. This implies that the dual map, S^*, of the isometric isomorphism $S\colon C_0(G,\omega^{-1}) \to C_0(G)$ is itself a linear isometric isomorphism of $M(G)$ onto $M(G,\omega)$. Validity of this definition of $M(G,\omega)$ thus requires that for each $\mu \in M(G)$, $\nu = S^*\mu \in M(G,\omega)$ is a complex Borel measure defined on all of $\mathfrak{B}(G)$ — the near-universal requirement of "Borel measures" in abstract harmonic analysis — satisfying

$$(1) \qquad \int \psi\,\mathrm{d}\nu = \langle \nu, \psi\rangle_\omega = \Big\langle \mu, \frac{\psi}{\omega}\Big\rangle = \int \frac{\psi}{\omega}\,\mathrm{d}\mu, \qquad \psi \in C_0(G,\omega^{-1}).$$

However, when ω is not bounded away from zero, it can happen that no such complex measure on $\mathfrak{B}(G)$ exists.

To see this, consider (G,ω) where $G = (\mathbb{Z},+)$ and $\omega(n) = 2^{-n}$, $n \in \mathbb{Z}$, and assume the above definition of $M(G,\omega)$ is sound. Since $\mu_1, \mu_2 \in l^1(\mathbb{Z})^+ = M(\mathbb{Z})^+$ and $\mu = \mu_1 - \mu_2 \in M(\mathbb{Z})$, where

$$\mu_1(n) = \begin{cases} 2^{-n}, & n \in 2\mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \mu_2(n) = \begin{cases} 2^{-n}, & n \in \mathbb{N}\backslash 2\mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

$\nu_1 = S^*(\mu_1)$, $\nu_2 = S^*(\mu_2)$, and $\nu = S^*(\mu) = \nu_1 - \nu_2$ are then required to be complex measures on $\mathfrak{B}(G) = \wp(\mathbb{Z})$ satisfying (1). Hence, for each $n \in \mathbb{Z}$,

$$\nu_1(\{n\}) = \int \chi_{\{n\}}\,\mathrm{d}\nu_1 = \Big\langle \mu_1, \frac{\chi_{\{n\}}}{\omega}\Big\rangle = \begin{cases} 1, & n \in 2\mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and}$$

$$\nu_2(\{n\}) = \begin{cases} 1, & n \in \mathbb{N}\backslash 2\mathbb{N}, \\ 0, & \text{otherwise}; \end{cases}$$

hence, $\nu_1(2\mathbb{N}) = \sum_{k\in\mathbb{N}} \nu_1(\{2k\}) = \infty$ and $\nu_2(\mathbb{N}\backslash 2\mathbb{N}) = \sum_{k\in\mathbb{N}} \nu_2(\{2k-1\}) = \infty$. Thus, ν_1, ν_2 do not map into $\mathbb{C}$. Moreover, (although ν_1, ν_2 can be viewed as positive measures), if $\nu = \nu_1 - \nu_2$ were a measure, additivity would give

$$\nu(\mathbb{N}) = \nu(2\mathbb{N}) + \nu(\mathbb{N}\backslash 2\mathbb{N}) = \nu_1(2\mathbb{N}) - \nu_2(\mathbb{N}\backslash 2\mathbb{N}) = \infty - \infty.$$

We conclude that functionals in $C_0(G,\omega^{-1})^*$ cannot necessarily be identified with complex Borel measures in the standard sense. It is perhaps for this reason that many authors assume the additional condition $\omega \geq 1$, since this guarantees containment of $M(G,\omega)$ in $M(G)$ and, thus, the essential properties of $M(G)$ also hold for $M(G,\omega)$, e.g., see [3]. Letting $\mathfrak{S}(G)$ denote the δ-ring of "compacted-Borel sets" — i.e., the δ-ring of all Borel subsets of G with compact closure — a *compacted-Borel measure* on G is a countably additive complex-valued function on $\mathfrak{S}(G)$ in the sense of [4, Definitions II.1.2 and II.8.2][1]. For non-compact G, there are positive regular measures μ, ν on $\mathfrak{B}(G)$ such that $\mu(G) = \nu(G) = \infty$ (e.g., Haar measures), and therefore $\mu - \nu$ is not defined on $\mathfrak{B}(G)$; however, these same measures are real-valued on $\mathfrak{S}(G)$, so $\mu - \nu$ is well-defined on $\mathfrak{S}(G)$. This is one benefit to studying measure theory over $\mathfrak{S}(G)$, rather than on all of $\mathfrak{B}(G)$.

The purpose of this article is to show that the theory of complex regular compacted-Borel measures, as developed in [4] (also see paragraph two of the "Notes and Remarks" section of Chapter II of [4] for additional references), can be used to provide a rigorous definition of $M(G,\omega)$, thus providing a solid foundation for all the papers in which $M(G,\omega)$ is employed without the requirement that $\omega \geq 1$; moreover, we hope this reduces the number of instances in which the $\omega \geq 1$ assumption is required going forward. To stress that we are using the theory of complex regular compacted-Borel measures, we will use the notation $\mathcal{M}(G,\omega)$ — inspired by [4] — rather than $M(G,\omega)$. Beyond identifying the correct collection of measures to employ, work is required to establish the needed theory. As measure theory can be quite finicky in general; because the study of compacted-Borel measures introduces different technicalities than those encountered in the Borel measure situation; and because a lot of research already depends on the results found herein, we have included a careful treatment of our development of $\mathcal{M}(G,\omega)$. There are numerous detailed classical expositions of the basic theory $M(G)$, and we believe the same is required for $\mathcal{M}(G,\omega)$.

We restrict ourselves to developing only the most standard properties of $\mathcal{M}(G,\omega)$: we provide a careful definition of its elements and show that with convolution product it is a dual Banach algebra containing a copy of the Beurling group algebra $L^1(G,\omega)$ as a closed ideal. Beyond this, we only show that $\mathcal{M}(G,\omega)$ embeds via a strict-to-weak* continuous isometric isomorphism as a subalgebra of the universal enveloping dual Banach algebra of $\mathcal{L}^1(G,\omega)$, $WAP(L^\infty(G,\omega))^*$, a result needed in [12]. The inspiration for this paper was our need to work with $\mathcal{M}(G,\omega)$ in [12].

1. $\mathcal{M}(G,\omega)$: definition and basic properties

Unless explicitly indicated otherwise, all references are to statements in Sections 1, 2, 5, 7–10 of Chapter II and Sections 10 of Chapter III of [4]. We will mostly adhere to the notation found therein. In particular, $\mathcal{M}(G)$ is the linear space composed of all regular complex compacted-Borel measures on G (Sections II.8 and III.10) and $\mathcal{M}_r(G)$ is the Banach space of *bounded* measures

[1]In [4], for the sake of brevity, the authors refer to compacted-Borel measures simply as Borel measures. To our knowledge, with the exception of [4], Borel measures in abstract harmonic analysis are always defined on $\mathfrak{B}(G)$.

in $\mathcal{M}(G)$ (Sections II.1 and II.8). Let $\mathfrak{C}(G)$ denote the directed set of compact subsets of G, and denote the space of continuous functions on G with compact support by $C_{00}(G)$, the space of continuous functions on G vanishing at infinity by $C_0(G)$, and the space of continuous functions on G supported on $K \in \mathfrak{C}(G)$ by $C_K(G)$; unless the context requires otherwise, these spaces are taken with the uniform norm $\|\cdot\|_\infty$.

Remark 1.1. (a) Let $\mu \in \mathcal{M}(G)$. A Borel subset A of G belongs to $\mathcal{E}_\mu$ if A is contained in some open set U such that

$$\sup\{|\mu|(A') \colon A' \in \mathfrak{S}(G) \text{ and } A' \subseteq U\} < \infty;$$

$\mathcal{E}_\mu$ is a δ-ring containing $\mathfrak{S}(G)$ and for $A \in \mathcal{E}_\mu$ putting

$$\mu_e(A) := \lim_C \mu(C), \qquad \text{where } C \in \mathfrak{C}(G),\ C \subseteq A, \tag{2}$$

we obtain a complex measure on $\mathcal{E}_\mu$ extending μ, called the maximal regular extension of μ (II.8.15). Observe that any Borel subset of a set in $\mathcal{E}_\mu$ is also in $\mathcal{E}_\mu$, from which it readily follows that $h\chi_E$ is locally μ_e-measurable whenever $E \in \mathcal{E}_\mu$ and h is a Borel-measurable function on G.

(b) When $\mu \in \mathcal{M}_r(G)$, $\mathcal{E}_\mu = \mathfrak{B}(G)$ and $\mu_e \in M(G)$, where $M(G)$ denotes the usual measure algebra of regular complex Borel measures $\mu\colon \mathfrak{B}(G) \to \mathbb{C}$, e.g., see [2], [9], [14]. Thus, the measures in $\mathcal{M}_r(G)$ are in one-to-one correspondence with measures in $M(G)$ via $\mu \mapsto \mu_e$; moreover, it is clear from the results in Section III.10 (or Theorem 1.5, below, in the non-weighted case) that $\mu \mapsto \mu_e$ is a weak*-continuous isometric algebra isomorphism of $\mathcal{M}_r(G)$ onto $M(G)$. Thus, for the purposes of abstract harmonic analysis on (non-weighted) G, $\mathcal{M}_r(G)$ can be used in place of the usual $M(G)$, and, as shown in [4], provides some advantages.

