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COMPLEX SYMMETRY OF TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACES

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Abstract. We give a concrete description of complex symmetric monomial Toeplitz operators $T_{z^p\overline{z}^q}$ on the weighted Bergman space $A^2(\Omega)$, where Ω denotes the unit ball or the unit polydisk. We provide a necessary condition for $T_{z^p\overline{z}^q}$ to be complex symmetric. When $p, q \in \mathbb{N}^2$, we prove that $T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\Omega)$ if and only if $p_1 = q_2$ and $p_2 = q_1$. Moreover, we completely characterize when monomial Toeplitz operators $T_{z^p\overline{z}^q}$ on $A^2(\mathbb{D}_n)$ are J_U-symmetric with the $n \times n$ symmetric unitary matrix U.

Keywords: complex symmetry; Toeplitz operator; weighted Bergman space

MSC 2020: 47B35, 32A36

1. Introduction

The general study of complex symmetric operators was initiated by Garcia and Putinar in [4] and followed up by many mathematicians over the past decade, see [5], [6], [7] for more details. In particular, weighted shifts play a basic role in exploring the structure of complex symmetric operators. By using Kakutani's unilateral weighted shift operator, Zhu et al. in [17] gave a negative answer to the question of whether or not the class of complex symmetric operators is norm closed. Garcia and Poore in [3] solved this problem via the unilateral shift and its adjoint to construct a different counterexample. Guo et al. in [8] characterized the weighted shifts with nonzero weights to be norm limits of complex symmetric operators.

As a natural extension of the classical weighted shift, monomial Toeplitz operators on weighted Bergman spaces enjoy interesting structure and properties. Inspired by Zhu and Li (see [16]), who completely determined when a weighted shift is complex symmetric, the aim of this paper is to give a characterization of monomial Toeplitz operators on weighted Bergman spaces being complex symmetric. Note

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that $\dim\ker T_{z^p\overline{z}^q} = \dim\ker T_{z^q\overline{z}^p}$ is a necessary condition for $T_{z^p\overline{z}^q}$ to be complex symmetric, see [5], Proposition 1 for more details. When $n = 1$, it is easy to check that $\dim\ker T_{z^p\overline{z}^q} = \dim\ker T_{z^q\overline{z}^p}$ if and only if $p = q$ (i.e., the symbol function is radial); it is often called the *trivial case*. In the higher dimensional case, as we all know, the problem will become more complicated and difficult. Indeed, when $n \geqslant 2$, $\dim\ker T_{z^p\overline{z}^q} = \dim\ker T_{z^q\overline{z}^p} = \infty$ will appear for many different pairs (p, q), thus it is worth to study the complex symmetry of Toeplitz operators $T_{z^p\overline{z}^q}$. Throughout this paper we consider the case $n \geqslant 2$. The reader is referred to [1], [9], [10], [12], [13], [14], [15] for more results about complex symmetric Toeplitz operators.

Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}_n$. For any $t > -1$, the weighted Lebesgue measure dv_t is defined by

$$dv_t(z) = \frac{\Gamma(n+t+1)}{n!\,\Gamma(t+1)}(1-|z|^2)^t dV(z),$$

where $dV(z)$ denotes the standard volume measure on $\mathbb{B}_n$. Another domain in $\mathbb{C}_n$ we consider is the unit polydisk $\mathbb{D}_n$, write

$$dv(z) = \prod_{i=1}^{n} dA(z_i),$$

where dA is the normalized area measure on the unit disk $\mathbb{D}$.

For the sake of simplicity, letting $[n] = \{1, 2, \ldots, n\}$, denote Ω as $\mathbb{B}_n$ or $\mathbb{D}_n$ and let $L^2(\Omega)$ be the square integrable function spaces equipped with the corresponding weighted measure $dv_t(z)$ or $dv(z)$. The weighted Bergman space $A^2(\Omega)$ is the closed subspace of $L^2(\Omega)$ consisting of all holomorphic functions on Ω. Let P be the projection from $L^2(\Omega)$ onto $A^2(\Omega)$. For $\varphi \in L^\infty(\Omega)$, the Toeplitz operator T_φ on $A^2(\Omega)$ is defined by $T_\varphi(f) = P(\varphi f)$ for all $f \in A^2(\Omega)$.

A conjugation on a complex Hilbert space $\mathcal{H}$ is an anti-linear operator $\mathcal{C}\colon \mathcal{H} \to \mathcal{H}$ such that $\mathcal{C}^2 = I$ and $\langle \mathcal{C}f, \mathcal{C}g\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$. A bounded linear operator T on $\mathcal{H}$ is called *complex symmetric* if there exists a conjugation $\mathcal{C}$ such that $T\mathcal{C} = \mathcal{C}T^*$ ($\mathcal{C}T\mathcal{C} = T^*$), we also say that T is a $\mathcal{C}$-symmetric operator.

Before stating our main theorems, we require a few notations.

Definition 1. Given a tuple $p, q \in \mathbb{N}^n$, we say a *pair* (p, q) *is in standard form* if (p, q) satisfies the following conditions:

$$(1) \qquad p - q = (a_1, \ldots, a_{k_1}, -a_{k_1+1}, \ldots, -a_{k_1+k_2}, a_{n-k_3+1}, \ldots, a_n),$$

where $a_i > 0$ for $1 \leqslant i \leqslant k_1 + k_2$, $a_i = 0$ for $n - k_3 + 1 \leqslant i \leqslant n$. Here k_1, k_2, k_3 may take 0.

For any fixed $m_i \in \mathbb{N}$, $i \in [n]$, let

$$I_{m_1,m_2,\ldots,m_n} = \overline{\operatorname{span}}\{z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n} :\ m_1 a_1 \leqslant \alpha_1 < (m_1+1)a_1, \ldots, m_{k_1+k_2} a_{k_1+k_2} \leqslant \alpha_{k_1+k_2} < (m_{k_1+k_2}+1) a_{k_1+k_2}, m_{n-k_3+1} \leqslant \alpha_{n-k_3+1} < m_{n-k_3+1}+1, \ldots, m_n \leqslant \alpha_n < m_n+1\}.$$

It follows that $\bigoplus_{m_1,m_2,\ldots,m_n} I_{m_1,m_2,\ldots,m_n} = A^2(\Omega)$.

Observe that for each pair (p,q) there is a permutation matrix V such that (Vp, Vq) is in standard form. For example, if $p = (1,3,4,6,0,1)$, $q = (3,2,5,4,0,2)$, let

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

then $Vp = (3,6,1,4,1,0)$, $Vq = (2,4,3,5,2,0)$ (i.e., (Vp, Vq) is in standard form). For a pair (p,q) in standard form, write $p = (p_1, \ldots, p_n) = (p_{(1)}, \ldots, p_{(3)})$ with $p_{(j)} = (p_{k_1+\ldots+k_{j-1}+1}, \ldots, p_{k_1+\ldots+k_j}) \in \mathbb{N}^{k_j}$ for $j = 1,2,3$. There is a similar notation for q.

