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MAXIMUM BIPARTITE SUBGRAPHS IN H-FREE GRAPHS

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Abstract. Given a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. Given a fixed graph H and a positive integer m, let $f(m,H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no copy of H. In this paper we prove that $f(m,\theta_{k,s}) \geqslant \frac{1}{2}m + \Omega(m^{(2k+1)/(2k+2)})$, which extends the results of N. Alon, M. Krivelevich, B. Sudakov. Write K_k' and $K_{t,s}'$ for the subdivisions of K_k and $K_{t,s}$. We show that $f(m,K_k') \geqslant \frac{1}{2}m + \Omega(m^{(5k-8)/(6k-10)})$ and $f(m,K_{t,s}') \geqslant \frac{1}{2}m + \Omega(m^{(5t-1)/(6t-2)})$, improving a result of Q. Zeng, J. Hou. We also give lower bounds on wheel-free graphs. All of these contribute to a conjecture of N. Alon, B. Bollobás, M. Krivelevich, B. Sudakov (2003).

Keywords: bipartite subgraph; H-free; partition

MSC 2020: 05C35, 05C70

1. Introduction

The Max-Cut problem asks for the largest bipartite subgraph of a graph. This problem has been widely studied in both computer science and combinatorics. Given a graph G, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. Given a positive integer m, let $f(m)$ denote the minimum value of $f(G)$, as G ranges over all graphs with m edges. In combinatorics, it is an important problem to estimate lower bounds on $f(m)$ in terms of m. It is well-known that $f(m) \geqslant \frac{1}{2}m$. Answering a conjecture of Erdős, Edwards in [8], [9] proved that

$$f(m) \geqslant \frac{m}{2} + \frac{\sqrt{8m+1}-1}{8} \tag{1}$$

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for every m and noted that equality holds for complete graphs with the odd order. More information on $f(m)$ can be found in [1], [3], [5], [6], [17].

One class which has drawn most of the attention is that of H-free graphs. Given a graph H, let $f(m, H)$ denote the minimum possible cardinality of $f(G)$, as G ranges over all graphs on m edges that contain no copy of H. The problem of estimating the error term more precisely is not easy, even for the relatively simple graph $H = C_3$. After a series of papers by various researchers (see [16], [18]), Alon in [1] proved that $f(m, C_3) = \frac{1}{2}m + \Theta(m^{4/5})$ for all m. Furthermore, Alon, Krivelevich and Sudakov in [4] studied the case that H is an even cycle and showed that for any integer $k \geqslant 2$, there is a positive constant $c(k)$ such that

$$f(m, C_{2k}) \geqslant \frac{m}{2} + c(k)m^{(2k+1)/(2k+2)} \tag{2}$$

for all m, and that this is tight up to the value of $c(k)$ for $k \in \{2, 3, 5\}$. The authors also studied $f(m, H)$ when H is a complete bipartite graph $K_{2,s}$, showing that for any integer $s \geqslant 2$, there is a positive constant $c(s)$ such that

$$f(m, K_{2,s}) \geqslant \frac{m}{2} + c(s)m^{5/6} \tag{3}$$

for all m and that this is tight up to the value of $c(s)$.

Throughout, all graphs are finite, undirected and have no loops or parallel edges. All logarithms are with the natural base e, unless otherwise indicated. We write $\theta_{k,s}$ for the (k, s)-theta-graph, obtained by joining two vertices by s internally vertex-independent paths of length k. Our first result is an extension of (2) and (3) to $\theta_{k,s}$-free graphs, by noting that $\theta_{k,2} = C_{2k}$ and $\theta_{2,s} = K_{2,s}$.

Theorem 1.1. *For any positive integers k and s and for all m, there is a constant $c(k, s) > 0$ such that*

$$f(m, \theta_{k,s}) \geqslant \frac{m}{2} + c(k, s)m^{(2k+1)/(2k+2)}.$$

In addition, Alon, Bollobás, Krivelevich and Sudakov in [2], [4] made a number of intriguing conjectures.

Conjecture 1.2 ([2]). *For any fixed graph H and all m, there exists a positive constant $\varepsilon = \varepsilon(H)$ such that*

$$f(m, H) \geqslant \frac{m}{2} + \Omega(m^{3/4+\varepsilon}).$$

Conjecture 1.3 ([4]). *For any integers $s \geqslant t \geqslant 2$ and all m,*

$$f(m, K_{t,s}) \geqslant \frac{m}{2} + \Omega(m^{(3t-1)/(4t-2)}).$$

We also consider the above conjectures. Note that the ℓ-subdivision of a graph H is the graph obtained from H by replacing the edges of H with internally disjoint paths of length $\ell+1$. When $\ell=1$, we simply talk about the subdivision of H. Our second result is the following theorem.