For $\mu \in \mathcal{M}(G)$, let I_μ denote the linear functional $I_\mu(f) = \int f \,\mathrm{d}\mu$ defined on $\mathcal{L}^1(\mu)$, or any subspace of $\mathcal{L}^1(\mu)$. Then

$$\mu \mapsto I_\mu \colon \mathcal{M}(G) \to \mathfrak{I} \tag{3}$$

is a linear bijection where $\mathfrak{I}$ is the set of all linear functionals I on $C_{00}(G)$ such that $I \in C_K(G)^*$ for each $K \in \mathfrak{C}(G)$; (3) maps $\mathcal{M}(G)^+$ onto $\mathfrak{I}^+$ and $\mathcal{M}_r(G)$ onto $C_{00}(G)^* = C_0(G)^*$ (II.8.12).

Remark 1.2. It should be noted that when μ is a complex measure on a δ-ring $\mathfrak{S}$, $f \in \mathcal{L}^1(\mu)$ requires that f vanishes off a countable union of sets in $\mathfrak{S}$ (II.2.5, paragraph 2). Thus, when $f \in \mathcal{L}^1(\mu)$ for $\mu \in \mathcal{M}(G)$, f must vanish off a σ-compact set, a technical issue requiring careful attention throughout this note. Consider the case when $\mu \in \mathcal{M}_r(G)$. Then any $\phi \in C_0(G)$ vanishes off a σ-compact set and since ϕ is continuous and bounded, it is easy to see that $\phi \in \mathcal{L}^1(\mu)$. Assuming further that $\mu \geq 0$ and $\phi \geq 0$ and taking an increasing sequence (ϕ_n) in $C_{00}(G)^+$ such that $\|\phi_n - \phi\|_\infty \to 0$, $\lim I_\mu(\phi_n) = \lim \int \phi_n \,\mathrm{d}\mu = \int \phi \,\mathrm{d}\mu = I_\mu(\phi)$ (e.g., by MCT II.7), so I_μ is the unique continuous extension of I_μ on $C_{00}(G)$ to $C_0(G)$. Thus, $C_0(G)^* = \{I_\mu \colon \mu \in \mathcal{M}_r(G)\}$, so — in this theory and as usual — we can identify $\mathcal{M}_r(G)$ and $C_0(G)^*$ through the pairing $\langle \mu, \phi \rangle = \int \phi \,\mathrm{d}\mu$.

Let $\nu \in \mathcal{M}(G)$, h a continuous function on G. Then h is locally ν-measurable (II.8.2) and for each $A \in \mathfrak{S}(G)$, $h\chi_A \in \mathcal{L}^1(\nu)$ since $|h|$ is bounded on A; i.e., h is locally ν-summable. Therefore,

$$h\nu(A) := \int h\chi_A \,\mathrm{d}\nu, \qquad A \in \mathfrak{S}(G),$$

defines a complex measure on $\mathfrak{S}(G)$ (see II.7.2, where the notation $h\,\mathrm{d}\nu$ rather than $h\nu$ is used); as $h\nu \ll \nu$ (II.7.8), $h\nu \in \mathcal{M}(G)$ (II.8.3). If $h > 0$, then $(1/h)(h\nu) \in \mathcal{M}(G)$ and a simple application of II.7.5 gives $(1/h)(h\nu) = \nu$.

Hence, $\omega\nu \sim \nu$ for each $\nu \in \mathcal{M}(G)$, and

$$\mathcal{M}(G) \to \mathcal{M}(G) \colon \nu \mapsto \omega\nu$$

defines a linear isomorphism with inverse $\nu \mapsto (1/\omega)\nu$. We can thus define

$$\mathcal{M}(G,\omega) := \{\nu \in \mathcal{M}(G) \colon \omega\nu \in \mathcal{M}_r(G)\}; \quad \text{letting } \|\nu\|_\omega = \|\omega\nu\|,\ \nu \in \mathcal{M}(G,\omega),$$

it follows that $\mathcal{M}(G,\omega)$ is a Banach space and $\nu \mapsto \omega\nu$ is an isometric linear isomorphism of $\mathcal{M}(G,\omega)$ onto $\mathcal{M}_r(G)$ with inverse map $\mu \mapsto (1/\omega)\mu$. (As shown in the introduction, this definition cannot, in general, be made with $M(G)$ replacing $\mathcal{M}_r(G)$.) Observe that by II.7.3, $\nu \in \mathcal{M}(G,\omega)$ exactly when $|\nu| \in \mathcal{M}(G,\omega)$, and $\|\nu\|_\omega = \||\nu|\|_\omega$.

Proposition 1.3. *For each $\nu \in \mathcal{M}(G,\omega)$, $I_\nu \in C_0(G,\omega^{-1})^*$ and $\|I_\nu\| = \|\nu\|_\omega$; moreover,*

$$C_0(G,\omega^{-1})^* = \{I_\nu \colon \nu \in \mathcal{M}(G,\omega)\}. \tag{4}$$

We can thus make the identification $\mathcal{M}(G,\omega) = C_0(G,\omega^{-1})^*$ through the pairing

$$\langle \nu, \psi \rangle_\omega = \int \psi \,\mathrm{d}\nu, \qquad \nu \in \mathcal{M}(G,\omega),\ \psi \in C_0(G,\omega^{-1}).$$

With respect to this identification, the inverse isometric isomorphisms

$$\mathcal{M}(G,\omega) \to \mathcal{M}_r(G) \colon \nu \mapsto \omega\nu \qquad \text{and} \qquad \mathcal{M}_r(G) \to \mathcal{M}(G,\omega) \colon \mu \mapsto \frac{1}{\omega}\mu$$

are weak-homeomorphisms.*

Proof: As noted above, $S\colon C_0(G,\omega^{-1}) \to C_0(G)\colon \psi \mapsto (\psi/\omega)$ is an isometric isomorphism, so $S^*\colon \mathcal{M}_r(G) = C_0(G)^* \to C_0(G,\omega^{-1})^*$ is also an isometric isomorphism. Let $\nu \in \mathcal{M}(G,\omega)$. Then $\omega\nu \in \mathcal{M}_r(G)$ and for $\psi \in C_0(G,\omega^{-1})$, $(\psi/\omega) \in C_0(G) \subseteq \mathcal{L}^1(\omega\nu)$, see Remark 1.2; therefore by II.7.5, $\psi = (\psi/\omega)\omega \in \mathcal{L}^1(\nu)$ and

$$\langle I_\nu, \psi\rangle = \int \psi \,\mathrm{d}\nu = \int \frac{\psi}{\omega}\,\mathrm{d}(\omega\nu) = \langle \omega\nu, S(\psi)\rangle = \langle S^*(\omega\nu), \psi\rangle.$$

Hence, $C_0(G,\omega^{-1}) \subseteq \mathcal{L}^1(\nu)$, $I_\nu = S^*(\omega\nu) \in C_0(G,\omega^{-1})^*$, and therefore $\|I_\nu\| = \|S^*(\omega\nu)\| = \|\omega\nu\| = \|\nu\|_\omega$; since $S^*(\mu) = I_{\omega^{-1}\mu}$ and S^* maps onto $C_0(G,\omega^{-1})^*$, we have (4). Making the identification of ν and I_ν, $\mu \mapsto (1/\omega)\mu = S^*(\mu)$ is weak*-continuous, with (weak*-continuous) inverse map $\nu \mapsto \omega\nu$. $\square$

In Lemma 1.4, X is a locally compact Hausdorff space, $h\colon X \to (0,\infty)$ is a continuous function, and $\mu \in \mathcal{M}(X)^+$ is such that $h\mu \in \mathcal{M}_r(X)$. Observe that $\mathcal{E}_\mu \subseteq \mathfrak{B}(X) = \mathcal{E}_{h\mu}$; see Remark 1.1.