Definition 2. Suppose that $U_1 = (b_{ij})_{1\leqslant i,j\leqslant n}$ is a symmetric permutation of $\operatorname{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\}$, where $\theta_i \in \mathbb{R}$, $i \in [n]$. We say matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ is mutually associated with U_1 if

$$a_{ij} = \begin{cases} 0 & \text{if } b_{ij} = 0, \\ 1 & \text{if } b_{ij} = e^{i\theta_j}, \end{cases}$$

where $j \in \mathbb{R}$, $i \in [n]$.

Definition 3. The *rising factorial* is defined by $x^{(n)} = x(x+1)(x+2)\ldots(x+n-1)$. The *falling factorial* is defined as $x_{(n)} = x(x-1)\ldots(x-n+1)$. Moreover, the rising factorial can be extended to real values of n using the gamma function provided x and $x+n$ are real numbers that are nonnegative integers:

$$x^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

And so can the falling factorial:

$$x_{(n)} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}.$$

The following theorem completely characterizes the complex symmetry of Toeplitz operators $T_{z^p\overline{z}^q}$ on the weighted Bergman space $A^2(\Omega)$.

Theorem 4. *Let $p, q \in \mathbb{N}^2$. Then $T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\Omega)$ if and only if $p_1 = q_2$ and $p_2 = q_1$.*

An exact description of complex symmetry of $T_{z^p\overline{z}^q}$ on $A^2(\Omega)$ (when $n \geqslant 3$) is difficult, though the property of conjugation $\mathcal{C}$ can be given, see Lemma 9. Following this idea, is it natural to ask when $T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\Omega)$ with some special conjugation? We will answer this question for the case of $A^2(\mathbb{D}_n)$.

Consider the anti-linear mapping $J_U\colon A^2(\mathbb{D}_n) \to A^2(\mathbb{D}_n)$, $(J_U f)(z) = \overline{f(U\overline{z})}$, where $f \in A^2(\mathbb{D}_n)$, $z \in \mathbb{D}_n$, U is a symmetric permutation of $\operatorname{diag}\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}\}$ with $\theta_i \in \mathbb{R}$, $i \in [n]$. Since U is symmetric and unitary, it is easy to check that J_U defines a conjugation on $A^2(\mathbb{D}_n)$.

The following theorem first investigates when a Toeplitz operator $T_{z^p\overline{z}^q}$ (the pair (p, q) is in standard form) is complex symmetric for some conjugation J_U.

Theorem 5. *Let $p, q \in \mathbb{N}^n$ and (p, q) is in standard form. Then the following statements are equivalent:*

(a) *$T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\mathbb{D}_n)$ for a conjugation J_U.*

(b) *$p = (p_{(1)}, p_{(2)}, p_{(3)})$, $q = (A_1 p_{(2)}, A_1^t p_{(1)}, p_{(3)})$, where $A_1 \in M_{k_1}(\mathbb{C})$ is a permutation matrix. Moreover, we get all possible matrices U such that $T_{z^p\overline{z}^q}$ is J_U-symmetric with*

$$U = \begin{pmatrix} & U_1 & \\ U_1^t & & \\ & & U_3 \end{pmatrix}, \tag{2}$$

where $U_1 \in M_{k_1}(\mathbb{C})$ is a permutation matrix of $\operatorname{diag}\{e^{i\theta_1}, \dots, e^{i\theta_{k_1}}\}$ with $(\theta_1, \dots, \theta_{k_1}) \in \mathbb{R}^{k_1}$ and $U_3 \in M_{k_3}(\mathbb{C})$ is a symmetric permutation matrix of $\operatorname{diag}\{e^{i\theta_{k_1+k_2+1}}, \dots, e^{i\theta_{k_1+k_2+k_3}}\}$ with $(\theta_{k_1+k_2+1}, \dots, \theta_{k_1+k_2+k_3}) \in \mathbb{R}^{k_3}$.

Note that Proposition 8 shows that $T_{z^p\overline{z}^q}$ is complex symmetric with J_U if and only if $T_{z^{Vp}\overline{z}^{Vq}}$ is complex symmetric with J_{V^tUV}, where V is a permutation matrix such that (Vp, Vq) is in standard form. For general $p, q \in \mathbb{N}^n$, combining Proposition 8 with Theorem 5 we have the following result.

Theorem 6. *Let $p, q \in \mathbb{N}^n$. Then the following statements are equivalent:*

(a) *$T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\mathbb{D}_n)$ for some conjugation J_U.*

(b) $\operatorname{card}\{1 \leqslant i \leqslant n\colon p_i < q_i\} = \operatorname{card}\{1 \leqslant j \leqslant n\colon p_j > q_j\}$ *and* $p = V^t A V q$, *where* V *is an* $n \times n$ *permutation matrix such that* (Vp, Vq) *is in standard form and*

$$A = \begin{pmatrix} & A_1 & \\ A_1^t & & \\ & & A_3 \end{pmatrix},$$

where $A_1 \in M_{k_1}(\mathbb{C})$ *is a permutation matrix and* $A_3 \in M_{k_3}(\mathbb{C})$ *is a symmetric permutation matrix.*

Moreover, in this case the symmetric unitary matrix $V^t U V$ *must be of the form* (2).

In fact, Theorem 6 provides an easy way to determine whether or not $T_{z^p \overline{z}^q}$ is complex symmetric on $A^2(\mathbb{D}_n)$ for a conjugation J_U. For example, let $p = (3,4,2,1,6,4)$, $q = (1,4,4,3,6,2)$, choose a permutation matrix V such that $Vp = (3,4,1,2,4,6)$, $Vq = (1,2,3,4,4,6)$, i.e., (Vp, Vq) is in standard form. Theorem 6 shows that $T_{z^p \overline{z}^q}$ is J_U symmetric if $V^t U V$ is one of the following forms:

$$\begin{pmatrix} 0 & 0 & e^{i\theta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_2} & 0 & 0 \\ e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix}, \begin{pmatrix} 0 & 0 & e^{i\theta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_2} & 0 & 0 \\ e^{i\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\theta_3} \\ 0 & 0 & 0 & 0 & e^{i\theta_4} & 0 \end{pmatrix},$$

where $\theta_i \in \mathbb{R}$ for $i \in [4]$.

2. Some propositions

For $p, q \in \mathbb{N}^n$, we write $p \succeq q$ if $p_i \geqslant q_i$ for all $i \in [n]$, $p \not\succeq q$ if $p_i < q_i$ for some $i \in [n]$, $p \succneqq q$ if $p_i \geqslant q_i$ for all $i \in [n]$ and there exists $j \in [n]$ such that $p_j > q_j$, $|p| = p_1 + \ldots + p_n$.