Theorem 1.4. *Let K'_k ($K'_{t,s}$, respectively) denote the subdivision of K_k ($K_{t,s}$, respectively).*

(i) *For any integer $k \geqslant 3$ and all m, there is a constant $c(k)$ such that*

$$f(m, K'_k) \geqslant \frac{m}{2} + c(k)m^{(5k-8)/(6k-10)}.$$

(ii) *For any integers $s \geqslant t \geqslant 2$ and all m, there is a constant $c(s,t)$ such that*

$$f(m, K'_{t,s}) \geqslant \frac{m}{2} + c(s,t)m^{(5t-1)/(6t-2)}.$$

Remark 1.5. Both (i) and (ii) slightly improve the result of Zeng and Hou in [21]: $f(m,H) \geqslant \frac{1}{2}m + \Omega(m^{5/6})$ for the bipartite graph $H = H[X,Y]$ with the vertex degree at most 2 for each vertex in Y.

Our last result gives some small progress towards Conjecture 1.2 for wheel graphs.

Theorem 1.6. *Let W_r denote the wheel graph obtained by connecting a single vertex to all vertices of a cycle of length r.*

(i) *For any even integer $r \geqslant 4$ and all m, there is a constant $c(r) > 0$ such that*

$$f(m, W_r) \geqslant \frac{m}{2} + c(r)m^{3/4}.$$

(ii) *For any odd integer $r \geqslant 3$ and all m, there is a constant $c'(r) > 0$ such that*

$$f(m, W_r) \geqslant \frac{m}{2} + c'(r)m^{2r/(3r+1)}(\log m)^{(r+1)/(3r+1)}.$$

2. Preliminaries

In this section we collect some lemmas that will be needed. Given a graph G, let $\chi(G)$ and $\alpha(G)$ denote the chromatic number and independence number of G, respectively. The following lower bound on $f(G)$ using the chromatic number plays the key role in our proofs.

Lemma 2.1 ([2]). *Let G be a graph with m edges and the chromatic number at most χ. Then*

$$f(G) \geqslant \frac{\chi+1}{2\chi}m.$$

Lemma 2.1 implies that graphs with small chromatic number must have large bipartite subgraphs. In the proof of Theorem 1.6 (ii), we mainly show that the chromatic number of a W_{2k+1}-free graph is relatively small. Indeed, the chromatic number of a graph is closely related to its independent number. A graph property is called *monotone* if it holds for all subgraphs of a graph which has this property. The next lemma on monotone properties, presented by Jensen and Toft (see [14]), gives an upper bound on the chromatic number of G with respect to $\alpha(G)$.

Lemma 2.2 ([14]). *For $s \geqslant 1$, let $\psi\colon [s,\infty) \to (0,\infty)$ be a positive continuous nondecreasing function. Suppose that $\mathcal{P}$ is a monotone class of graphs such that $\alpha(G) \geqslant \psi(|V(G)|)$ for every $G \in \mathcal{P}$ with $|V(G)| \geqslant s$. Then for every such G with $|V(G)| \geqslant s$,*

$$\chi(G) \leqslant s + \int_s^{|V(G)|} \frac{1}{\psi(x)}\,\mathrm{d}x.$$

In order to bound $\chi(G)$ through Lemma 2.2, we need to bound $\alpha(G)$. Following well-known Turán's lower bound and another two lemmas from [15], [19] are crucial to us.

Lemma 2.3 (Turán's lower bound, [20]). *Let G be a graph on n vertices with average degree at most d. Then*

$$\alpha(G) \geqslant \frac{n}{1+d}.$$

Lemma 2.4 ([15]). *Let G be a graph on n vertices with the average degree at most d. If the average degree of the subgraph induced by the neighbourhood of any vertex is at most a, then*

$$\alpha(G) \geqslant nF_{a+1}(d),$$

where

$$F_a(x) = \int_0^1 \frac{(1-t)^{1/a}}{a+(x-a)t}\,\mathrm{d}t > \frac{\log(x/a)-1}{x} \quad (x > 0).$$

Lemma 2.5 ([19]). *For any fixed integer $k \geqslant 1$, let G be a K_{2k+1}-free graph with n vertices and the average degree $d > \mathrm{e}$. Then there exists a constant $a_k \in (0, \frac{1}{4})$ such that*

$$\alpha(G) \geqslant \frac{a_k n \log d}{d \log\log d}.$$

We also need the following two lemmas, which establish the lower bounds on $f(G)$ for graphs G in terms of different parameters.