Lemma 1.4. *The function h is locally μ_e-summable and for any set $A \in \mathcal{E}_\mu$, $h(\mu_e)(A) = (h\mu)_e(A)$.*

Proof: Let $A \in \mathcal{E}_\mu$. Take $(C_n)_n$ to be an increasing sequence of compact subsets of A such that $\mu_e(A) = \lim_n \mu(C_n)$ and let $D = \bigcup_n C_n$. Observe that D, $A\backslash D \in \mathcal{E}_\mu$ and $\mu_e(D) = \lim \mu_e(C_n) = \lim \mu(C_n) = \mu_e(A)$; hence

$$\mu_e(A\backslash D) = 0. \tag{5}$$

It follows that for any compact subset C of $A\backslash D$, $\mu(C) = 0$ and therefore, since h is locally μ-summable and bounded on C, $h\mu(C) = 0$. Hence,

$$\lim (h\mu)_e(A\backslash D) = \lim\{(h\mu)(C)\colon C \in \mathfrak{C}(X),\ C \subseteq A\backslash D\} = 0. \tag{6}$$

As noted in Remark 1.1, $h\chi_{A\backslash D}$ is locally μ_e-measurable and it follows from (5) and II.2.7 that

$$\int h\chi_{A\backslash D}\,\mathrm{d}\mu_e = \lim_n \int (h\wedge n)\chi_{A\backslash D}\,\mathrm{d}\mu_e = 0. \tag{7}$$

Also, since $h\mu$ is bounded, $\lim \int h\chi_{C_n}\,\mathrm{d}\mu_e = \lim \int h\chi_{C_n}\,\mathrm{d}\mu = \sup (h\mu)(C_n) < \infty$ (using II.8.15 Remark 3), and therefore by II.2.7,

$$\int h\chi_D\,\mathrm{d}\mu_e = \lim \int h\chi_{C_n}\,\mathrm{d}\mu_e = \lim (h\mu)(C_n) = \lim (h\mu)_e(C_n) = (h\mu)_e(D). \tag{8}$$

From (7) and (8), $h\chi_{A\backslash D}, h\chi_D \in \mathcal{L}^1(\mu_e)$, whence $h\chi_A \in \mathcal{L}^1(\mu_e)$. Hence, h is locally μ_e summable. Moreover, (8), (7) and (6) yield $h(\mu_e)(A) = (h\mu)_e(A)$. $\square$

Let $p\colon G \times G \to G\colon (s,t) \mapsto st$. Following III.10.2, we say that $\mu, \nu \in \mathcal{M}(G)$ are convolvable, or that $\mu * \nu$ exists, if p is $\mu \times \nu$-proper in the sense of II.10.3, i.e., if $p^{-1}(A) \in \mathcal{E}_{\mu\times\nu}$ whenever $A \in \mathfrak{S}(G)$. In this case, $\mu * \nu \in \mathcal{M}(G)$, where for $A \in \mathfrak{S}(G)$,

$$\begin{aligned}\mu * \nu(A) &= p_*((\mu \times \nu)_e)(A) = (\mu \times \nu)_e(p^{-1}(A))\\ &= \lim\{(\mu \times \nu)(C)\colon C \subseteq p^{-1}(A),\ C \in \mathfrak{C}(G \times G)\};\end{aligned}$$

see III.10.2, II.10.3, II.10.5, II.10.1. Equivalently, one can check that $\mu * \nu$ exists if and only if

$$\sup\{(|\mu| \times |\nu|)(C)\colon C \subseteq p^{-1}(D),\ C \in \mathfrak{C}(G \times G)\} < \infty$$

for every compact subset D of G. (In our context, the definition of $\mu \times \nu \in \mathcal{M}(G \times G)$ and its properties are found in Section II.9.)

Theorem 1.5. *With respect to convolution product, $\mathcal{M}(G,\omega) = C_0(G,\omega^{-1})^*$ is a Banach algebra, i.e., $(\mu,\nu) \mapsto \mu * \nu$ is a well-defined associative operation on $\mathcal{M}(G,\omega)$ satisfying $\|\mu * \nu\|_\omega \le \|\mu\|_\omega \|\nu\|_\omega$. Moreover, for $\mu, \nu \in \mathcal{M}(G,\omega)$ and $\psi \in C_0(G,\omega^{-1})$,*

$$\begin{aligned}\langle \mu * \nu, \psi\rangle_\omega &= \int \psi(st)\,\mathrm{d}(\mu \times \nu)_e(s,t)\\ &= \iint \psi(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t) = \iint \psi(st)\,\mathrm{d}\nu(t)\,\mathrm{d}\mu(s).\end{aligned} \tag{9}$$

Proof: Let $\mu, \nu \in \mathcal{M}(G,\omega)$, with $\mu, \nu \ge 0$. Let D be a compact subset of G, C a compact subset of $p^{-1}(D)$. The functions $1_C(x,y)$ and $g(x,y) = \big(1/(\omega(x)\omega(y))\big)1_C(x,y)$ are Borel measurable functions, and are therefore locally $(\sigma \times \varrho)$-measurable for any pair of measures $\sigma, \varrho \in \mathcal{M}(G)$; moreover, since they are nonnegative, bounded and vanish off C, $1_C, g \in \mathcal{L}^1(\sigma \times \varrho)$. Applying the Fubini theorem (II.9.8) to these functions, and using II.7.5 twice — which also applies by II.9.8 — we obtain

$$\begin{aligned}\mu \times \nu(C) &= \iint 1_C(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \iint g(x,y)\omega(x)\,\mathrm{d}\mu(x)\omega(y)\,\mathrm{d}\nu(y)\\ &= \iint g(x,y)\,\mathrm{d}\omega\mu(x)\,\mathrm{d}\omega\nu(y)\\ &= \int_{G\times G} \frac{1}{\omega(x)\omega(y)}\,1_C(x,y)\,\mathrm{d}(\omega\mu \times \omega\nu)(x,y)\end{aligned}$$

$$\leq \int_{G\times G} \frac{1}{\omega(xy)}\, 1_C(x,y)\, \mathrm{d}(\omega\mu\times\omega\nu)(x,y)$$
$$\leq \int_{G\times G} M_D 1_C(x,y)\, \mathrm{d}(\omega\mu\times\omega\nu)_e(x,y),$$

where $M_D = \sup_{z\in D}\omega(z)^{-1}$, since $C \subseteq p^{-1}(D)$, and we have used II.8.15 Remark 3. Observe that $p^{-1}(D) \in \mathfrak{B}(G\times G) = \mathcal{E}_{\omega\mu\times\omega\nu}$, since $\omega\mu\times\omega\nu \in \mathcal{M}_r(G\times G)$ — see II.9.14 — so

$$\mu\times\nu(C) \leq \int_{G\times G} M_D 1_{p^{-1}(D)}\, \mathrm{d}(\omega\mu\times\omega\nu)_e \leq M_D\|\omega\mu\times\omega\nu\|$$
$$= M_D\|\omega\mu\|\|\omega\nu\| = M_D\|\mu\|_\omega\|\nu\|_\omega.$$

Hence, $\mu*\nu$ exists. We now show $\mu*\nu \in \mathcal{M}(G,\omega)$ and $\|\mu*\nu\|_\omega \leq \|\mu\|_\omega\|\nu\|_\omega$. Let $A \in \mathfrak{S}(G)$. Since ω is continuous on G and $\mu*\nu \in \mathcal{M}(G)$, ω is locally $\mu*\nu$-summable and $\omega(\mu*\nu) \in \mathcal{M}(G)$. Hence, $\omega\chi_A \in \mathcal{L}^1(\mu*\nu) = \mathcal{L}^1(p_*(\mu\times\nu)_e)$. Therefore, II.10.2 gives $(\omega\chi_A)\circ p \in \mathcal{L}^1((\mu\times\nu)_e)$ and

$$\omega(\mu*\nu)(A) = \int \omega\chi_A\, \mathrm{d}(p_*((\mu\times\nu)_e)) = \int (\omega\chi_A)\circ p\ \mathrm{d}(\mu\times\nu)_e$$
$$= \int \omega\circ p\,\chi_{p^{-1}(A)}\, \mathrm{d}(\mu\times\nu)_e \leq \int (\omega\times\omega)\chi_{p^{-1}(A)}\, \mathrm{d}(\mu\times\nu)_e,$$

where $(\omega\times\omega)(s,t) = \omega(s)\omega(t)$. By II.9.9 and II.9.3, $(\omega\times\omega)(\mu\times\nu) = \omega\mu\times\omega\nu$, which belongs to $\mathcal{M}_r(G\times G)$ by II.9.14. Observe that $\omega\times\omega$ is locally $(\mu\times\nu)_e$-summable, by Lemma 1.4, and $p^{-1}(A) \in \mathcal{E}_{\mu\times\nu}$, since $\mu*\nu$ exists. Hence, the above inequality and Lemma 1.4 yield

$$\omega(\mu*\nu)(A) \leq (\omega\times\omega)(\mu\times\nu)_e(p^{-1}(A)) = ((\omega\times\omega)(\mu\times\nu))_e(p^{-1}(A))$$
$$= (\omega\mu\times\omega\nu)_e(p^{-1}(A)) \leq \|\omega\mu\times\omega\nu\| = \|\omega\mu\|\|\omega\nu\| = \|\mu\|_\omega\|\nu\|_\omega.$$

Hence, $\omega(\mu*\nu)$ is bounded, i.e., $\mu*\nu \in \mathcal{M}(G,\omega)$, and $\|\mu*\nu\|_\omega = \|\omega(\mu*\nu)\| \leq \|\mu\|_\omega\|\nu\|_\omega$.