Let $\{e^\alpha\colon \alpha \in \mathbb{N}^n\}$ be an orthonormal basis on $A^2(\Omega)$. Then

$$e^\alpha = \begin{cases} \Big[\dfrac{\Gamma(n+|\alpha|+t+1)}{\alpha!\,\Gamma(n+t+1)}\Big]^{1/2} z^\alpha & \text{if } \Omega = \mathbb{B}_n, \\ \prod_{i=1}^n \sqrt{\alpha_i+1}\, z^\alpha & \text{if } \Omega = \mathbb{D}_n \end{cases}$$

for any $\alpha \in \mathbb{N}^n$, $z \in \Omega$.

The next lemma will be essential for our main results.

Lemma 7. *Let* $p, q \in \mathbb{N}^n$. *Then on* $A_t^2(\Omega)$ *the following conclusions hold.*

(i) *If* $\Omega = \mathbb{B}_n$ *for each* $\alpha \in \mathbb{N}^n$ *we have*

$$T_{z^p\overline{z}^q}(\mathrm{e}^\alpha) = \begin{cases} C_\alpha^{p,q}\mathrm{e}^{\alpha+p-q}, & \alpha+p \succeq q, \\ 0, & \alpha+p \not\succeq q, \end{cases}$$

where

$$C_\alpha^{p,q} = \frac{(\alpha+p)!}{\Gamma(n+|\alpha|+|p|+t+1)}\sqrt{\frac{\Gamma(n+|\alpha|+t+1)}{\alpha!}}\sqrt{\frac{\Gamma(n+|\alpha|+|p|-|q|+t+1)}{(\alpha+p-q)!}}.$$

(ii) *If* $\Omega = \mathbb{D}_n$ *for each* $\alpha \in \mathbb{N}^n$ *we have*

$$T_{z^p\overline{z}^q}(\mathrm{e}^\alpha) = \begin{cases} H_{p,q}(\alpha)\mathrm{e}^{\alpha+p-q}, & \alpha+p \succeq q, \\ 0, & \alpha+p \not\succeq q, \end{cases}$$

where

$$H_{p,q}(\alpha) = \prod_{i=1}^n \frac{\sqrt{(\alpha_i+1)(\alpha_i+p_i-q_i+1)}}{\alpha_i+p_i+1}.$$

P r o o f. (i) It follows immediately from [2], Lemma 4 and the fact that $\|z^\alpha\|^2_{\mathbb{B}_n} = \alpha!\,\Gamma(n+t+1)/\Gamma(n+|\alpha|+t+1)$.

(ii) The proof is obvious from [11], Lemma 2.1 and the fact that $\|z^\alpha\|^2_{\mathbb{D}_n} = \prod_{i=1}^n 1/(\alpha_i+1)$. □

The next proposition provides a unitary equivalence relation for $\mathcal{C}$-symmetric Toeplitz operators.

Proposition 8. *Let* $p, q \in \mathbb{N}^n$ *and* V *be an* $n\times n$ *permutation matrix. Then* $T_{z^p\overline{z}^q}$ *is complex symmetric on* $A^2(\Omega)$ *with respect to conjugation* $\mathcal{C}$ *if and only if* $T_{z^{Vp}\overline{z}^{Vq}}$ *is complex symmetric with* $C_{V^t}\mathcal{C}C_V$*.*

P r o o f. Let $C_V\colon A^2(\Omega) \to A^2(\Omega)$ be defined by $(C_Vf)(z) = f(Vz)$ for all $f \in A^2(\Omega)$. Since V is a permutation matrix, a direct calculation shows that

$$(Vz)^\alpha = z^{V^t\alpha} \quad \forall\,\alpha \in \mathbb{N}^n.$$

For the case $\Omega = \mathbb{B}_n$, a computation using Lemma 7 (i) shows that

$$\begin{aligned} C_{V^t}T_{z^p\overline{z}^q}C_V(z^\alpha) &= C_{V^t}T_{z^p\overline{z}^q}(z^{V^t\alpha}) \\ &= C_{V^t}\frac{\Gamma(n+|\alpha|+|p|-|q|+t+1)(V^t\alpha+p)!}{\Gamma(n+|\alpha|+|p|+t+1)(V^t\alpha+p-q)!}z^{V^t\alpha+p-q} \\ &= \frac{\Gamma(n+|\alpha|+|p|-|q|+t+1)(V^t\alpha+p)!}{\Gamma(n+|\alpha|+|p|+t+1)(V^t\alpha+p-q)!}z^{\alpha+V(p-q)} \quad \forall\,V^t\alpha+p \succeq q \end{aligned}$$

and

$$T_{z^{Vp}\overline{z}^{Vq}}(z^{\alpha}) = \frac{\Gamma(n+|\alpha|+|p|-|q|+t+1)(\alpha+Vp)!}{\Gamma(n+|\alpha|+|p|+t+1)(\alpha+Vp-Vq)!} z^{\alpha+V(p-q)} \quad \forall\, \alpha+Vp \succeq Vq.$$

Since $V^t\alpha + p \succeq q$ is equivalent to $\alpha + Vp \succeq Vq$, we get that

$$C_{V^t} T_{z^p\overline{z}^q} C_V = T_{z^{Vp}\overline{z}^{Vq}}.$$

Note that C_V is a unitary operator on $A_t^2(\mathbb{B}_n)$ and $C_{V^t}\mathcal{C}C_V$ is also a conjugation on $A_t^2(\mathbb{B}_n)$, the desired result then follows from [4], page 1291. For the case of $A_t^2(\mathbb{D}_n)$, the proof is similar. □

When $p \succeq q \succeq 0$ or $q \succeq p \succeq 0$, it follows from [10], Corollary 9 that if $T_{z^p\overline{z}^q}$ is $\mathcal{C}$-symmetric on $A^2(\Omega)$, then $p = q$. If $p = q$, we know that $T_{z^p\overline{z}^p}$ is complex symmetric. In fact, consider the conjugation $Jf(z) = \overline{f(\overline{z})}$, $f \in A^2(\Omega)$, $z \in \Omega$. It is easy to check that $JT_{z^p\overline{z}^p} z^{\alpha} = T_{z^p\overline{z}^p} J z^{\alpha}$ for all $\alpha \in \mathbb{N}^n$. In the case $p \not\succeq q$ and $p \not\preceq q$ it remains to be found precisely when $T_{z^p\overline{z}^q}$ is complex symmetric. Thus, the use of our work will focus on $p \not\succeq q$ and $p \not\preceq q$ (i.e., $k_1, k_2 \geqslant 1$ in Definition 1) in Section 3.