Lemma 2.6 ([11]). *Let G be a graph on n vertices with m edges and a positive minimum degree. Then*

$$f(G) \geqslant \frac{m}{2} + \frac{n}{6}.$$

Lemma 2.7 ([1]). *Let $G = (V, E)$ be a graph with m edges. Suppose $U \subset V$ and let G' be the induced subgraph of G on U. If G' has m' edges, then*

$$f(G) \geqslant f(G') + \frac{m - m'}{2}.$$

We end this section with a result of Alon, Krivelevich and Sudakov (see [4]), which provides a useful lower bound on the size of a maximum bipartite subgraph in a graph each vertex of which has a sparse neighbourhood.

Lemma 2.8 ([4]). *There exists an absolute positive constant ε such that for every positive constant C there is a $\delta = \delta(C) > 0$ with the following property. Let G be a graph on n vertices (with positive degrees) with m edges and the degree sequence $d_1, d_2, \ldots, d_n$. Suppose, further, that the induced subgraph on any set of $d \geqslant C$ vertices, all of which have a common neighbour, contains at most $\varepsilon d^{3/2}$ edges. Then*

$$f(G) \geqslant \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i}.$$

3. Bipartite subgraphs of sparse graphs

In this section, we present proofs of Theorems 1.1 and 1.4 using the idea from [1].

3.1. Theta-graph. To prove Theorem 1.1, we employ the following upper bound, obtained by Faudree and Simonovits (see [12]), on the maximum number of edges in $\theta_{k,s}$-free graphs.

Lemma 3.1 ([12]). *Let k and t be two positive integers and let G be a graph on n vertices. If G is $\theta_{k,s}$-free, then there exists a constant $b(k, s)$ such that*

$$e(G) \leqslant b(k, s) n^{1+1/k}.$$

P r o o f of Theorem 1.1. Let G be a $\theta_{k,s}$-free graph with n vertices and m edges. We assume that m is sufficiently large in view of (1). Note that a graph is *D-degenerate* if every subgraph contains a vertex of degree at most D. Now let $D = \mu m^{1/(k+1)}$, where $\mu = \mu(k, s) > 1$ will be chosen later.

Claim 3.2. *G is D-degenerate.*

P r o o f. Otherwise, suppose that G contains a subgraph G' with the minimum degree greater than D. Note that the number of vertices of G' is $N \leqslant 2m/D = 2D^k/\mu^{k+1}$. Thus, the number of edges of G' is

$$e(G') \geqslant \frac{1}{2}DN \geqslant \Big(\frac{1}{2}\mu N\Big)^{1+1/k}.$$

Since G' is $\theta_{k,s}$-free, by Lemma 3.1, there is a constant $b = b(k,s)$ such that $e(G') \leqslant (b\,N)^{1+1/k}$, which is a contradiction by choosing $\mu > 2b$. This completes the proof of Claim 3.2. $\square$

Claim 3.3. *There exists a positive constant ℓ such that the neighbourhood of any vertex of degree $d > (\ell/\varepsilon)^2$ in G induces a subgraph with at most $\varepsilon d^{3/2}$ edges, where ε is a constant defined as in Lemma* 2.8.

P r o o f. Note that a *spider* is a rooted tree in which each vertex has degree one or two, except for the root. A *leg* of a spider is a path from the root to a vertex of degree one. Since G is $\theta_{k,s}$-free, the neighbourhood of any vertex of degree d in G cannot contain a spider of s legs such that every leg has length $k-1$. As is well-known, there is a constant $\ell = \ell(k,s)$ such that the induced subgraph of G on the neighbourhood of any vertex with degree d can span at most ℓd edges, which is smaller than $\varepsilon d^{3/2}$ for all $d > (\ell/\varepsilon)^2$. This completes the proof of Claim 3.3. $\square$

By Claim 3.2, it is easy to see that there exists a labelling $v_1, v_2, \ldots, v_n$ of the vertices of G such that $d_i^+ \leqslant D$ for every i, where d_i^+ denotes the number of neighbours v_j of v_i with $j < i$ in G. (Indeed, let v_n be the vertex of the minimal degree in G. Thus, the degree of v_n is at most D. Delete it from G and repeat the process.) Clearly, $\sum\limits_{i=1}^{n} d_i^+ = m$. Let d_i be the degree of v_i in G for each $1 \leqslant i \leqslant n$. Then

$$\sum_{i=1}^{n} \sqrt{d_i} \geqslant \sum_{i=1}^{n} \sqrt{d_i^+} \geqslant \frac{\sum_{i=1}^{n} d_i^+}{\sqrt{q}} = \frac{m}{\sqrt{q}} = \frac{1}{\sqrt{\mu}}m^{(2k+1)/(2k+2)}.$$

This, together with Lemma 2.8 by choosing $C = (\ell/\varepsilon)^2$ and Claim 3.3, implies that

$$f(G) \geqslant \frac{m}{2} + \delta\sum_{i=1}^{n}\sqrt{d_i} \geqslant \frac{m}{2} + \frac{\delta}{\sqrt{\mu}}m^{(2k+1)/(2k+2)},$$

where $\delta = \delta(k,s)$ is a constant, as required. This completes the proof of Theorem 1.1. $\square$

3.2. The subdivision of K_k and $K_{t,s}$. In this short section, we prove Theorem 1.4. The following two lemmas are needed.