Assume now that μ,ν are any two measures in $\mathcal{M}(G,\omega)$. As we have noted, $\sigma \in \mathcal{M}(G,\omega)$ exactly when $|\sigma| \in \mathcal{M}(G,\omega)$ and $\|\sigma\|_\omega = \||\sigma|\|_\omega$, so it follows from III.10.3 and the positive case that $\mu*\nu$ exists and $|\mu*\nu| \leq |\mu|*|\nu|$. Hence, $\omega|\mu*\nu| \leq \omega|\mu|*|\nu|$, so $\mu*\nu \in \mathcal{M}(G,\omega)$ and

$$\|\mu*\nu\|_\omega = \|\omega|\mu*\nu|\| \leq \|\omega|\mu|*|\nu|\| = \||\mu|*|\nu|\|_\omega \leq \||\mu|\|_\omega\||\nu|\|_\omega = \|\mu\|_\omega\|\nu\|_\omega.$$

Associativity of convolution in $\mathcal{M}(G,\omega)$ is now an immediate consequence of III.10.10. Since any $\psi \in C_0(G,\omega^{-1})$ vanishes off a σ-compact subset of G and any $\mu,\nu \in \mathcal{M}(G,\omega)$ are σ-bounded — since $\omega\mu$ and $\omega\nu$ are so, and $\omega\mu \sim \mu$, $\omega\nu \sim \nu$ — Remark III.10.8 applies to give (9). $\square$

Let $\lambda = \lambda_G$ be a fixed left Haar measure on G, $\mathcal{L}^1(G) = \mathcal{L}^1(\lambda)$. Then $\lambda \in \mathcal{M}(G)$ (Section III.7), so $\omega\lambda \in \mathcal{M}(G)$ as well and, since $\omega > 0$, $\omega\lambda \sim \lambda$, from which it follows that g is locally $\omega\lambda$-measurable and vanishes off a σ-compact set if and only if $g\omega$ is locally λ-measurable and vanishes off a σ-compact set. Hence, if we define $\mathcal{L}^1(G,\omega) := \mathcal{L}^1(\omega\lambda)$, $g \in \mathcal{L}^1(G,\omega)$ exactly when $g\omega \in \mathcal{L}^1(G)$, and in this case $\int g \,\mathrm{d}(\omega\lambda) = \int g\omega \,\mathrm{d}\lambda$, by II.7.5. Thus,

$$\mathcal{L}^1(G,\omega) = \{g\colon g\omega \in \mathcal{L}^1(G)\} \qquad \text{and} \qquad \|g\|_\omega := \|g\|_{\mathcal{L}^1(\omega\lambda)} = \|g\omega\|_1$$

defines a Banach space norm on $\mathcal{L}^1(G,\omega)$. Moreover, $T\colon \mathcal{L}^1(G,\omega) \to \mathcal{L}^1(G)\colon g \mapsto g\omega$ is an isometric linear isomorphism, with inverse $f \mapsto (1/\omega)f$, so $T^*\colon \mathcal{L}^\infty(G) = \mathcal{L}^1(G)^* \to \mathcal{L}^1(G,\omega)^*$ is a weak*-continuous isometric isomorphism given by $\langle T^*\phi, g\rangle_\omega = \langle \phi, \omega g\rangle = \int (\phi\omega) g \,\mathrm{d}\lambda$. Letting

$$\mathcal{L}^\infty(G,\omega^{-1}) := \{\phi\omega\colon \phi \in \mathcal{L}^\infty(G)\} = \Big\{\psi\colon \frac{\psi}{\omega} \in \mathcal{L}^\infty(G)\Big\}, \qquad \text{where } \|\psi\|_{\infty,\omega^{-1}} := \Big\|\frac{\psi}{\omega}\Big\|_\infty,$$

we can hence identify $\mathcal{L}^1(G,\omega)^*$ with $\mathcal{L}^\infty(G,\omega^{-1})$ via the pairing $\langle \psi, g\rangle_\omega = \int \psi g \,\mathrm{d}\lambda$. Observe that $S = (T^*)^{-1}\colon \mathcal{L}^\infty(G,\omega^{-1}) \to \mathcal{L}^\infty(G)\colon \psi \mapsto \psi/\omega$ is a weak*-homeomorphic isometric isomorphism. We note that $\mathcal{L}^\infty(G,\omega^{-1})$ is not usually the same space as $\mathcal{L}^\infty(\omega\lambda)$ ($= \mathcal{L}^\infty(\lambda)$ because $\omega\lambda \sim \lambda$), which can also be identified with $\mathcal{L}^1(\omega\lambda)^* = \mathcal{L}^1(G,\omega)^*$ in the usual way by II.7.11. Note that because T^{-1} maps $C_{00}(G)$ onto itself, $C_{00}(G)$ is dense in $\mathcal{L}^1(G,\omega)$.

Let $g \in \mathcal{L}^1(G,\omega) = \mathcal{L}^1(\omega\lambda)$, $A \in \mathfrak{S}(G)$. Then $\omega g \in \mathcal{L}^1(\lambda)$ and $1/\omega$ is bounded on A, so $\chi_A g = ((1/\omega)\chi_A)\omega g \in \mathcal{L}^1(\lambda)$; hence, $g\lambda \in \mathcal{M}(G)$ is well-defined (II.7.2). Also, $\omega(g\lambda) = (\omega g)\lambda \in \mathcal{M}(G)$ by II.7.5 and, by II.7.9/III.11.3, $\|f\|_1 = \|f\lambda\|$ for $f \in \mathcal{L}^1(G)$ and

$$\mathcal{M}_a(G) = \{\mu \in \mathcal{M}_r(G)\colon \mu \ll \lambda\} = \{f\lambda\colon f \in \mathcal{L}^1(G)\} = \{(\omega g)\lambda\colon g \in \mathcal{L}^1(G,\omega)\}.$$

Since $\omega\nu \sim \nu$ for any $\nu \in \mathcal{M}(G)$, it readily follows that $g \mapsto g\lambda\colon \mathcal{L}^1(G,\omega) \to \mathcal{M}_a(G,\omega)$ is a surjective linear isometry, where $\mathcal{M}_a(G,\omega) := \{\nu \in \mathcal{M}(G,\omega)\colon \nu \ll \lambda\}$. We can thus identify $\mathcal{L}^1(G,\omega)$ with $\mathcal{M}_a(G,\omega)$ via $g \mapsto g\lambda$.

Proposition 1.6. *The Banach space $\mathcal{L}^1(G,\omega) = \mathcal{M}_a(G,\omega)$ is a closed ideal in $\mathcal{M}(G,\omega)$ and has a contractive approximate identity. Moreover, if $g \in \mathcal{L}^1(G,\omega)$ and $\nu \in \mathcal{M}(G,\omega)$, then $\nu * g$, $g * \nu \in \mathcal{L}^1(G,\omega)$ are given by the formulas, which hold for locally λ-almost all $t \in G$,*

$$(10)\quad \nu * g(t) = \int g(s^{-1}t)\,\mathrm{d}\nu(s) \qquad \textit{and} \qquad g * \nu(t) = \int \Delta(s^{-1}) g(ts^{-1})\,\mathrm{d}\nu(s);$$

thus, $\mathcal{L}^1(G,\omega)$ is a Banach algebra with respect to the convolution product

$$f * g(t) = \int f(s)g(s^{-1}t)\,\mathrm{d}\lambda(s). \tag{11}$$

Proof: We have already noted that g is locally λ-summable and vanishes off a σ-compact set, and $(g\lambda)*\nu$, $\nu*(g\lambda)$ exist in $\mathcal{M}(G,\omega)$ by Theorem 1.5. Letting $h(t)$ and $k(t)$ be defined by the respective integral formulas on the left and right of (10), $\nu*(g\lambda) = h\lambda$ and $(g\lambda)*\nu = k\lambda$ by III.11.5. Thus, $h\lambda, k\lambda \in \mathcal{M}_a(G,\omega) = \{f\lambda\colon f \in \mathcal{L}^1(G,\omega)\}$, so the uniqueness part of the Radon–Nikodym theorem — see Remark 1 of II.7.8 — implies that $h, k \in \mathcal{L}^1(G,\omega)$. The formula (11) now follows quickly (or directly from III.11.6). Let $\mathcal{I}$ be the neighbourhood system at e_G and for each $\alpha \in \mathcal{I}$, let $f_\alpha \in C_{00}(G)$ be chosen with $f_\alpha \geq 0$, $\|f_\alpha\|_1 = 1$ and support contained in α. Then $(f_\alpha)_\alpha$ is a bounded approximate identity for $\mathcal{L}^1(G)$. Letting $e_\alpha = \omega^{-1}f_\alpha$, $\|e_\alpha\|_\omega = 1$ and $\|e_\alpha\|_1 \to 1$, from which it easily follows that $(e_\alpha)_\alpha$ is also a bounded approximate identity for $\mathcal{L}^1(G)$; the proof of Lemma 2.1 in [6] now shows that $(e_\alpha)_\alpha$ is a contractive approximate identity for $\mathcal{L}^1(G,\omega)$. $\square$

Remark 1.7. Every Borel measurable function is locally λ-measurable and every $f \in L^1(G,\omega)$ — where $L^1(G,\omega)$ is defined in the usual sense (as in the introduction) — vanishes off a σ-compact set. It follows that the Banach algebra $\mathcal{L}^1(G,\omega)$, as we have defined it, exactly coincides with the usual definition of the Beurling group algebra $L^1(G,\omega)$, which, as noted in the introduction, is always valid. Going forward, we can therefore use any known result about $L^1(G,\omega) = \mathcal{L}^1(G,\omega)$ that was proved independently of $M(G,\omega)$.

2. The dual Banach algebra $\mathcal{M}(G,\omega)$ and the embedding map

The *support* of μ in $\mathcal{M}(G)$ is the set $\mathrm{s}(\mu) = G\backslash\bigcup\{U \in \mathfrak{S}(G)\colon U$ is open and $|\mu|(U) = 0\}$, see II.8.9. Let $\mathcal{M}_{cr}(G) = \{\mu \in \mathcal{M}(G)\colon \mathrm{s}(\mu)$ is compact$\}$.