Lemma 9. *Let $p, q \in \mathbb{N}^n$ and (p, q) be in standard form. If $T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\Omega)$ with respect to a conjugation $\mathcal{C}$, then we have*

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,\dots,m_n}\Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}$$

for any $l_1, l_2 \in \mathbb{N}$.

P r o o f. For the sake of convenience denote $T_{z^p\overline{z}^q} = T$, $T_{\overline{z}^p z^q} = T^*$. By Lemma 7, we obtain

$$T(I_{m_1,m_2,\dots,m_n}) = \begin{cases} I_{m_1+1,\dots,m_{k_1}+1,m_{k_1+1}-1,\dots,m_{k_1+k_2}-1,m_{k_1+k_2+1},\dots,m_n}, & m_{k_1+1},\dots,m_{k_1+k_2} > 0,\\ 0, & m_{k_1+1},\dots, \text{ or } m_{k_1+k_2} = 0 \end{cases}$$

and

$$T^*(I_{m_1,m_2,\dots,m_n}) = \begin{cases} I_{m_1-1,\dots,m_{k_1}-1,m_{k_1+1}+1,\dots,m_{k_1+k_2}+1,m_{k_1+k_2+1},\dots,m_n}, & m_1,\dots,m_{k_1} > 0,\\ 0, & m_1,\dots, \text{ or } m_{k_1} = 0. \end{cases}$$

It follows that

$$\ker T = \bigoplus_{\min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0} I_{m_1,m_2,\dots,m_n}, \quad \ker T^* = \bigoplus_{\min\{m_1,\dots,m_{k_1}\}=0} I_{m_1,m_2,\dots,m_n}.$$

Suppose that T is complex symmetric with conjugation $\mathcal{C}$. Then we have

$$(3) \quad T\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,\dots,m_n} \Bigg) = \mathcal{C}T^*\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n} \Bigg) = 0$$

and

$$(4) \quad T^*\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n} \Bigg) = \mathcal{C}T\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n} \Bigg) = 0.$$

By (3) and (4), we get

$$(5) \quad \mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n} \Bigg) \subset \ker T \cap \ker T^* = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n}.$$

Combining (5) with the property that $\mathcal{C}^2 = I$, we conclude that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n} \Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=0 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=0}} I_{m_1,m_2,\dots,m_n}.$$

Let $L \in \mathbb{N}$ be fixed and, in order to use induction, assume that

$$(6) \quad \mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n} \Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2 \\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}$$

for any $l_1 + l_2 \leqslant L$. We seek to show that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) = \bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_2 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\ldots,m_n}$$

for any $l_1 + l_2 = L + 1$.

Let $l_1, l_2 \in \mathbb{N}$ be fixed and note that $T\mathcal{C} = \mathcal{C}T^*$ implies $T^{l+1}\mathcal{C} = \mathcal{C}(T^*)^{l+1}$ for any $l \in \mathbb{N}$. Some calculations give that

$$T^{l_1+1}\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) = \mathcal{C}(T^*)^{l_1+1}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) = 0$$

and

$$(T^*)^{l_2+1}\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) = \mathcal{C}T^{l_2+1}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) = 0.$$

Therefore, we deduce that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) \subset \ker T^{l_1+1} \cap \ker(T^*)^{l_2+1} = \bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}\leqslant l_2 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}\leqslant l_1}} I_{m_1,m_2,\ldots,m_n}. \tag{7}$$

Thus, for any l_1, l_2 with $l_1 + l_2 = L + 1$, we get

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}=l_1 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\ldots,m_n} \Bigg) \subset \bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}\leqslant l_2 \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}\leqslant l_1}} I_{m_1,m_2,\ldots,m_n} \subset \bigoplus_{\substack{\min\{m_1,\ldots,m_{k_1}\}+ \\ \min\{m_{k_1+1},\ldots,m_{k_1+k_2}\}\leqslant L+1}} I_{m_1,m_2,\ldots,m_n}. \tag{8}$$

Observe that

$$\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) \perp \Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}+\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}\leqslant L}} I_{m_1,m_2,\dots,m_n}\Bigg)$$

for any l_1, l_2 with $l_1 + l_2 = L + 1$. Since $\mathcal{C}$ preserves orthogonality, we have

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) \perp \mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}+\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}\leqslant L}} I_{m_1,m_2,\dots,m_n}\Bigg),$$

then by the induction hypothesis (6) we have

(9)
$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) \perp \Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}+\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}\leqslant L}} I_{m_1,m_2,\dots,m_n}\Bigg).$$

By (8) and (9), we get

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) \subset \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}+\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=L+1}} I_{m_1,m_2,\dots,m_n}.$$

Now by using (7), we obtain that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) \subset \Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}+\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=L+1}} I_{m_1,m_2,\dots,m_n}\Bigg)$$
$$\cap \Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}\leqslant l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}\leqslant l_1}} I_{m_1,m_2,\dots,m_n}\Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}.$$

Similarly, we deduce that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}\Bigg) \subset \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}$$

for any l_1, l_2 with $l_1 + l_2 = L + 1$. Thus,

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}$$

for any l_1, l_2 with $l_1 + l_2 = L + 1$. Hence, an induction argument shows that

$$\mathcal{C}\Bigg(\bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_1\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_2}} I_{m_1,m_2,\dots,m_n}\Bigg) = \bigoplus_{\substack{\min\{m_1,\dots,m_{k_1}\}=l_2\\ \min\{m_{k_1+1},\dots,m_{k_1+k_2}\}=l_1}} I_{m_1,m_2,\dots,m_n}$$

for any $l_1, l_2 \in \mathbb{N}$. This completes the proof. □

3. Proofs of main results

This entire section is devoted to the proofs of Theorems 4 and 5.

3.1. Proof of Theorem 4. Now we are ready to prove Theorem 4 for the case of the unit ball $\mathbb{B}_n$.

Proof. Since $p, q \in \mathbb{N}^2$, $p \not\succeq q$ and $q \not\succeq p$, by Proposition 8 there is no loss of generality in assuming that $p_1 - q_1 > 0$, $q_2 - p_2 > 0$ and we proceed under this assumption. Suppose that $T_{z^p\overline{z}^q}$ is complex symmetric with conjugation $\mathcal{C}$. By Lemma 9 we have

$$\mathcal{C}I_{l_1,l_2} = I_{l_2,l_1} \quad \forall\, l_1, l_2 \in \mathbb{N}.$$