Lemma 3.4 ([13]). *Let $k \geqslant 3$ be an integer and let G be a graph on n vertices. If G is K_k'-free, then there exists a constant $b(k)$ such that*

$$e(G) \leqslant b(k)n^{3/2-1/(4k-6)}.$$

Lemma 3.5 ([7]). *Let $s \geqslant t \geqslant 2$ be two integers and let G be a graph on n vertices. If G is $K_{t,s}'$-free, then there exists a constant $b(t,s)$ such that*

$$e(G) \leqslant b(t,s)n^{3/2-1/(2t)}.$$

The proof of Theorem 1.4 is similar to that of Theorem 1.1.

Proof of Theorem 1.4. Let $\mathscr{H} = \{K_k', K_{t,s}'\}$. For every $H \in \mathscr{H}$, let G be an H-free graph with n vertices and m edges. Define $D = \mu m^{\alpha}$ for some fixed real $\alpha \in (0,1)$, where $\mu = \mu(H) > 1$ will be chosen later. In the case of (i), we set $\alpha = (k-2)/(3k-5)$, while in the case of (ii), we set $\alpha = (t-1)/(3t-1)$. As before, we can claim that G is D-degenerate. If not, suppose that G contains a subgraph G' with the minimum degree greater than D. It follows that the number of vertices of G' is $N \leqslant 2m^{1-\alpha}/\mu$. But this is impossible; in view of Lemma 3.4 (or Lemma 3.5), proceed as in the proof of Claim 3.2 for a suitable chosen value of $\mu = \mu(H)$. Note that the neighbourhood of a vertex in G is also H-free. Thus, by Lemma 3.4 (or Lemma 3.5), the induced subgraph of G on any set of common neighbours of a vertex with degree d can span less than $\varepsilon d^{3/2}$ edges. We apply, again, Lemma 2.8 and conclude that

$$f(G) \geqslant \frac{m}{2} + \delta \sum_{i=1}^{n} \sqrt{d_i} \geqslant \frac{m}{2} + \frac{\delta m}{\sqrt{D}} \geqslant \frac{m}{2} + \frac{\delta}{\sqrt{\mu}} m^{1-\alpha/2},$$

where $\delta = \delta(H)$ is a constant, as required. This completes the proof of Theorem 1.4. □

4. Bipartite subgraphs of graphs without wheels

In this section, we prove Theorem 1.6. The lower bound for Theorem 1.6 (ii) appears in Section 4.1 and the lower bound for (i) appears in Section 4.2.

4.1. Odd wheels. Here we study the maximum bipartite subgraphs in graphs without odd wheels. Let P_t stand for a simple path with t vertices. The following two results are needed. The first one is the well-known upper bound of Erdős and Gallai (see [10]) and another one is from [22].

Lemma 4.1 ([10]). *Let $t \geqslant 2$ be an integer and let G be a graph on n vertices. If G is P_t-free, then $e(G) \leqslant \frac{1}{2}n(t-2)$.*

Lemma 4.2 ([22]). *Let $k \geqslant 2$ be an integer and let G be a graph on n vertices. If G is C_{2k+1}-free, then*

$$\alpha(G) \geqslant \frac{1}{5k^2}(n^k \log n)^{1/(k+1)}.$$

First, we apply Lemmas 2.3 and 2.4 to give a lower bound on the independence number of a W_{2k+1}-free graph.

Lemma 4.3. *For any integer $k \geqslant 2$, let G be a W_{2k+1}-free graph on n vertices. Then*

$$\alpha(G) \geqslant \frac{1}{15k^2}(n^k \log^{k+1} n)^{1/(2k+1)}.$$

Proof. Let G be a graph with m edges and the maximum degree Δ. Let G' be the subgraph of G induced by the neighbourhood of any vertex of G with the maximum degree Δ, and let G'' be the subgraph of G' induced by the neighbourhood of any vertex of G' with the maximum degree Δ' in G'. Clearly, G' is C_{2k+1}-free and G'' is P_{2k}-free.