Remark 2.1. 1. Observe that $\mathrm{s}(\mu) = \mathrm{s}(\mu_e) = G\backslash\bigcup\{V \in \mathcal{E}_\mu\colon V$ is open and $|\mu_e|(V) = 0\}$.

2. Since ω and $1/\omega$ are bounded on any set A in $\mathfrak{S}(G)$, $\mathrm{s}(\mu) = \mathrm{s}(\omega\mu) = \mathrm{s}((1/\omega)\mu)$ for any $\mu \in \mathcal{M}(G)$.

3. By III.10.16, $\mathcal{M}_{cr}(G)$ is a dense subalgebra of $\mathcal{M}_r(G)$. From 2 above, the inverse linear isometries $\nu \mapsto \omega\nu$ and $\mu \mapsto (1/\omega)\mu$ between $\mathcal{M}(G,\omega)$ and $\mathcal{M}_r(G)$ map $\mathcal{M}_{cr}(G)$ onto itself, so $\mathcal{M}_{cr}(G)$ is also a dense subalgebra of $\mathcal{M}(G,\omega)$.

A measure σ on a δ-ring $\mathfrak{S}$ is *concentrated* on a set F if for each $A \in \mathfrak{S}$, $A \cap F, A\backslash F \in \mathfrak{S}$ and $\sigma(A) = \sigma(A \cap F)$ or, equivalently, $\sigma(A\backslash F) = 0$. For $\mu \in \mathcal{M}(G)$ and a Borel set F, $A\cap F, A\backslash F \in \mathfrak{S}(G)$ ($A\cap F, A\backslash F \in \mathcal{E}_\mu$, respectively) is automatic for any $A \in \mathfrak{S}(G)$ ($A \in \mathcal{E}_\mu$), and it is clear from (2) that μ is concentrated on F if and only if μ_e is concentrated on F. A function $\psi \in LUC(G,\omega^{-1})$ may fail to vanish off a σ-compact set and therefore, as noted in Remark 1.2, in this theory we cannot integrate ψ with respect to any μ in $\mathcal{M}(G)$. Lemma 2.2 allows us to move past this issue.

Lemma 2.2. (a) *Every μ in $\mathcal{M}(G)$ is concentrated on its support,* $\mathrm{s}(\mu)$.

(b) *Let $\mu \in \mathcal{M}_r(G)$. Then μ (and therefore μ_e) is concentrated on a σ-compact subset F of G and, for any such F and any Borel measurable function $f \in \mathcal{L}^1(\mu_e)$, $f\chi_F \in \mathcal{L}^1(\mu) \cap \mathcal{L}^1(\mu_e)$ and*

$$\int f \,\mathrm{d}\mu_e = \int f\chi_F \,\mathrm{d}\mu_e = \int f\chi_F \,\mathrm{d}\mu.$$

(c) *Any $\nu \in \mathcal{M}(G,\omega)$ is concentrated on a σ-compact set.*

Proof: (a) Let $A \in \mathfrak{S}(G)$. Any compact subset of $A\backslash\mathrm{s}(\mu)$ is covered by the collection of open sets $U \in \mathfrak{S}(G)$ with $|\mu|(U) = 0$, and is therefore $|\mu|$-null; by regularity of μ (II.8.2(II)), $|\mu|(A\backslash\mathrm{s}(\mu)) = 0$.

(b) Take $(C_n)_n$ to be an increasing sequence of compact subsets of $\mathrm{s}(\mu)$ such that $|\mu|(C_n) > \|\mu\| - 1/n$ and let $F = \bigcup C_n$, where we have used (b). Then μ is concentrated on F because for $A \in \mathfrak{S}(G)$,

$$\begin{aligned}|\mu|(A\backslash F) &= |\mu|((A\backslash F) \cap \mathrm{s}(\mu)) \le |\mu|_e(\mathrm{s}(\mu)\backslash F) = |\mu|_e(\mathrm{s}(\mu)) - |\mu_e|(F)\\ &= \|\mu\| - \lim |\mu|(C_n) = 0.\end{aligned}$$

Suppose $\mu \ge 0$, F is any σ-compact set on which μ is concentrated, and $f \in \mathcal{L}^1(\mu_e)$ is a nonnegative Borel-measurable function. It is then clear (from II.2.2 and II.2.5) that $f\chi_F \in \mathcal{L}^1(\mu_e)$ and $\int f \,\mathrm{d}\mu_e = \int f\chi_F \,\mathrm{d}\mu_e$. Also, $f\chi_F$ is locally μ-measurable (II.8.2), vanishes off the σ-compact set F and, taking any sequence of nonnegative $\mathfrak{S}(G)$-simple functions such that $h_n \uparrow f\chi_F$, II.2.2 gives

$$\int f\chi_f \,\mathrm{d}\mu_e = \lim \int h_n \,\mathrm{d}\mu_e = \lim \int h_n \,\mathrm{d}\mu = \int f\chi_F \,\mathrm{d}\mu.$$

(c) Since $\omega\nu \in \mathcal{M}_r(G)$ and $\nu \sim \omega\nu$, this follows from (b). □

Since $\mathcal{L}^1(G,\omega)$ is a closed ideal in $\mathcal{M}(G,\omega)$, through the operations

$$\langle \psi \cdot \nu, g\rangle = \langle \psi, \nu * g\rangle \quad \text{and} \quad \langle \nu \cdot \psi, g\rangle = \langle \psi, g * \nu\rangle$$
$$\text{for } \psi \in \mathcal{L}^\infty(G,\omega^{-1}),\ \nu \in \mathcal{M}(G,\omega),\ g \in \mathcal{L}^1(G,\omega),$$

$\mathcal{L}^\infty(G,\omega^{-1}) = \mathcal{L}^1(G,\omega)^*$ is a dual $\mathcal{M}(G,\omega)$-module. Observe that for $\psi \in \mathcal{L}^\infty(G,\omega^{-1})$ and $s \in G$,

$$\psi \cdot \delta_s(t) = \psi \cdot s(t) := \psi(st) \quad \text{and} \quad \delta_s \cdot \psi(t) = s \cdot \psi(t) := \psi(ts), \qquad t \in G.$$

Recall that ψ belongs to $LUC(G,\omega^{-1})$ $[RUC(G,\omega^{-1})]$ when ψ/ω belongs to $LUC(G)$ $[RUC(G)]$. For $LUC(G,\omega^{-1})$, the following is [8, Proposition 1.3] and [3, Propositions 7.15 and 7.17], (where no restrictions are needed on the weight ω); symmetric arguments establish the $RUC(G,\omega^{-1})$ case.

Lemma 2.3. *The following statements are equivalent:*

(a) $\psi \in LUC(G,\omega^{-1})$ $[RUC(G,\omega^{-1})]$;
(b) $\psi \in l^\infty(G,\omega^{-1})$ *and the map* $G \to (l^\infty(G,\omega^{-1}), \|\cdot\|_{\infty,\omega^{-1}})\colon s \mapsto \psi\cdot s$ $[s\cdot\psi]$ *is continuous;*
(c) $\psi \in \mathcal{L}^\infty(G,\omega^{-1})$ *and the map* $G \to (\mathcal{L}^\infty(G,\omega^{-1}), \|\cdot\|_{\infty,\omega^{-1}})\colon s \mapsto \psi \cdot s$ $[s \cdot \psi]$ *is continuous;*
(d) $\psi \in \mathcal{L}^\infty(G,\omega^{-1}) \cdot \mathcal{L}^1(G,\omega)$ $[\psi \in \mathcal{L}^1(G,\omega) \cdot \mathcal{L}^\infty(G,\omega^{-1})]$.

Remark 2.4. 1. Observe that condition (b) implies ψ is continuous on G, whence $\psi \in \mathcal{L}^\infty(G,\omega^{-1})$.

2. In the proof of [3, Proposition 7.15], the authors establish continuity of a function ψ satisfying (c) via Ascoli's theorem. An alternative approach is to establish (i) and (ii) as follows:

(i) If $\phi \in \mathcal{L}^\infty(G,\omega^{-1})$ and $g \in \mathcal{L}^1(G,\omega)$, then $\phi \cdot g$ can be identified with the continuous function

$$(\phi \cdot g)(t) = \langle \phi, g * \delta_t \rangle \qquad \text{for every } \; t \in G. \tag{12}$$

[Note that $H \in l^\infty(G,\omega^{-1})$ where $H(t) := \langle \phi, g * \delta_t \rangle$ and, since $t \mapsto g * \delta_t \colon G \to (\mathcal{L}^1(G,\omega), \|\cdot\|_\omega)$ is continuous — e.g., see [19, Lemma 3.1.5], which holds for any weight ω — H is continuous on G (and satisfies Lemma 2.3 (c)); in a standard way, one can check that for $f \in \mathcal{L}^1(G,\omega)$, $\langle \phi \cdot g, f \rangle = \langle H, f \rangle$.]

(ii) If ψ satisfies (c) and (e_i) is a bounded approximate identity for $\mathcal{L}^1(G,\omega)$, then $\|\psi \cdot e_i - \psi\|_{\infty,\omega^{-1}} \to 0$; since $CB(G,\omega^{-1})$ is closed in $\mathcal{L}^\infty(G,\omega^{-1})$, $\psi \in CB(G,\omega^{-1})$.

