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REMARKS ON MONOTONICALLY STAR COMPACT SPACES

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Abstract. A space X is said to be monotonically star compact if one assigns to each open cover $\mathcal{U}$ a subspace $s(\mathcal{U}) \subseteq X$, called a kernel, such that $s(\mathcal{U})$ is a compact subset of X and $\mathrm{St}(s(\mathcal{U}),\mathcal{U}) = X$, and if $\mathcal{V}$ refines $\mathcal{U}$ then $s(\mathcal{U}) \subseteq s(\mathcal{V})$, where $\mathrm{St}(s(\mathcal{U}),\mathcal{U}) = \bigcup\{U \in \mathcal{U} \colon U \cap s(\mathcal{U}) \neq \emptyset\}$. We prove the following statements:

(1) The inverse image of a monotonically star compact space under the open perfect map is monotonically star compact.
(2) The product of a monotonically star compact space and a compact space is monotonically star compact.
(3) If X is monotonically star compact space with $e(X) < \omega$, then $A(X)$ is monotonically star compact, where $A(X)$ is the Alexandorff duplicate of space X.

The above statement (2) gives an answer to the question of Song (2015).

Keywords: monotonically star compact; regular closed; perfect; star-compact; covering; star-covering; topological space

MSC 2020: 54D20, 54D30, 54D40

1. Introduction and preliminaries

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let A be a subset of X and $\mathcal{U}$ be a collection of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \colon U \cap A \neq \emptyset\}$. We usually write $\mathrm{St}(x,\mathcal{U}) = \mathrm{St}(\{x\},\mathcal{U})$.

Definition 1.1 ([1], [2], [3], [8]). Let $\mathcal{P}$ be a topological property. A space X is said to be *star* $\mathcal{P}$ if, whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with the property $\mathcal{P}$ such that $\mathrm{St}(A,\mathcal{U}) = X$. The set A is called a star kernel of the cover $\mathcal{U}$.

The term star $\mathcal{P}$ was coined in [1], [2], [3], [8] but certain star properties, specifically those corresponding to "$\mathcal{P}$=finite" and "$\mathcal{P}$=countable" were first studied by

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van Douwen et al. in [7] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [6], [7]. The author believes that the terminology from [1], [2], [3], [8] used in this paper is simple and logical.

Definition 1.2 ([9]). Let $\mathcal{P}$ be a topological property. A space X is said to be *monotonically star* $\mathcal{P}$ if one assigns to each open cover $\mathcal{U}$ a subspace $s(\mathcal{U}) \subseteq X$, called a kernel, such that $s(\mathcal{U})$ has the property $\mathcal{P}$ and $\mathrm{St}(s(\mathcal{U}),\mathcal{U}) = X$, and if $\mathcal{V}$ refines $\mathcal{U}$ then $s(\mathcal{U}) \subseteq s(\mathcal{V})$.

From the definitions above, it is clear that every monotonically star compact space is star compact and every monotonically star finite space is monotonically star compact, but the converse need not be true (see [10], Example 2.1 and [10], Example 2.2).

Song in [10] studied the monotonically star compact spaces and related spaces, and investigated the topological properties of monotonically star compact spaces. He showed that the product of a monotonically star compact space with a countably compact space or with a Lindelöf space is not monotonically star compact and asked whether there exist a monotonically star compact space X and a compact space Y such that $X \times Y$ is not monotonically star compact (see [10], Remark 2.17).

The purpose of this paper is to prove the statements clamed in the abstract, which gives an answer to the above question of Song (see [10]).

All notations and terminology not explained in the paper are given in [5].

2. Results

Theorem 2.1. *Let f be an open perfect map from a space X onto a monotonically star compact space Y. Then X is a monotonically star compact space.*

Proof. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}$ be any refinement of $\mathcal{U}$. Then for each $y \in Y$, there is a finite subset $\mathcal{V}_y$ of $\mathcal{V}$ such that $f^{-1}\{y\} \subset \bigcup \mathcal{V}_y$ and $V \cap f^{-1}\{y\} \neq \emptyset$ for each $V \in \mathcal{V}_y$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, thus for each $V \in \mathcal{V}_y$, there is $U_V \in \mathcal{U}$ such that $V \subset U_V$. Let $\mathcal{U}_y = \{U_V \colon V \in \mathcal{V}_y\}$. Then $\mathcal{U}_y$ is a finite subset of $\mathcal{U}$ such that $\bigcup \mathcal{V}_y \subset \bigcup \mathcal{U}_y$. Thus for each $y \in Y$, $f^{-1}\{y\} \subset \bigcup \mathcal{U}_y$, and $U \cap f^{-1}\{y\} \neq \emptyset$ for each $U \in \mathcal{U}_y$. Choose $V_y = Y \setminus f(X \setminus \bigcup \mathcal{U}_y)$ to be an open neighborhood of y in Y such that

$$f^{-1}(V_y) \subset \bigcup \{U \colon U \in \mathcal{U}_y\},$$

and also suppose that $V_y \subset \bigcap \{f(U) \colon U \in \mathcal{U}_y\}$ as f is open. Let $\mathcal{U}' = \{V_y \colon y \in Y\}$ be an open cover of Y. Since Y is a monotonically star compact space, thus there exists a monotonically star compact operator s_Y such that $Y = \mathrm{St}(s_Y(\mathcal{U}'),\mathcal{U}')$. Let $s_X(\mathcal{U}) = f^{-1}(s_Y(\mathcal{U}'))$. Then $s_X(\mathcal{U})$ is a compact subset of X. Now we have to show that $X \subseteq \mathrm{St}(s_X(\mathcal{U}),\mathcal{U})$. Let $x \in X$, then there exists $y \in Y$ such that $y = f(x) \in V_y \in \mathcal{U}'$ and $V_y \cap s_Y(\mathcal{U}') \neq \emptyset$ since $x \in f^{-1}(V_y) \subset \bigcup\{U\colon U \in \mathcal{U}_y\}$. We can choose $U \in \mathcal{U}_y$ with $x \in U$ such that $V_y \subset f(U)$ and hence $U \cap s_X(\mathcal{U}) \neq \emptyset$. Therefore $x \in \mathrm{St}(s_X(\mathcal{U}),\mathcal{U})$ and thus