Claim 4.4.

$$\Delta \leqslant (n^{k+1} \log^k n)^{1/(2k+1)}.$$

Proof. Otherwise, suppose that $\Delta > (n^{k+1} \log^k n)^{1/(2k+1)}$. By Lemma 4.2,

$$\alpha(G) \geqslant \alpha(G') \geqslant \frac{1}{5k^2}(\Delta^k \log \Delta)^{1/(k+1)} \geqslant \frac{1}{15k^2}(n^k \log^{k+1} n)^{1/(2k+1)}.$$

This gives the desired result and completes the proof of Claim 4.4. $\square$

Claim 4.5.

$$\Delta' \leqslant \frac{1}{3k}(n^k \log^{k+1} n)^{1/(2k+1)}.$$

Proof. Otherwise, suppose that $\Delta' > (1/3k)(n^k \log^{k+1} n)^{1/(2k+1)}$. Note that G'' is P_{2k}-free. By Lemma 4.1, $e(G'') \leqslant (2k-2)\Delta'/2$ and hence the average degree of G'' is at most $2k-2$. It then follows from Lemma 2.3 that

$$\alpha(G) \geqslant \alpha(G'') \geqslant \frac{\Delta'}{2k-2+1} > \frac{1}{3k(2k-1)}(n^k \log^{k+1} n)^{1/(2k+1)}.$$

Hence, we get the desired result and complete the proof of Claim 4.5. $\square$

Claim 4.6.

$$\frac{k+1}{2k+1}\log n - \log(\Delta'+1) - 1 \geqslant \frac{1}{15k^2}\log n.$$

P r o o f. It is trivial when $\Delta' \leqslant 1$, so assume that $\Delta' \geqslant 2$. It follows that $\Delta'+1 \leqslant \frac{3}{2}\Delta'$. Thus, it suffices to show that

$$\log n^{(k+1)/(2k+1)} - \log n^{1/(15k^2)} \geqslant \log\Bigl(\frac{3}{2}\mathrm{e}\Delta'\Bigr).$$

By Claim 4.5, it suffices to verify

$$n^{1/(2k+1)-1/(15k^2)} \geqslant \frac{\mathrm{e}}{2k}(\log n)^{(k+1)/(2k+1)},$$

which is equivalent to $\log n \leqslant l_k n^{q_k}$, where $l_k = (2k/\mathrm{e})^{(2k+1)/(k+1)}$ and $q_k = (15k^2-2k-1)/(15k^2(k+1))$. Let $g(x) = \log x - l_k x^{q_k}$. Obviously, $g(1) < 0$. Assume that $x \geqslant 2$. Note that $g'(x) = x^{-1}(1 - l_k q_k x^{q_k})$. So we obtain the stationary point $x(k) = (l_k q_k)^{-1/q_k}$. If $k = 2$, then one can get the value of $x(2)$. It yields that the maximum value of the function $g(x) < 0$. If $k \geqslant 3$, then we have $x(k) < 2$. It follows that the function $g(x)$ is monotonically decreasing over the interval $[2,\infty)$ and thus $g(x) < 0$. Hence, we have $\log n \leqslant l_k n^{q_k}$ for all n and complete the proof of Claim 4.6. $\square$

By Claim 4.4, the average degree of G is at most $(n^{k+1}\log^k n)^{1/(2k+1)}$. Combining with Lemma 2.4 and Claim 4.6, we have

$$\begin{aligned}\alpha(G) &\geqslant nF_{\Delta'+1}((n^{k+1}\log^k n)^{1/(2k+1)}) \geqslant n\frac{(k+1)/(2k+1)\log n - \log(\Delta'+1) - 1}{(n^{k+1}\log^k n)^{1/(2k+1)}}\\ &\geqslant \frac{1}{15k^2}(n^k \log^{k+1} n)^{1/(2k+1)}.\end{aligned}$$

This completes the proof of Lemma 4.3. $\square$

Next, we use Lemmas 2.2 and 4.3 to obtain an upper bound on the chromatic number of a W_{2k+1}-free graph G in terms of $|V(G)|$.

Lemma 4.7. *For any integer $k \geqslant 2$, let G be a W_{2k+1}-free graph with n vertices. Then*

$$\chi(G) \leqslant 50k^2\Bigl(\frac{n}{\log n}\Bigr)^{(k+1)/(2k+1)}.$$

P r o o f. It is trivial if $n < \mathrm{e}^2$, so we assume that $n \geqslant \mathrm{e}^2$. Let

$$\psi(x) = \frac{1}{15k^2}(x^k \log^{k+1} x)^{1/(2k+1)} \quad \text{and} \quad \tau(x) = \frac{k+1}{2k+1}\Bigl(1 - \frac{1}{\log x}\Bigr).$$