Let $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}\setminus\{0\}$ be fixed and arbitrarily choose an $\mathrm{e}^\alpha \in I_{l_1,l_2}$. By Lemma 7 we have $T_{z^p\overline{z}^q}\mathrm{e}^\alpha = C_\alpha^{p,q}\mathrm{e}^{\alpha+p-q}$. This gives that $\|T_{z^p\overline{z}^q}\mathrm{e}^\alpha\|^2 = (C_\alpha^{p,q})^2$. Hence, we have

$$\|T_{z^p\overline{z}^q}|_{I_{l_1,l_2}}\|^2 = \max\{(C_\alpha^{p,q})^2 \colon\ l_1a_1 \leqslant \alpha_1 < (l_1+1)a_1,\ l_2a_2 \leqslant \alpha_2 < (l_2+1)a_2\}.$$

Similarly, we obtain

$$\|T_{z^q\overline{z}^p}|_{I_{l_2,l_1}}\|^2 = \max\{(C_\alpha^{q,p})^2 \colon\ l_2a_1 \leqslant \alpha_1 < (l_2+1)a_1,\ l_1a_2 \leqslant \alpha_2 < (l_1+1)a_2\}.$$

Since $\mathcal{C}T_{z^p\overline{z}^q} = T_{z^q\overline{z}^p}\mathcal{C}$ and $\mathcal{C}$ is isometric, it holds that

$$\|T_{z^p\overline{z}^q}|_{I_{l_1,l_2}}\|^2 = \|T_{z^q\overline{z}^p}|_{I_{l_2,l_1}}\|^2 \quad \forall\, l_1 \in \mathbb{N},\ l_2 \in \mathbb{N}\setminus\{0\}. \tag{10}$$

Let

$$F(\alpha) = \frac{\prod_{i=1}^2 \alpha_{i_{(p_i)}}\alpha_{i_{(q_i)}}}{(2+|\alpha|+t)_{(|p|)}(2+|\alpha|+t)_{(|q|)}}, \tag{11}$$

where $\alpha_i \geqslant p_i$, $\alpha_i \geqslant q_i$, $i = 1, 2$. Then we have $F(\alpha+p) = (C_\alpha^{p,q})^2$, $F(\alpha+q) = (C_\alpha^{q,p})^2$.

Set $S(\alpha) = \prod\limits_{i=1}^{2} \alpha_{i\,(p_i)}\alpha_{i\,(q_i)}$ and $Q(\alpha) = (2+|\alpha|+t)_{(|p|)}(2+|\alpha|+t)_{(|q|)}$. Obviously, they are two polynomials in α_1, α_2 and

$$\frac{\partial F(\alpha)}{\partial \alpha_i} = \frac{(\partial S(\alpha)/\partial\alpha_i)Q(\alpha) - S(\alpha)(\partial Q(\alpha)/\partial\alpha_i)}{Q(\alpha)^2}, \quad i = 1, 2.$$

Note that the numerator of $\partial F(\alpha)/\partial\alpha_i$ is also a polynomial. Then we have $\partial F(\alpha)/\partial\alpha_i \geqslant 0$ or $\partial F(\alpha)/\partial\alpha_i \leqslant 0$ for α_i large enough, i.e., $F(\alpha)$ is a monotone function of α_i for α_i large enough, where $i = 1, 2$.

Next we will break the discussion into four cases.

Case 1: For α_1, α_2 large enough, the function $F(\alpha)$ is monotone increasing in α_1, α_2, respectively. It follows from (10) that $\|T_{z^p\overline{z}^q}|_{I_{l_1,l_2}}\|^2 = \|T_{z^q\overline{z}^p}|_{I_{l_2,l_1}}\|^2$ for all $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}\setminus\{0\}$, i.e.,

$$\begin{aligned}\max\{(C_\alpha^{p,q})^2\colon\ & l_1a_1 \leqslant \alpha_1 < (l_1+1)a_1,\, l_2a_2 \leqslant \alpha_2 < (l_2+1)a_2\}\\ &= \max\{(C_\alpha^{q,p})^2\colon\ l_2a_1 \leqslant \alpha_1 < (l_2+1)a_1,\, l_1a_2 \leqslant \alpha_2 < (l_1+1)a_2\}.\end{aligned}$$

Note that $F(\alpha+p) = (C_\alpha^{p,q})^2$, $F(\alpha+q) = (C_\alpha^{q,p})^2$ and the function $F(\alpha)$ is monotone increasing in α_1, α_2 when α_1, α_2 are large enough. Thus, we have

$$F(l_1a_1+p_1, l_2a_2+p_2) = F(l_2a_1+q_1, l_1a_2+q_2) \tag{12}$$

and

$$\begin{aligned}F((l_1+1)a_1-1+p_1,\,&(l_2+1)a_2-1+p_2)\\ &= F((l_2+1)a_1-1+q_1,(l_1+1)a_2-1+q_2)\end{aligned} \tag{13}$$

for any l_1, l_2 large enough. Since $F(\alpha) = S(\alpha)/Q(\alpha)$, where $S(\alpha)$ and $Q(\alpha)$ are two polynomials in α_1, α_2, we can deduce that (12) and (13) hold for any $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}\setminus\{0\}$.

Combining (11) with (12), we have

$$\begin{aligned}&\frac{\prod_{i=1}^{2}[(l_ia_i+1)^{(p_i)}(l_ia_i+p_i)_{(q_i)}]}{(3+l_1a_1+l_2a_2+t)^{(|p|)}(2+l_1a_1+l_2a_2+t+|p|)_{(|q|)}}\\ &\qquad= \frac{(l_2a_1+1)^{(q_1)}(l_1a_2+1)^{(q_2)}(l_2a_1+q_1)_{(p_1)}(l_1a_2+q_2)_{(p_2)}}{(3+l_2a_1+l_1a_2+t)^{(|q|)}(2+l_2a_1+l_1a_2+t+|q|)_{(|p|)}}\end{aligned}$$

for any $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N}\setminus\{0\}$.

Comparing the highest degree of l_1 and the coefficients of the highest degree of l_1 on each side of the above equation, respectively, we have

$$p_1 + q_1 = p_2 + q_2 \tag{14}$$

and

$$\frac{[(l_2a_2+p_2)!]^2}{a_1^{p_2+q_2}(l_2a_2)!\,(l_2a_2+p_2-q_2)!} = \frac{[(l_2a_1+q_1)!]^2}{a_2^{p_1+q_1}(l_2a_1)!\,(l_2a_1+q_1-p_1)!}$$

for any $l_2 \in \mathbb{N} \setminus \{0\}$ or write

$$\tag{15} \frac{[(l_2a_2 + a_2 + p_2)!]^2}{a_1^{p_2+q_2}(l_2a_2 + a_2)!\,(l_2a_2 + a_2 + p_2 - q_2)!} = \frac{[(l_2a_1 + a_1 + q_1)!]^2}{a_2^{p_1+q_1}(l_2a_1 + a_1)!\,(l_2a_1 + a_1 + q_1 - p_1)!}$$