Proposition 2.5. *The spaces* $LUC(G,\omega^{-1})$ *and* $RUC(G,\omega^{-1})$ *are* $\mathcal{M}(G,\omega)$-*submodules of* $\mathcal{L}^\infty(G,\omega^{-1})$. *Moreover, for* $\nu \in \mathcal{M}(G,\omega)$, $\psi \in LUC(G,\omega^{-1})$

$[\psi \in RUC(G,\omega^{-1})]$ and for every $s \in G$,

$$(\nu \cdot \psi)(s) = \int (\psi \cdot s)\chi_{F_s}\,\mathrm{d}\nu = \int \frac{\psi \cdot s}{\omega}\,\chi_{F_s}\,\mathrm{d}(\omega\nu) = \int \frac{\psi \cdot s}{\omega}\,\mathrm{d}(\omega\nu)_e$$
$$\left[(\psi \cdot \nu)(s) = \int (s \cdot \psi)\chi_{F_s}\,\mathrm{d}\nu = \int \frac{s \cdot \psi}{\omega}\,\chi_{F_s}\,\mathrm{d}(\omega\nu) = \int \frac{s \cdot \psi}{\omega}\,\mathrm{d}(\omega\nu)_e\right],$$

where F_s is any σ-compact set on which ν is concentrated; F_s can be chosen to vary with $s \in G$.

Proof: Letting $\nu \in \mathcal{M}(G,\omega)$, $\psi \in LUC(G,\omega^{-1})$, it is clear from Lemma 2.3 (d) that $\psi \cdot \nu$, $\nu \cdot \psi \in LUC(G,\omega^{-1})$. Since $\psi \cdot s/\omega \in LUC(G)$ and $\omega\nu \in \mathcal{M}_r(G)$,

$$H(s) = H_{\nu,\psi}(s) := \int \frac{\psi \cdot s}{\omega}\,\mathrm{d}(\omega\nu)_e = \int \frac{\psi \cdot s}{\omega}\,\chi_{F_s}\,\mathrm{d}(\omega\nu)$$

is well-defined, where we have used Lemma 2.2. The function $(\psi \cdot s)\chi_{F_s} \in l^\infty(G,\omega^{-1})$ is Borel measurable — and therefore locally ν-measurable — and vanishes off the σ-compact set F_s, so $(\psi \cdot s/\omega)\chi_{F_s} \in \mathcal{L}^1(\omega\nu)$. Therefore, by II.7.5, $(\psi \cdot s)\chi_{F_s} \in \mathcal{L}^1(\nu)$ and

$$\int (\psi \cdot s)\chi_{F_s}\,\mathrm{d}\nu = \int \frac{\psi \cdot s}{\omega}\,\chi_{F_s}\,\omega\,\mathrm{d}\nu = \int \frac{\psi \cdot s}{\omega}\,\chi_{F_s}\,\mathrm{d}(\omega\nu) = H(s).$$

Since $|H(s)| \le \|\psi \cdot s/\omega\|_\infty \|\omega\nu\| \le \omega(s)\|\psi\|_{\infty,\omega^{-1}}\|\nu\|_\omega$, $H = H_{\nu,\psi} \in l^\infty(G,\omega^{-1})$ with $\|H_{\nu,\psi}\|_{\infty,\omega^{-1}} \le \|\psi\|_{\infty,\omega^{-1}}\|\nu\|_\omega$. Hence, if $s_i \to s$ in G,

$$\|(H_{\nu,\psi})\cdot s_i - (H_{\nu,\psi})\cdot s\|_{\infty,\omega^{-1}} = \|H_{\nu,\psi\cdot s_i - \psi\cdot s}\|_{\infty,\omega^{-1}} \le \|\psi\cdot s_i - \psi\cdot s\|_{\infty,\omega^{-1}}\|\nu\|_\omega \to 0;$$

by Lemma 2.3, $H_{\nu,\psi} \in LUC(G,\omega^{-1})$. To show that $H_{\nu,\psi} = \nu \cdot \psi$, we can assume $\nu \ge 0$, $\psi \ge 0$ and take $F = F_s$ for each $s \in G$. Let $g \ge 0$ be a function in the dense subspace $C_{00}(G)$ of $\mathcal{L}^1(G,\omega)$. Since the maps $(s,t) \mapsto \psi(t)\Delta(s^{-1})g(ts^{-1})\chi_F(s)$, $\psi(ts)g(t)\chi_F(s)$ are Borel measurable — hence locally $(\nu\times\lambda)$-measurable — and vanish off a σ-compact subset of $G\times G$, our applications of the Fubini theorem (II.9.8) are valid in the following calculation. Using (10):

$$\begin{aligned}
\langle \nu \cdot \psi, g\rangle &= \langle \psi, g * \nu\rangle = \int \psi(t) \int \Delta(s^{-1})g(ts^{-1})\,\mathrm{d}\nu(s)\,\mathrm{d}\lambda(t) \\
&= \iint \psi(t)\Delta(s^{-1})g(ts^{-1})\chi_F(s)\,\mathrm{d}\nu(s)\,\mathrm{d}\lambda(t) \\
&= \iint \psi(t)\Delta(s^{-1})g(ts^{-1})\chi_F(s)\,\mathrm{d}\lambda(t)\,\mathrm{d}\nu(s)
\end{aligned}$$

$$= \iint \psi(ts)g(t)\chi_F(s)\,\mathrm{d}\lambda(t)\,\mathrm{d}\nu(s)$$
$$= \iint \psi\cdot t(s)\chi_F(s)\,\mathrm{d}\nu(s)\,g(t)\,\mathrm{d}\lambda(t) = \langle H_{\nu,\lambda}, g\rangle;$$

since both functions are continuous, $\nu\cdot\lambda = H_{\nu,\lambda}$. □

Corollary 2.6. *The space $C_0(G,\omega^{-1})$ is a $\mathcal{M}(G,\omega)$-submodule of $\mathcal{L}^\infty(G,\omega^{-1})$, and for $\nu\in\mathcal{M}(G,\omega)$, $\psi\in C_0(G,\omega^{-1})$ and $s\in G$,*

$$(13)\quad \nu\cdot\psi(s) = \int \psi\cdot s\,\mathrm{d}\nu = \langle \nu, \psi\cdot s\rangle_\omega \qquad \textit{and} \qquad \psi\cdot\nu(s) = \int s\cdot\psi\,\mathrm{d}\nu = \langle\nu, s\cdot\psi\rangle_\omega.$$

Proof: Let $\psi\in C_0(G,\omega^{-1})$ and let F be a σ-compact set on which ν is concentrated. Taking A_s to be a σ-compact set off which $\psi\cdot s$ and $s\cdot\psi$ vanish, and putting $F_s = F\cup A_s$, Proposition 2.5 gives $\nu\cdot\psi$, $\psi\cdot\nu\in(LUC\cap RUC)(G,\omega^{-1})$ and

$$\nu\cdot\psi(s) = \int(\psi\cdot s)\chi_{F_s}\,\mathrm{d}\nu = \int\psi\cdot s\,\mathrm{d}\nu \quad\text{and}\quad \psi\cdot\nu(s) = \int(s\cdot\psi)\chi_{F_s}\,\mathrm{d}\nu = \int s\cdot\psi\,\mathrm{d}\nu.$$

Observe that $\nu\cdot\psi$ is supported on $\mathrm{s}(\psi)\mathrm{s}(\nu)^{-1}$, which is compact when ν belongs to the dense subspace $\mathcal{M}_{cr}(G)$ of $\mathcal{M}(G,\omega)$ and ψ belongs to the dense subspace $C_{00}(G)$ of $C_0(G,\omega^{-1})$. It follows that $C_0(G,\omega^{-1})$ is a left (and similarly, right) $\mathcal{M}(G,\omega)$-submodule of $\mathcal{L}^\infty(G,\omega^{-1})$. □

It follows that $\mathcal{M}(G,\omega) = C_0(G,\omega^{-1})^*$ is a dual $\mathcal{M}(G,\omega)$-module with respect to the operations

$$\langle\mu\cdot_r\nu,\psi\rangle_\omega = \langle\mu,\nu\cdot\psi\rangle_\omega \quad\text{and}\quad \langle\mu\cdot_l\nu,\psi\rangle_\omega = \langle\nu,\psi\cdot\mu\rangle_\omega, \qquad \mu,\nu\in\mathcal{M}(G,\omega),\ \psi\in C_0(G,\omega^{-1}).$$

However, from (9) and (13),

$$(14)\qquad \mu\cdot_r\nu = \mu*\nu = \mu\cdot_l\nu,$$

so $(\mu,\nu)\mapsto\mu*\nu$ is separately weak*-continuous on $\mathcal{M}(G,\omega)$. Hence:

Corollary 2.7. *The Beurling measure algebra $\mathcal{M}(G,\omega)$ is a dual Banach algebra.*