Then $\alpha(G) \geqslant \psi(n)$ by Lemma 4.3. Note that $\psi(x)$, $\tau(x)$ are continuous non-decreasing functions and $\tau(x) \geqslant \frac{1}{4}$ for $x \geqslant \mathrm{e}^2$. Thus, by Lemma 2.2,

$$\begin{aligned}\chi(G) &\leqslant \mathrm{e}^2 + \int_{\mathrm{e}^2}^{n} \frac{1}{\psi(x)}\,\mathrm{d}x \leqslant \mathrm{e}^2 + \frac{15k^2}{\tau(\mathrm{e}^2)} \int_{\mathrm{e}^2}^{n} \frac{\tau(x)}{(x^k \log^{k+1} x)^{1/(2k+1)}}\,\mathrm{d}x \\ &< \mathrm{e}^2 + 50k^2\Big(\frac{x}{\log x}\Big)^{(k+1)/(2k+1)}\Big|_{\mathrm{e}^2}^{n} \leqslant 50k^2\Big(\frac{n}{\log n}\Big)^{(k+1)/(2k+1)}.\end{aligned}$$

This completes the proof of Lemma 4.7. □

However, we need the upper bound on the chromatic number of a W_{2k+1}-free graph G in terms of $e(G)$. Finally, we establish the following theorem, which plays a key role in the proof of Theorem 1.6 (ii). The approach we take is an extension of that by [16].

Theorem 4.8. *For any integer $k \geqslant 2$, let G be a W_{2k+1}-free graph with $m > 1$ edges and let a_k be an integer described as in Lemma 2.5. Then*

$$\chi(G) \leqslant 100(k^2 + a_k^{-1})\Big(\frac{m \log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}.$$

Proof. Let G be a W_{2k+1}-free graph on n vertices with $m > 1$ edges and let

$$n' = \Big(\frac{(m \log\log m)^{2k+1}}{\log^k m}\Big)^{1/(3k+2)}.$$

We prove it by a sequence of claims.

Claim 4.9. $n > n'$.

Proof. For otherwise, suppose that $n \leqslant n'$. It follows from $m > 1$ that $n \geqslant 3 > \mathrm{e}$. Note that the function $g(x) = x/\log x$ is monotonically increasing over the interval (e, ∞). Thus, by Lemma 4.7,

$$\begin{aligned}\chi(G) &\leqslant 50k^2\Big(\frac{n}{\log n}\Big)^{(k+1)/(2k+1)} \leqslant 50k^2\Big(\frac{n'}{\log n'}\Big)^{(k+1)/(2k+1)} \\ &\leqslant 100k^2\Big(\frac{m \log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}.\end{aligned} \tag{4}$$

We get the desired result and complete the proof of Claim 4.9. □

We can delete all the vertices with degree zero or one in G, that is, we can assume that $m \geqslant n$. Now we construct a graph sequence $\{G_i\}_{i\geqslant 0}$ on G according to the following greedy iterative procedure. Start with $i = 0$, $G_0 = G$ and $n_0 = |V(G_0)|$.

If $n_i > n'$ we do the following iterative procedure, otherwise we stop. Choose S_i to be the maximum independent set of G_i. Then set $G_{i+1} = G_i \setminus S_i$, $n_{i+1} = |V(G_{i+1})|$, and an increment i. Let G_t be the graph in the end of the process. Clearly, $\chi(G) \leqslant \chi(G_t) + t$.

Claim 4.10.

$$\chi(G_t) \leqslant 100k^2 \Big(\frac{m \log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}.$$

P r o o f. Note that G_t is W_{2k+1}-free and $n_t \leqslant n'$. It is trivial for $n_t \leqslant 2$. If $n_t \geqslant 3$, then by Lemma 4.7 and (4), the desired result follows. This completes the proof of Claim 4.10. □

In the following, it is sufficient to bound the value of t. We first bound the value of $|S_i|$. Let $l = \lceil \frac{n}{n'} \rceil$. By Claim 4.9, $l \geqslant 2$. Let $I = \{0, 1, \ldots, t-1\}$. For each $i \in I$, we have $n_i > n' \geqslant n/l$ by the definition of l. Let $v_1, \ldots, v_{n_0}$ be a labelling of the vertices of G_0 such that $S_i = \{v_p \colon n_{i+1} < p \leqslant n_i\}$ for each $i \in I$. Let S be the union of S_i for all $i \in I$ and let $J = \{2, 3, \ldots, l\}$. Thus for each $j \in J$, put

$$V_j = \Big\{v_p \in S \colon \frac{n}{j} < p \leqslant \frac{n}{j-1}\Big\} \quad \text{and} \quad I_j = \Big\{i \in I \colon n_i > \frac{n}{j}\Big\}.$$

Observe that $S \setminus S_{t-1} \subseteq \bigcup\limits_{j \in J} V_j \subseteq S$ and $I_2 \subseteq I_3 \subseteq \ldots \subseteq I_l$. Hence for each $v \in V_j$, there exists an $i \in I_j$ such that $v \in S_i$. In addition,

$$|V_j| \leqslant \Big\lceil \frac{n}{j-1} - \frac{n}{j} \Big\rceil \leqslant \frac{2n}{j^2}. \tag{5}$$

Claim 4.11. *For each* $i \in I_j \neq \emptyset$,

$$|S_i| \geqslant \frac{a_k n^2}{2j^2 m} Q\Big(\frac{2jm}{n}\Big),$$

where $Q(x) = (\log x)/(\log\log x)$ *for* $x > \mathrm{e}$.