for any $l_2 \in \mathbb{N}$. By (11) and (13), we see that

$$\frac{\prod_{i=1}^{2}[(l_ia_i + a_i)^{(p_i)}(l_ia_i + a_i + p_i)_{(q_i)}]}{(1 + l_1a_1 + l_2a_2 + a_1 + a_2 + t)^{(|p|)}(l_1a_1 + l_2a_2 + a_1 + a_2 + |p| + t)_{(|q|)}} = \frac{(l_2a_1 + a_1)^{(q_1)}(l_1a_2 + a_2)^{(q_2)}(l_2a_1 + a_1 + q_1)_{(p_1)}(l_1a_2 + a_2 + q_2)_{(p_2)}}{(1 + l_2a_1 + l_1a_2 + a_1 + a_2 + t)^{(|p|)}(l_2a_1 + l_1a_2 + a_1 + a_2 + |q| + t)_{(|p|)}}$$

for any $l_1 \in \mathbb{N}$, $l_2 \in \mathbb{N} \setminus \{0\}$. Then comparing the coefficients of $l_1^{p_2+q_2}$ on each side of the above equation we obtain

$$\tag{16} \frac{[(l_2a_2 + a_2 + p_2 - 1)!]^2}{a_1^{p_2+q_2}(l_2a_2 + a_2 - 1)!\,(l_2a_2 + a_2 + p_2 - q_2 - 1)!} = \frac{[(l_2a_1 + a_1 + q_1 - 1)!]^2}{a_2^{p_1+q_1}(l_2a_1 + a_1 - 1)!}$$

for any $l_2 \in \mathbb{N} \setminus \{0\}$. It follows from (15) and (16) that

$$\frac{(l_2a_2 + a_2 + p_2)^2}{(l_2a_2 + a_2)l_2a_2} = \frac{(l_2a_1 + a_1 + q_1)^2}{(l_2a_1 + a_1)l_2a_1} \quad \forall\, l_2 \in \mathbb{N} \setminus \{0\}.$$

Straightforward computation shows that

$$0 = a_2(l_2a_1 + a_1 + q_1) - a_1(l_2a_2 + a_2 + p_2) = a_2q_1 - a_1p_2 = q_1q_2 - p_1p_2,$$

that is $p_1p_2 = q_1q_2$. Now substituting $q_2 = p_1p_2/q_1$ back into (14), we have

$$0 = p_1q_1 + q_1^2 - p_2q_1 - p_1p_2 = (p_1 + q_1)(q_1 - p_2),$$

which means that $p_1 = q_2$, $p_2 = q_1$.

Case 2: For α_1, α_2 large enough, the function $F(\alpha)$ is monotone decreasing in α_1, α_2, respectively. In this case, by (10) we see that

$$F((l_1+1)a_1 - 1 + p_1, (l_2+1)a_2 - 1 + p_2) = F((l_2+1)a_1 - 1 + q_1, (l_1+1)a_2 - 1 + q_2)$$

and

$$F(l_1a_1 + p_1, l_2a_2 + p_2) = F(l_2a_1 + q_1, l_1a_2 + q_2)$$

for any l_1, l_2 large enough. Just like in Case 1, we also get $p_1 = q_2$, $p_2 = q_1$.

Case 3: For α_1, α_2 large enough, the function $F(\alpha)$ is monotone increasing in α_1 and $F(\alpha)$ is monotone decreasing in α_2. Then (10) implies that

$$F((l_1+1)a_1-1+p_1, l_2a_2+p_2) = F((l_2+1)a_1-1+q_1, l_1a_2+q_2)$$

and

$$F(l_1a_1+p_1, (l_2+1)a_2-1+p_2) = F(l_2a_1+q_1, (l_1+1)a_2-1+q_2)$$

for any l_1, l_2 large enough. Using a similar argument as in Case 1, one can prove that $p_1 = q_2$, $p_2 = q_1$.

Case 4: For α_1, α_2 large enough, the function $F(\alpha)$ is monotone decreasing in α_1 and $F(\alpha)$ is monotone increasing in α_2. In this case, we have

$$F(l_1a_1+p_1, (l_2+1)a_2-1+p_2) = F(l_2a_1+q_1, (l_1+1)a_2-1+q_2)$$

and

$$F((l_1+1)a_1-1+p_1, l_2a_2+p_2) = F((l_2+1)a_1-1+q_1, l_1a_2+q_2)$$

for any l_1, l_2 large enough. So it follows from Case 3 that $p_1 = q_2$, $p_2 = q_1$.

Conversely, if $p_1 = q_2$, $p_2 = q_1$, then consider the conjugation $J_U f(z) = \overline{f(U\overline{z})}$ for all $f \in A_t^2(\mathbb{B}_n)$, $z \in \mathbb{B}^n$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is easy to check that $J_U T_{z^p\overline{z}^q} z^\alpha = T_{z^q\overline{z}^p} J_U z^\alpha$ for all $\alpha \in \mathbb{N}^n$, thus $T_{z^p\overline{z}^q}$ is complex symmetric. $\square$

Now we will give the proof of Theorem 4 in the case of the unit polydisk.

P r o o f. First we show that $H^2_{p,q}(\alpha)$, $H^2_{q,p}(\alpha)$ are monotone increasing functions of α_1, α_2 when α_1, α_2 are large enough. In fact,

$$\frac{\partial H^2_{p,q}(\alpha)}{\partial \alpha_1} = \frac{(\alpha_2+1)(\alpha_2+p_2-q_2+1)[(p_1+q_1)\alpha_1+p_1^2+p_1-p_1q_1+q_1]}{(\alpha_2+p_2+1)^2(\alpha_1+p_1+1)^3},$$
$$\frac{\partial H^2_{p,q}(\alpha)}{\partial \alpha_2} = \frac{(\alpha_1+1)(\alpha_1+p_1-q_1+1)[(p_2+q_2)\alpha_2+p_2^2+p_2-p_2q_2+q_2]}{(\alpha_1+p_1+1)^2(\alpha_2+p_2+1)^3}.$$

Thus, $\partial H^2_{p,q}(\alpha)/\partial\alpha_i \geqslant 0$ for α_i large enough, where $i = 1, 2$, therefore, $H^2_{p,q}(\alpha)$ has the desired property, the result for $H^2_{q,p}(\alpha)$ can be proved in a similar manner.