$$X = \mathrm{St}(s_X(\mathcal{U}),\mathcal{U}).$$

If $\mathcal{V} \prec \mathcal{U}$, then for each $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subset U$, $\bigcup \mathcal{V}_y \subset \bigcup \mathcal{U}_y$ (see the definition of $\mathcal{U}_y$), which implies $V_y = Y \setminus f(X \setminus \bigcup \mathcal{V}_y) \subset Y \setminus f(X \setminus \bigcup \mathcal{U}_y) = U_y$. Thus for each $V_y \in \mathcal{V}'$ there is some $U_y \in \mathcal{U}'$ such that $V_y \subset U_y$, thus $\mathcal{V}' \prec \mathcal{U}'$, hence $s_Y(\mathcal{U}') \subseteq s_Y(\mathcal{V}')$ and $s_X(\mathcal{U}) \subseteq s_X(\mathcal{V})$, thus s_X is a monotonically star compact operator for X. This completes the proof. □

By Theorem 2.1 above, we have the following corollary.

Corollary 2.2. *If X is the monotonically star compact space and Y is a compact space, then $X \times Y$ is monotonically star compact.*

R e m a r k 2.3. In [10], Song gave Example 2.14 and Example 2.16, which show that the compact space Y in Corollary 2.2 cannot be replaced by a countably compact space or by a Lindelöf space.

From the definition of monotonically star compact spaces, it is clear that if a space X has a compact dense subset, then X is monotonically star compact.

A space X is said to be *metaLindelöf* if every open cover of X has a point-countable open refinement.

Theorem 2.4 ([11], Theorem 3.10)**.** *If X is a T_1 regular star compact metaLindelöf space, then X is compact.*

Since every compact space is monotonically star compact, it is natural to ask under which condition monotonically star compactness implies compactness. The following observation gives a partial answer to this question.

Corollary 2.5. *If X is a T_1 metaLindelöf monotonically star compact regular space, then X is compact.*

P r o o f. Since every monotonically star compact space is star compact, thus the result follows from Theorem 2.4. □

Corollary 2.6. *A paraLindelöf monotonically star compact regular space is compact.*

Corollary 2.7 ([10])**.** *Every clopen subset of a monotonically star compact space is monotonically star compact.*

Now we consider the Alexandorff duplicate $A(X) = X \times \{0,1\}$ of a space X. The basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X and all points $\langle x, 1\rangle \in X \times \{1\}$ are isolated points.

In [10], Example 2.10, Song gave an example of a Tychonoff monotonically star compact space X such that $A(X)$ is not monotonically star compact. Now it is natural to ask under which condition it holds that if X is monotonically star compact, then $A(X)$ is monotonically star compact. The following result gives a partial answer to this question. For showing the result we need a lemma from [4].

Lemma 2.8 ([4])**.** *For T_1-space X, $e(X) = e(A(X))$.*

Theorem 2.9. *If X is a monotonically star compact space with $e(X) < \omega$, then $A(X)$ is monotonically star compact.*

Proof. Let $\mathcal{U}$ be an open cover of $A(X)$. Let $\mathcal{U}' = \{U \cap (X \times \{0\}) \colon U \in \mathcal{U}\}$. Then $\mathcal{U}'$ is an open cover of $X \times \{0\}$. Since $X \times \{0\}$ is homeomorphic to X and X is monotonically star compact, thus $X \times \{0\}$ is monotonically star compact, hence there exists a compact subset $s(\mathcal{U}')$ of X such that $\mathrm{St}(s(\mathcal{U}'), \mathcal{U}') = X \times \{0\}$. Let $s(\mathcal{U})' = A(s(\mathcal{U}'))$. Then $s(\mathcal{U})'$ is a compact subset of $A(X)$. Let $A_{\mathcal{U}} = A(X) \setminus \mathrm{St}(s(\mathcal{U}'), \mathcal{U}')$. Then $A_{\mathcal{U}}$ is a discrete closed subset of $A(X)$. By Lemma 2.8, the set $A_{\mathcal{U}}$ is finite, then $s(\mathcal{U}) = s(\mathcal{U})' \cup A_{\mathcal{U}}$. Then $s(\mathcal{U})$ is a compact subset of $A(X)$ and $\mathrm{St}(s(\mathcal{U}), \mathcal{U}) = A(X)$.

If $\mathcal{V} \prec \mathcal{U}$, then $\mathcal{V}' \prec \mathcal{U}'$, hence $s(\mathcal{U}') \subseteq s(\mathcal{V}')$ and $A_{\mathcal{U}} \subseteq A_{\mathcal{V}}$, thus $s(\mathcal{U}) \subseteq s(\mathcal{V})$. Therefore s is a monotonically star compact operator for X. $\square$

Theorem 2.10 ([10], Theorem 2.12)**.** *If X is a T_1-space and $A(X)$ is a monotonically star compact space, then $e(X) < \omega$.*

We have the following corollary from Theorem 2.9 and Theorem 2.10.

Corollary 2.11. *If X is a T_1 monotonically star compact space, then $A(X)$ is monotonically star compact if and only if $e(X) < \omega$.*

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