P r o o f. For each $i \in I$, we let d_i denote the average degree of G_i. For each $i \in I_j$, observe that $d_i \leqslant 2m/n_i \leqslant 2jm/n$. If $d_i > \mathrm{e}$, the function $Q(x)/x$ is decreasing over the interval (e, ∞). Note that G_i is W_{2k+1}-free, so G_i is also K_{2k+1}-free. Thus, from Lemma 2.5 and the fact $d_i \leqslant 2jm/n$, we have

$$|S_i| \geqslant \frac{a_k n_i Q(d_i)}{d_i} \geqslant \frac{a_k n^2}{2j^2 m} Q\Big(\frac{2jm}{n}\Big).$$

Otherwise, $d_i \leqslant \mathrm{e}$. By Lemma 2.3, $|S_i| \geqslant n_i/(1+\mathrm{e}) \geqslant \frac{1}{4} n_i \geqslant n/(4j)$. This, together with the fact that $x \geqslant Q(x)$ for $x > \mathrm{e}$ and $a_k < \frac{1}{4}$, implies the required lower bound as well. We complete the proof of Claim 4.11. □

Then, for each $v \in S_i$ and $i \in I$, let $w(v) = |S_i|^{-1}$. Hence for each $v \in S_i \subset V_j$, Claim 4.11 gives that

$$w(v) = |S_i|^{-1} \leqslant \frac{2j^2 m}{a_k n^2 Q(2jm/n)} \leqslant \frac{2j^2 m \log\log m}{a_k n^2 \log(2jm/n)},$$

where the last inequality holds because $j \leqslant l \leqslant \frac{1}{2}n$ by the definitions of l and n'. Combining the above inequality, the definition of $w(v)$ and (5), we have

(6)

$$t-1 = \sum_{i\in I\setminus\{t-1\}} \sum_{v\in S_i} w(v) \leqslant \sum_{j\in J}\sum_{v\in V_j} w(v) \leqslant \sum_{j=2}^{l} |V_j| w(v) \leqslant \frac{4m}{a_k n}\sum_{j=2}^{l} \frac{\log\log m}{\log j + \log(m/n)}.$$

Claim 4.12.

$$t-1 \leqslant \frac{48}{a_k}\Big(\frac{m\log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}.$$

P r o o f. Recall that $l = \lceil \frac{n}{n'} \rceil$, so $l-1 < n/n' \leqslant l$. Thus, by the definition of n', we have

$$l\frac{m}{n} \geqslant \frac{n}{n'}\frac{m}{n} = \frac{m}{n'} = \Big(\frac{m^{k+1}\log^k m}{\log^{2k+1}\log m}\Big)^{1/(3k+2)}. \tag{7}$$

It follows that $\max\{l, m/n\} \geqslant m^{(k+1)/(2(3k+2))}$ and hence

$$\max\Big\{\log l, \log\frac{m}{n}\Big\} \geqslant \frac{k+1}{2(3k+2)}\log m > \frac{\log m}{6}. \tag{8}$$

If $l < m/n$, then combining (6), (7), (8) and the fact $l-1 < n/n'$, we have

$$\begin{aligned} t-1 &\leqslant \frac{4m}{a_k n}\sum_{j=2}^{l}\frac{\log\log m}{\log(m/n)} \leqslant \frac{4m(l-1)}{a_k n}\frac{\log\log m}{\log(m/n)} \\ &\leqslant \frac{4m}{a_k n'}\frac{\log\log m}{\log(m/n)} \leqslant \frac{24}{a_k}\Big(\frac{m\log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}. \end{aligned}$$

Otherwise, $l \geqslant m/n$. Delete the second term of the denominator in (6) and obtain

$$t-1 \leqslant \frac{4m}{a_k n}\sum_{j=2}^{l}\frac{\log\log m}{\log j} \leqslant \frac{8m\log\log m}{a_k n'\log l} < \frac{48}{a_k}\Big(\frac{m\log\log m}{\log^2 m}\Big)^{(k+1)/(3k+2)}.$$

Note that the second inequality holds due to the fact that

$$\sum_{j=2}^{l}\frac{1}{\log j} \leqslant \int_2^l \frac{1}{\log x}\,\mathrm{d}x \leqslant \frac{2(l-1)}{\log l} < \frac{2n}{n'\log l}$$

and the last inequality follows from (7) and (8). This completes the proof of Claim 4.12. $\square$