Suppose that $T_{z^p\overline{z}^q}$ is $\mathcal{C}$-symmetric. Applying the same reasoning as in Theorem 4 (see the first paragraph and Case 1 of the proof of Theorem 4) for any $l_1, l_2 \in \mathbb{N}\setminus\{0\}$ we have

$$H^2_{p,q}(l_1a_1, l_2a_2) = H^2_{q,p}(l_2a_1, l_1a_2) \tag{17}$$

and

$$H^2_{p\cdot q}((l_1+1)a_1-1, (l_2+1)a_2-1) = H^2_{q,p}((l_2+1)a_1-1, (l_1+1)a_2-1). \tag{18}$$

From Lemma 7 (ii) and (18) we obtain that

$$\frac{[(l_1+1)a_1][(l_1+2)a_1][(l_2+1)a_2]l_2a_2}{[(l_1+1)a_1+p_1]^2[(l_2+1)a_2+p_2]^2} = \frac{[(l_2+1)a_1]l_2a_1[(l_1+1)a_2][(l_1+2)a_2]}{[(l_2+1)a_1+q_1]^2[(l_1+1)a_2+q_2]^2} \quad \forall\, l_1, l_2 \in \mathbb{N}\setminus\{0\}.$$

After eliminating the molecules, this simplifies to

$$\begin{aligned} 0 &= [(l_1+1)a_1+p_1][(l_2+1)a_2+p_2] - [(l_2+1)a_1+q_1][(l_1+1)a_2+q_2] \\ &= l_1(a_1p_2-a_2q_1) + l_2(a_2p_1-a_1q_2) + a_1p_2 - a_2q_1 + a_2p_1 - a_1q_2 + p_1p_2 - q_1q_2 \\ &= l_1(p_1p_2-q_1q_2) + l_2(q_1q_2-p_1p_2) + p_1p_2 - q_1q_2 \quad \forall\, l_1, l_2 \in \mathbb{N}\setminus\{0\}, \end{aligned}$$

which yields that $p_1p_2 = q_1q_2$.

On the other hand, Lemma 7 (ii), along with (17), gives that

$$\frac{(l_1a_1+1)(l_1a_1+a_1+1)(l_2a_2+1)(l_2a_2-a_2+1)}{(l_1a_1+p_1+1)^2(l_2a_2+p_2+1)^2} = \frac{(l_2a_1+1)(l_2a_1-a_1+1)(l_1a_2+1)(l_1a_2+a_2+1)}{(l_2a_1+q_1+1)^2(l_1a_2+q_2+1)^2} \quad \forall\, l_1, l_2 \in \mathbb{N}\setminus\{0\}.$$

By eliminating the denominators, we have

$$\begin{aligned} &[l_1^2a_1^2 + (2a_1+a_1^2)l_1 + 1 + a_1][l_1^2a_2^2 + 2(q_2+1)a_2l_1 + (q_2+1)^2] \\ &\quad \times [l_2^2a_2^2 + (2a_2-a_2^2)l_2 + 1 - a_2][l_2^2a_1^2 + 2(q_1+1)a_1l_2 + (q_1+1)^2] \\ &= [l_1^2a_2^2 + (2a_2+a_2^2)l_1 + 1 + a_2][l_1^2a_1^2 + 2(p_1+1)a_1l_1 + (p_1+1)^2] \\ &\quad \times [l_2^2a_1^2 + (2a_1-a_1^2)l_1 + 1 - a_1][l_2^2a_2^2 + 2(p_2+1)a_2l_1 + (p_2+1)^2]. \end{aligned} \tag{19}$$

Comparing the coefficients of l_1^4 and l_2^4 in (19) we obtain that

$$(1-a_1)(p_2+1)^2 = (1-a_2)(q_1+1)^2 \tag{20}$$

and

$$(1+a_1)(q_2+1)^2 = (1+a_2)(p_1+1)^2, \tag{21}$$

where $a_1 = p_1 - q_1$, $a_2 = q_2 - p_2$.

Subtracting Equations (20) and (21) we get

$$(1-p_1+q_1)(p_2+1)^2 - (1+p_1-q_1)(q_2+1)^2 = (1-q_2+p_2)(q_1+1)^2 - (1+q_2-p_2)(p_1+1)^2.$$

By substituting $q_2 = p_1p_2/q_1$ into this equation and simplifying, we have

$$\begin{aligned}
0 = &-p_1^3p_2^2 + p_1^3p_2q_1 + p_1^2p_2^2q_1 - p_1^2p_2q_1^2 - p_1p_2^2q_1^2 + p_1p_2q_1^3 \\
&+ p_2^2q_1^3 - p_2q_1^4 - p_1^2p_2^2 + p_1^2q_1^2 + p_2^2q_1^2 - q_1^4 \\
= {}& p_1^3p_2(q_1 - p_2) + p_1^2p_2q_1(p_2 - q_1) - p_1p_2q_1^2(p_2 - q_1) \\
&+ p_2q_1^3(p_2 - q_1) + p_1^2(q_1^2 - p_2^2) + q_1^2(p_2^2 - q_1^2) \\
= {}& p_1^2p_2(q_1 - p_2)(p_1 - q_1) - p_2q_1^2(p_2 - q_1)(p_1 - q_1) + (q_1^2 - p_2^2)(p_1^2 - q_1^2) \\
= {}& (q_1 - p_2)(p_1 - q_1)(p_1^2p_2 + p_2q_1^2 + p_1q_1 + q_1^2 + p_1p_2 + p_2q_1).
\end{aligned}$$

Since $(p_1 - q_1)(p_1^2p_2 + p_2q_1^2 + p_1q_1 + q_1^2 + p_1p_2 + p_2q_1) \neq 0$, it then follows that

$$p_1 = q_2, \quad p_2 = q_1.$$

Conversely, the idea of the proof is the same as in the unit ball, so we omit the details. □

3.2. Proof of Theorem 5.

Proof. First we show (a) implies (b). Suppose that $T_{z^p\overline{z}^q}$ is J_U-symmetric, where U is a symmetric permutation of $\operatorname{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\}$, where $\theta_i \in \mathbb{R}$, $i \in [n]$, then we have

$$J_U T_{z^p\overline{z}^q} z^\alpha = T_{z^q\overline{z}^p} J_U z^\alpha \quad \forall\, \alpha \in \mathbb{N}^n.$$

Notice that $J_U z^\alpha = e^{i\theta\cdot\alpha} z^{A\alpha}$, where matrix A is mutually associated with U.