Let A be a Banach algebra. Recall that a closed submodule $\mathcal{S}(A^*)$ of the dual A-bimodule A^* is left [right] introverted if for each $\mu\in\mathcal{S}(A^*)^*$ and $\phi\in\mathcal{S}(A^*)$, $\mu\,\square\,\phi\in\mathcal{S}(A^*)$ [$\phi\diamond\mu\in\mathcal{S}(A^*)$] where $\mu\,\square\,\phi$, $\phi\diamond\mu\in A^*$ are defined by

$$\langle\mu\,\square\,\phi, a\rangle_{A^*-A} = \langle\mu,\phi\cdot a\rangle_{\mathcal{S}^*-\mathcal{S}} \qquad\text{and}\qquad \langle\phi\diamond\mu, a\rangle_{A^*-A} = \langle\mu, a\cdot\phi\rangle_{\mathcal{S}^*-\mathcal{S}};$$

in this case, $\mathcal{S}(A^*)^*$ is a Banach algebra with respect to its left [right] Arens product

$$\langle \mu \,\Box\, \nu, \phi\rangle = \langle \mu, \nu \,\Box\, \phi\rangle \qquad [\langle \mu \diamond \nu, \phi\rangle = \langle \nu, \phi \diamond \mu\rangle], \qquad \mu, \nu \in \mathcal{S}(A^*)^*,\ \phi \in \mathcal{S}(A^*).$$

The map $\eta_{\mathcal{S}}\colon A \to \mathcal{S}(A^*)^*$ defined by $\langle \eta_{\mathcal{S}}(a), \phi\rangle = \langle \phi, a\rangle$ is a bounded homomorphism with weak*-dense range and, when A is left introverted, $\eta_{\mathcal{S}}$ maps into the topological centre of $(\mathcal{S}(A^*)^*, \Box)$, $Z_t(\mathcal{S}(A^*)^*) = \{\mu \in \mathcal{S}(A^*)^*\colon \nu \mapsto \mu \,\Box\, \nu$ is wk* – wk* continuous on $\mathcal{S}(A^*)^*\}$. For this see, e.g., [3].

Proposition 2.8. *The subspace $C_0(G, \omega^{-1})$ of $\mathcal{L}^\infty(G, \omega^{-1}) = \mathcal{L}^1(G, \omega)^*$ is left and right introverted and $\mu{*}\nu = \mu \,\Box\, \nu = \mu \diamond \nu$ for $\mu, \nu \in \mathcal{M}(G, \omega) = C_0(G, \omega^{-1})^*$.*

Proof: By Corollary 2.6, $C_0(G, \omega^{-1})$ is a $\mathcal{L}^1(G, \omega)$-submodule of $\mathcal{L}^\infty(G, \omega^{-1})$. Let $\mu, \nu \in \mathcal{M}(G, \omega)$, $\psi \in C_0(G, \omega^{-1})$. For $g \in \mathcal{L}^1(G, \omega)$, equation (14) gives

$$\langle \nu \,\Box\, \psi, g\rangle_{\mathcal{L}^\infty - \mathcal{L}^1} = \langle \nu, \psi \cdot g\rangle_\omega = \langle g * \nu, \psi\rangle_\omega = \langle g, \nu \cdot \psi\rangle_\omega = \langle \nu \cdot \psi, g\rangle_{\mathcal{L}^\infty - \mathcal{L}^1}.$$

Hence, $C_0(G, \omega)$ is left introverted and $\langle \mu \,\Box\, \nu, \psi\rangle = \langle \mu, \nu \,\Box\, \psi\rangle = \langle \mu, \nu \cdot \psi\rangle = \langle \mu{*}\nu, \psi\rangle$, where we have again used (14). Similarly, $C_0(G, \omega^{-1})$ is right introverted and $\mu * \nu = \mu \diamond \nu$. $\Box$

Let $\mathcal{S}(\omega^{-1})$ be a left introverted subspace of $\mathcal{L}^\infty(G, \omega^{-1})$ such that

$$C_0(G, \omega^{-1}) \preceq \mathcal{S}(\omega^{-1}) \preceq LUC(G, \omega^{-1})$$

and define

$$\text{(15)}\quad \Theta\colon \mathcal{M}(G, \omega) \to \mathcal{S}(\omega^{-1})^* \qquad \text{by } \langle \Theta(\nu), \psi\rangle_{\mathcal{S}^* - \mathcal{S}} = (\nu \cdot \psi)(e_G) = \int \psi\, \chi_{F_\nu}\, \mathrm{d}\nu,$$

where $\nu \in \mathcal{M}(G, \omega)$, $\psi \in \mathcal{S}(\omega^{-1})$ and F_ν is any σ-compact set on which ν is concentrated. By Proposition 2.5, Θ is well-defined and $|\langle \Theta(\nu), \psi\rangle| \le \|\nu \cdot \psi\|_{\infty, \omega^{-1}} \le \|\nu\|_\omega \|\psi\|_{\infty, \omega^{-1}}$, so $\|\Theta(\nu)\| \le \|\nu\|_\omega$; by equation (13), $\Theta(\nu)|_{C_0(G, \omega^{-1})} = \nu$, so $\|\Theta(\nu)\| = \|\nu\|_\omega$. Thus, Θ is a linear isometry.

Let so_l and so_r denote the left and right strict topologies on $\mathcal{M}(G, \omega)$ taken with respect to the ideal $\mathcal{L}^1(G, \omega)$, i.e., the locally convex topologies respectively generated by the semi-norms $p_g(\nu) = \|g * \nu\|$ and $q_g(\nu) = \|\nu * g\|$ for $g \in \mathcal{L}^1(G, \omega)$, $\nu \in \mathcal{M}(G, \omega)$. Since $\mathcal{L}^1(G, \omega)$ has a contractive approximate identity, (the unit ball of) $\mathcal{L}^1(G, \omega)$ is so_l/so_r-dense in (the unit ball of) $\mathcal{M}(G, \omega)$. Observe that when $\mathcal{S}(\omega^{-1}) \preceq LUC(G, \omega^{-1})$ is a $\mathcal{L}^1(G, \omega)$-submodule of $\mathcal{L}^\infty(G, \omega^{-1})$, by Lemma 2.3 (d) and the Cohen factorization theorem [1, Theorem 11.10], $\mathcal{S}(\omega^{-1}) = \mathcal{S}(\omega^{-1}) \cdot \mathcal{L}^1(G, \omega)$. Also note that $LUC(G, \omega^{-1})$ is always left introverted in $\mathcal{L}^\infty(G, \omega^{-1})$ by Lemma 2.3 and [3, Proposition 5.9]. In the non-weighted case and when $\omega \ge 1$, the final statement in Proposition 2.9, which simplifies Arens product calculations, is [13, Lemma 3] and [3, Proposition 7.21], respectively.

Proposition 2.9. *Suppose that $\mathcal{S}(\omega^{-1})$ is a left [right] introverted subspace of $\mathcal{L}^\infty(G,\omega^{-1}) = \mathcal{L}^1(G,\omega)^*$ and*

$$C_0(G,\omega^{-1}) \preceq \mathcal{S}(\omega^{-1}) \preceq LUC(G,\omega^{-1})\ [RUC(G,\omega^{-1})].$$

Then $\Theta\colon \mathcal{M}(G,\omega) \hookrightarrow \mathcal{S}(\omega^{-1})^*$ is a so_l-weak* [so_r-weak*] continuous isometric homomorphic embedding into $Z_t(\mathcal{S}(\omega^{-1})^*)$ that extends $\eta_{\mathcal{S}}\colon \mathcal{L}^1(G,\omega) \to \mathcal{S}(\omega^{-1})^*$. Moreover, $(n \,\square\, \psi)(s) = \langle n, \psi\cdot s\rangle$ for any $n \in \mathcal{S}(\omega^{-1})^*$, $\psi \in \mathcal{S}(\omega^{-1})$ and $s \in G$; hence, $\mathcal{S}(\omega^{-1})$ is introverted as a subspace of $l^\infty(G,\omega^{-1}) = l^1(G,\omega)^*$, the Arens product on $\mathcal{S}(\omega^{-1})^*$ agrees under either interpretation, and Θ also extends $\eta_{\mathcal{S}}$: $l^1(G,\omega) \hookrightarrow \mathcal{S}(\omega^{-1})^*$.

Proof: If $g \in \mathcal{L}^1(G,\omega) = \mathcal{L}^1(\omega\lambda)$, g vanishes off a σ-compact set F_g, and therefore $g = g\lambda \in \mathcal{M}(G,\omega)$ is concentrated on F_g; hence, for $\psi \in \mathcal{S}(\omega^{-1})$,

$$\langle \Theta(g),\psi\rangle_{\mathcal{S}^*-\mathcal{S}} = \int \psi\,\chi_{F_g}\,\mathrm{d}(g\lambda) = \int \psi g\,\mathrm{d}\lambda = \langle \psi, g\rangle_{\mathcal{L}^\infty-\mathcal{L}^1} = \langle \eta_{\mathcal{S}}(g),\psi\rangle_{\mathcal{S}^*-\mathcal{S}}.$$