Combining Claims 4.10 and 4.12 and noting that $\chi(G) \leqslant \chi(G_t) + t$, we get the desired result and complete the proof of Theorem 4.8. $\square$

P r o o f of Theorem 1.6 (ii). Let $r \geqslant 3$ be a fixed integer and let G be a W_r-free graph with m edges. Note that $W_3 = K_4$. Thus, the lower bound for the case $r = 3$ was already proved in [21]. So we assume that $r \geqslant 5$. Set $r = 2k+1$, $a_k = a$ and $c'(r) = \frac{1}{50}(r^2 + 4a^{-1})^{-1}$. The desired result follows from Lemma 2.1 and Theorem 4.8. $\square$

4.2. Even wheels. In this subsection, we prove Theorem 1.6 (i). We need a result of Shearer, see [18].

Lemma 4.13 ([18]). *For any graph G with m edges and vertex degrees $d_1, \ldots, d_n$, we have* $\sum\limits_{i=1}^{n} \sqrt{d_i} \geqslant m^{3/4}$.

The following fact is also needed. See, e.g., [4] for a proof.

Lemma 4.14. *Let $G = (V, E)$ be a graph with m edges and the minimum degree at least m^{θ} for some fixed real $\theta \in (0, 1)$. Suppose that m is sufficiently large and the induced subgraph on the neighbourhood of any vertex $v \in V$ of degree d contains fewer than $sd^{3/2}$ edges for some positive constant s. Then for every constant $\eta \in (0, 1)$, there exists an induced subgraph $G' = (V', E')$ of G satisfying the following properties:*

(i) *G' contains at least $\frac{1}{2}\eta^2 m$ edges.*
(ii) *Every vertex v of degree d in G that lies in V' has the degree at least $\frac{1}{2}\eta d$ in G'.*
(iii) *Every neighbourhood of the vertex v in V' contains at most $2\eta^2 sd^{3/2}$ edges in G'.*

P r o o f of Theorem 1.6 (i). Let $r = 2k \geqslant 4$ be a fixed integer. Let G be a W_{2k}-free graph with m edges and vertex degrees $d_1, d_2, \ldots, d_n$. If $k > 2$, then note that $C_{2k} = \theta_{k,2}$, and the neighbourhood of any vertex of G contains no copy of C_{2k}. By Lemma 3.1, there is a constant $b = b(k)$ such that the neighbourhood of any vertex of degree d in G spans at most $bd^{1+1/k}$ edges. Combining Lemmas 2.8 and 4.13, the desired result follows.

If $k = 2$, we aim to employ Lemma 2.8 to get the desired result, but there is an extra twist. We tacitly assume that m is sufficiently large. If $n \geqslant \frac{1}{2}m^{3/4}$, then by Lemma 2.6, we are done. So assume that $n < \frac{1}{2}m^{3/4}$. As long as there is a vertex of the degree smaller than $m^{1/4}$ in G, delete it. Note that $n < \frac{1}{2}m^{3/4}$, hence this process terminates after deleting fewer than $m^{1/4}n < \frac{1}{2}m$ edges. It then terminates with an induced subgraph $G^* = (V^*, E^*)$ of G with at least $\frac{1}{2}m$ edges and the minimum degree at least $m^{1/4}$. Clearly, G^* is W_4-free. Hence, the induced subgraph (of G^*) on the neighbourhood of any vertex v of degree d in G^* is C_4-free. By Lemma 3.1, there exists a constant $b' > 1$ such that this induced subgraph spans at most $b'd^{3/2}$ edges. Now we apply Lemma 4.14 on G^* by choosing $\eta < \varepsilon^2/32b'^2$, where ε is the constant from Lemma 2.8. Therefore, we obtain an induced subgraph $G' = (V', E')$ of G^* (and hence of G) with n' vertices and at least $\frac{1}{4}\eta^2 m$ edges such that the induced subgraph on all the neighbours of any vertex of degree d' in G' contains at most $\varepsilon(d')^{3/2}$ edges in G'. From Lemmas 2.8 and 4.13,

$$f(G') \geqslant \frac{e(G')}{2} + \delta \sum_{i=1}^{n'} \sqrt{d_i} \geqslant \frac{e(G')}{2} + \delta(e(G'))^{3/4} \geqslant \frac{e(G')}{2} + \delta\Big(\frac{\eta^2}{4}\Big)^{3/4} m^{3/4},$$

where $\delta = \delta(G')$ is a constant, as needed. Thus, by Lemma 2.7, we conclude that

$$f(G) \geqslant f(G') + \frac{m - e(G')}{2} \geqslant \frac{m}{2} + cm^{3/4},$$

where $c = \delta(\frac{1}{4}\eta^2)^{3/4}$. This completes the proof of Theorem 1.6 (i). $\square$

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