Using Lemma 7 (ii), some elementary calculations give us that

$$\text{(22)}\qquad \begin{aligned}
J_U T_{z^p\overline{z}^q} z^\alpha &= J_U \left(\prod_{i=1}^n \frac{\alpha_i + p_i - q_i + 1}{\alpha_i + p_i + 1}\right) z^{\alpha+p-q} \\
&= e^{-i\theta\cdot(\alpha+p-q)} \left(\prod_{i=1}^n \frac{\alpha_i + p_i - q_i + 1}{\alpha_i + p_i + 1}\right) z^{A(\alpha+p-q)} \quad \forall\, \alpha + p \succeq q
\end{aligned}$$

and

$$\text{(23)}\qquad \begin{aligned}
T_{z^q\overline{z}^p} J_U z^\alpha &= e^{-i\theta\cdot\alpha} T_{z^q\overline{z}^p} z^{A\alpha} \\
&= e^{-i\theta\cdot\alpha} \left(\prod_{i=1}^n \frac{\alpha_{\sigma(i)} + q_i - p_i + 1}{\alpha_{\sigma(i)} + q_i + 1}\right) z^{A\alpha+q-p} \quad \forall\, A\alpha + q \succeq p,
\end{aligned}$$

where $A\alpha = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, σ is a permutation of $[n]$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_{(2)} = q_{(2)} - p_{(2)}$, $\alpha_j = 0$ for $j \in [n] \setminus \{k_1+1, \ldots, k_1+k_2\}$, this gives that $\alpha + p - q = (p_{(1)} - q_{(1)}, 0, \ldots, 0)$. Comparing the degree of z in (22) and (23), we get that

$$A(p_{(1)} - q_{(1)}, p_{(2)} - q_{(2)}, 0, \ldots, 0) = (q_{(1)} - p_{(1)}, q_{(2)} - p_{(2)}, 0, \ldots, 0). \tag{24}$$

From Equation (24) we see that

$$\prod_{i=1}^{n} (\alpha_i + p_i - q_i + 1) = \prod_{i=1}^{n} (\alpha_{\sigma(i)} + q_i - p_i + 1). \tag{25}$$

Let $V_1 = (p_{(1)} - q_{(1)}, p_{(2)} - q_{(2)}, 0, \ldots, 0)$, $V_2 = (q_{(1)} - p_{(1)}, q_{(2)} - p_{(2)}, 0, \ldots, 0)$, note that A is a permutation matrix, thus V_2 is an elementary row transformation of V_1, then we have

$$\operatorname{card}\{1 \leqslant i \leqslant n\colon\ p_i < q_i\} = \operatorname{card}\{1 \leqslant j \leqslant n\colon\ p_j > q_j\}.$$

Note that A is a symmetric permutation matrix by Definition 2, this, together with (24), implies that A must have the following form:

$$A = \begin{pmatrix} & A_1 & \\ A_1^t & & \\ & & A_3 \end{pmatrix}, \tag{26}$$

where $A_1 \in M_{k_1}(\mathbb{C})$ is a permutation matrix and $A_3 \in M_{k_3}(\mathbb{C})$ is a symmetric permutation matrix. Moreover, we get all possible forms of U such that $T_{z^p\bar{z}^q}$ is J_U-symmetric:

$$U = \begin{pmatrix} & U_1 & \\ U_1^t & & \\ & & U_3 \end{pmatrix},$$

where $U_1 \in M_{k_1}(\mathbb{C})$ is a permutation of $\operatorname{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_{k_1}}\}$ with $(\theta_1, \ldots, \theta_{k_1}) \in \mathbb{R}^{k_1}$ and $U_3 \in M_{k_3}(\mathbb{C})$ is a symmetric permutation of $\operatorname{diag}\{e^{i\theta_{k_1+k_2+1}}, \ldots, e^{i\theta_{k_1+k_2+k_3}}\}$ with $(\theta_{k_1+k_2+1}, \ldots, \theta_{k_1+k_2+k_3}) \in \mathbb{R}^{k_3}$.

From (24) and (26), we obtain that $A_1^t(p_{(1)} - q_{(1)}) = q_{(2)} - p_{(2)}$. In addition, by (26) we have $A_1^t\theta_{(1)} = \theta_{(2)}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_{(2)} \succeq q_{(2)} - p_{(2)}$, $\alpha_j = 0$ for $j \in [n] \setminus \{k_1+1, \ldots, k_1+k_2\}$, this gives that $\alpha + p \succeq q$ and $A\alpha + q \succeq p$. Since $A_1^t(p_{(1)} - q_{(1)}) = q_{(2)} - p_{(2)}$, we have

$$\begin{aligned} e^{-i\theta\cdot(\alpha+p-q)} &= e^{-i\theta\cdot\alpha + i\theta_{(1)}\cdot(q_{(1)}-p_{(1)}) + iA_1\theta_{(2)}\cdot(A_1(q_{(2)}-p_{(2)}))} \\ &= e^{-i\theta\cdot\alpha + i\theta_{(1)}\cdot(q_{(1)}-p_{(1)}) + i\theta_{(1)}\cdot(p_{(1)}-q_{(1)})} = e^{-i\theta\cdot\alpha}. \end{aligned}$$

Then comparing the coefficients of $z^{A\alpha+q-p}$ in (22) and (23), we see that

$$\prod_{i=1}^{n}(\alpha_i+p_i+1)=\prod_{i=1}^{n}(\alpha_{\sigma(i)}+q_i+1) \tag{27}$$

holds for an infinite $\alpha\in\mathbb{N}^n$. Observe that both sides of (27) are polynomials in α_i, $i\in[n]$, thus Equation (27) holds for any $\alpha\in\mathbb{N}^n$, then it is clear that

$$q_{(1)}=A_1p_{(2)},\quad q_{(2)}=A_1^tp_{(1)},$$

thus, we get $p=(p_{(1)},p_{(2)},p_{(3)})$, $q=(A_1p_{(2)},A_1^tp_{(1)},p_{(3)})$, as desired.

Conversely, if $p=(p_{(1)},p_{(2)},p_{(3)})$, $q=(A_1p_{(2)},A_1^tp_{(1)},p_{(3)})$, where A_1 is a $k_1\times k_1$ permutation matrix.

Let

$$A=\begin{pmatrix} & A_1 & \\ A_1^t & & \\ & & A_3\end{pmatrix},$$

where $A_1\in M_{k_1}(\mathbb{C})$ is a permutation matrix and $A_3\in M_{k_3}(\mathbb{C})$ is a symmetric permutation matrix. It is easy to check that $\alpha+p\succeq q$ is equivalent to $A\alpha+q\succeq p$.

Now let U be a symmetric permutation of $\operatorname{diag}\{e^{i\theta_1},e^{i\theta_2},\dots,e^{i\theta_n}\}$ such that the above matrix A is mutually associated with U, where $\theta_i\in\mathbb{R}$, $i\in[n]$. From the sufficiency part of the proof of the theorem,

$$J_UT_{z^p\overline{z}^q}z^\alpha=T_{z^q\overline{z}^p}J_Uz^\alpha\quad\forall\,\alpha\in\mathbb{N}^n.$$

This completes the proof. □

The results in this paper lead us to consider the following problem:

Open question. Let $p,q\in\mathbb{N}^n$ with $n\geqslant 3$. If $T_{z^p\overline{z}^q}$ is complex symmetric on $A^2(\Omega)$ with respect to a conjugation $\mathcal{C}$, what is the relationship between p and q?

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