For $f \in \mathcal{L}^1(G,\omega)$, $\nu \in \mathcal{M}(G,\omega)$ and $\psi \in \mathcal{S}(\omega^{-1})$,

$$\begin{aligned}\langle \Theta(f)\,\square\,\Theta(\nu),\psi\rangle_{\mathcal{S}^*-\mathcal{S}} &= \langle \eta_{\mathcal{S}}(f), \Theta(\nu)\,\square\,\psi\rangle_{\mathcal{S}^*-\mathcal{S}} = \langle \Theta(\nu)\,\square\,\psi, f\rangle_{\mathcal{L}^\infty-\mathcal{L}^1}\\ &= \langle \Theta(\nu), \psi\cdot f\rangle_{\mathcal{S}^*-\mathcal{S}} = \nu\cdot(\psi\cdot f)(e_G) = (\nu\cdot\psi)\cdot f(e_G)\\ &= \langle \nu\cdot\psi, f*\delta_{e_G}\rangle_{\mathcal{L}^\infty-\mathcal{L}^1} = \langle \psi, f*\nu\rangle_{\mathcal{L}^\infty-\mathcal{L}^1}\\ &= \langle \eta_{\mathcal{S}}(f*\nu),\psi\rangle_{\mathcal{S}^*-\mathcal{S}} = \langle \Theta(f*\nu),\psi\rangle_{\mathcal{S}^*-\mathcal{S}},\end{aligned}$$

where we have used (12). Suppose that $\nu_i \to \nu$ with respect to so_l. Writing $\psi \in \mathcal{S}(\omega^{-1})$ as $\psi = \phi\cdot g$ for some $\phi \in \mathcal{S}(\omega^{-1})$ and $g \in \mathcal{L}^1(G,\omega)$,

$$\langle \Theta(\nu_i) - \Theta(\nu),\psi\rangle_{\mathcal{S}^*-\mathcal{S}} = \langle \Theta(g)\,\square\,\Theta(\nu_i-\nu),\phi\rangle_{\mathcal{S}^*-\mathcal{S}} = \langle \Theta(g*(\nu_i-\nu)),\phi\rangle_{\mathcal{S}^*-\mathcal{S}} \to 0.$$

Hence, Θ is so_l-weak* continuous. Let $\mu,\nu \in \mathcal{M}(G,\omega)$ and let (h_i) be a net in $\mathcal{L}^1(G,\omega)$ such that $so_l - \lim h_i = \mu$. Then $so_l - \lim h_i * \nu = \mu*\nu$, so

$$\Theta(\mu)\,\square\,\Theta(\nu) = \mathrm{wk}^* - \lim \Theta(h_i)\,\square\,\Theta(\nu) = \mathrm{wk}^* - \lim \Theta(h_i*\nu) = \Theta(\mu*\nu).$$

Identify the Banach algebra $\mathcal{M}(G,\omega)$ with its copy $\Theta(\mathcal{M}(G,\omega))$ in $\mathcal{S}(\omega^{-1})^*$. Since $\mathcal{S}(\omega^{-1}) = \mathcal{S}(\omega^{-1})\cdot\mathcal{L}^1(G,\omega)$ is a right $\mathcal{M}(G,\omega)$-module, $\mathcal{S}(\omega^{-1})^*$ is a left dual $\mathcal{M}(G,\omega)$-module, and the proof of [7, Lemma 1.4] shows that $\mu\,\square\,n = \mu\cdot n$ for $\mu \in \mathcal{M}(G,\omega)$ and $n \in \mathcal{S}(\omega^{-1})^*$; hence, Θ maps into $Z_t(\mathcal{S}(\omega^{-1})^*)$. For $n \in \mathcal{S}(\omega^{-1})^*$, $\psi \in \mathcal{S}(\omega^{-1})$ and $s \in G$, $(n\,\square\,\psi)(s) = \langle \delta_s, n\,\square\,\psi\rangle = \langle \delta_s\,\square\, n,\psi\rangle = \langle \delta_s\cdot n,\psi\rangle = \langle n, \psi\cdot\delta_s\rangle = \langle n,\psi\cdot s\rangle$. The final line is now easily verified. $\square$

For a Banach algebra A, the space $WAP(A^*)$ of weakly almost periodic functionals on A is a left and right introverted subspace of A^* such that for every

$m, n \in WAP(A^*)^*$, $m \,\square\, n = m \diamond n$ [3, Proposition 3.11]. Thus, $WAP(A^*)^*$ is a dual Banach algebra. Moreover, $WAP(A^*)^*$ satisfies the following universal property [16, Theorem 4.10].

Theorem 2.10 (Runde)**.** *If $\mathfrak{B}$ is a dual Banach algebra and $\varphi\colon A \to \mathfrak{B}$ is a continuous algebra homomorphism, then there is a unique weak*-weak* continuous algebra homomorphism $\varphi_{WAP}\colon WAP(A^*)^* \to \mathfrak{B}$ such that $\varphi_{WAP} \circ \eta_{WAP} = \varphi$.*

Taking $A_\omega = \mathcal{L}^1(G,\omega)$, it follows that the embedding $\mathrm{id}\colon \mathcal{L}^1(G,\omega) \hookrightarrow \mathcal{M}(G,\omega)$ determines a unique weak*-weak* continuous homomorphism $P\colon WAP(A_\omega^*)^* \to \mathcal{M}(G,\omega)$ such that $P \circ \eta_{WAP} = \mathrm{id}$. Letting $P_*\colon C_0(G,\omega^{-1}) \to WAP(A_\omega^*) \preceq \mathcal{L}^\infty(G,\omega^{-1})$ be the predual mapping of P, $\langle P_*\psi, g\rangle_{\mathcal{L}^\infty-\mathcal{L}^1} = \langle P \circ \eta_{WAP}(g), \psi\rangle = \langle \psi, g\rangle_{\mathcal{L}^\infty-\mathcal{L}^1}$ for $\psi \in C_0(G,\omega^{-1})$, $g \in \mathcal{L}^1(G,\omega)$. Hence, $C_0(G,\omega^{-1}) \preceq WAP(A_\omega^*)$. Moreover, by [3, Proposition 3.12] and Lemma 2.3,

$$WAP(A_\omega^*) \preceq (LUC \cap RUC) \times (G,\omega^{-1}).$$

Hence, we have the following immediate corollary to Proposition 2.9.

Corollary 2.11. *The map $\Theta\colon \mathcal{M}(G,\omega) \hookrightarrow WAP(A_\omega^*)^*$, as defined in* (15)*, is a so_l-weak* and so_r-weak* continuous isometric homomorphic embedding that extends $\eta_{WAP}\colon \mathcal{L}^1(G,\omega) \hookrightarrow WAP(A_\omega^*)^*$.*

As shown in [3], $WAP(A_\omega^*)$ may fail to equal $WAP(G,\omega^{-1}) = \{f\colon f/\omega \in WAP(G)\}$. Our final two results are needed in [12]. Corollary 2.12 improves [10, Theorem 5.6] in the case of $\mathcal{L}^1(G,\omega)$:

Corollary 2.12. *Let $\mathfrak{B}$ be a dual Banach algebra, $\varphi\colon \mathcal{L}^1(G,\omega) \to \mathfrak{B}$ a bounded homomorphism. Then there is a unique so_l-weak* and so_r-weak* continuous homomorphic extension $\widetilde{\varphi}\colon \mathcal{M}(G,\omega) \to \mathfrak{B}$ of φ.*

PROOF: Letting $\varphi_{WAP}\colon WAP(A_\omega^*)^* \to \mathfrak{B}$ be the weak*-weak* continuous extension of φ from Theorem 2.10 and $\Theta\colon \mathcal{M}(G,\omega) \hookrightarrow WAP(A_\omega^*)^*$ the so_l/so_r-weak* continuous embedding from Corollary 2.11, $\widetilde{\varphi} := \varphi_{WAP} \circ \Theta$ is the desired extension; uniqueness follows from the so_l-density of $\mathcal{L}^1(G,\omega)$ in $\mathcal{M}(G,\omega)$. $\square$

Corollary 2.13. *Let $\mathfrak{B}$ be a dual Banach algebra, $\varphi\colon \mathcal{M}(G,\omega) \to \mathfrak{B}$ a bounded homomorphism that is so_l-weak* continuous on the unit ball of $\mathcal{M}(G,\omega)$. Then φ is so_l-weak* and so_r-weak* continuous on all of $\mathcal{M}(G,\omega)$.*

PROOF: By Corollary 2.12, the restriction, φ_1, of φ to $\mathcal{L}^1(G,\omega)$ has a so_l/so_r-weak* continuous extension $\widetilde{\varphi_1}\colon \mathcal{M}(G,\omega) \to \mathfrak{B}$. As noted before, $\mathcal{L}^1(G,\omega)_{\|\cdot\|\le 1}$ is so_l-dense in $\mathcal{M}(G,\omega)_{\|\cdot\|\le 1}$, so $\varphi = \widetilde{\varphi_1}$ on $\mathcal{M}(G,\omega)_{\|\cdot\|\le 1}$ and therefore on $\mathcal{M}(G,\omega)$. $\square$

Remark 2.14. Suppose that (H, ω_H) is another weighted locally compact group and $\varphi\colon \mathcal{M}(G,\omega) \to \mathcal{M}(H,\omega_H)$ is a bounded algebra isomorphism. By [7, Lemma 3.3] — which applies, as written, to $\mathcal{M}(G,\omega)$ — φ is so_l-weak* continuous on bounded subsets of $\mathcal{M}(G,\omega)$. By Corollary 2.13, φ is so_l/so_r-weak* continuous on all of $\mathcal{M}(G,\omega)$.

Acknowledgement. The author is grateful to Fereidoun Ghahramani for helpful discussions regarding the topic of this paper.

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(*Received* February 16, 2021, *revised* April 14, 2021)