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REMARKS ON THE A PRIORI BOUND FOR THE VORTICITY OF THE AXISYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. We study the axisymmetric Navier-Stokes equations. In 2010, Loftus-Zhang used a refined test function and re-scaling scheme, and showed that

$$|\omega^r(x,t)| + |\omega^z(r,t)| \leqslant \frac{C}{r^{10}}, \quad 0 < r \leqslant \frac{1}{2}.$$

By employing the dimension reduction technique by Lei-Navas-Zhang, and analyzing ω^r, ω^z and ω^θ/r on different hollow cylinders, we are able to improve it and obtain

$$|\omega^r(x,t)| + |\omega^z(r,t)| \leqslant \frac{C|\ln r|}{r^{17/2}}, \quad 0 < r \leqslant \frac{1}{2}.$$

Keywords: axisymmetric Navier-Stokes equations; weighted a priori bounds

MSC 2020: 35B65, 35Q35, 76D03

1. Introduction

We investigate the axisymmetric solutions to the Navier-Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\Pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0}, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u^1, u^2, u^3) = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3$ is the fluid velocity field, Π is a scalar pressure. In cylindrical coordinates r, θ, z with

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z,$$

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the axisymmetric solutions are of the form

$$\boldsymbol{u} = u^r(r,z,t)\boldsymbol{e}_r + u^z(r,z,t)\boldsymbol{e}_\theta + u^z(r,z,t)\boldsymbol{e}_z, \tag{1.2}$$

where

$$\boldsymbol{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big) = (\cos\theta, \sin\theta, 0), \quad \boldsymbol{e}_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big) = (-\sin\theta, \cos\theta, 0),$$
$$\boldsymbol{e}_z = (0,0,1)$$

are the basis of $\mathbb{R}^3$ in the cylindrical coordinates. Here, u^r, u^θ and u^z are called the radial, swirl (or azimuthal) and axial components of $\boldsymbol{u}$, respectively. In terms of u^r, u^θ and u^z, the system (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t u^r + (\boldsymbol{b}\cdot\nabla)u^r - \Big(\Delta - \dfrac{1}{r^2}\Big)u^r - \dfrac{(u^\theta)^2}{r} + \partial_r \Pi = 0, \\ \partial_t u^\theta + (\boldsymbol{b}\cdot\nabla)u^\theta - \Big(\Delta - \dfrac{1}{r^2}\Big)u^\theta + \dfrac{u^r u^\theta}{r} = 0, \\ \partial_t u^z + (\boldsymbol{b}\cdot\nabla)u^z - \Delta u^z + \partial_z \Pi = 0, \\ \partial_r(ru^r) + \partial_z(ru^z) = 0, \end{cases} \tag{1.3}$$

where $\boldsymbol{b} = u^r\boldsymbol{e}_r + u^z\boldsymbol{e}_z$.

Applying the curl operator to $(1.1)_1$ and putting

$$\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u} = \omega^r\boldsymbol{e}_r + \omega^\theta\boldsymbol{e}_\theta + \omega^z\boldsymbol{e}_z \tag{1.4}$$

with

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r} = \frac{1}{r}\partial_r(ru^\theta), \tag{1.5}$$

we may rewrite the vorticity equation

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{\omega} = \boldsymbol{0} \tag{1.6}$$

as

$$\begin{cases} \partial_t\omega^r + (\boldsymbol{b}\cdot\nabla)\omega^r - \Big(\Delta - \dfrac{1}{r^2}\Big)\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\ \partial_t\omega^\theta + (\boldsymbol{b}\cdot\nabla)\omega^\theta - \Big(\Delta - \dfrac{1}{r^2}\Big)\omega^\theta - \dfrac{2u^\theta\partial_z u^\theta}{r} - \dfrac{u^r\omega^\theta}{r} = 0, \\ \partial_t\omega^z + (\boldsymbol{b}\cdot\nabla)\omega^z - \Delta\omega^z - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0. \end{cases} \tag{1.7}$$

A key property of (1.3) is that the quantity $\Theta = ru^\theta$ obeys the equation

$$\partial_t\Theta + (\boldsymbol{b}\cdot\nabla)\Theta - \Delta\Theta + \frac{2}{r}\partial_r\Theta = 0.$$

Applying the maximum principle as in [1], Proposition 1, we have

$$\|\Theta(t)\|_{L^\infty} \leqslant \|\Theta_0\|_{L^\infty}. \tag{1.8}$$

When $u_0^\theta = 0$, we deduce from (1.8) that u^θ vanishes for all later times and (1.3) is exactly two-dimensional, and its global well-posedness problem has been solved, see [9], [13], [18]. However, when the initial swirl does not vanish, the global regularity is open, even if (1.3) can be rewritten as a two dimensional form, that is, $(1.3)_2$ and $(1.7)_2$, with u^r, u^z being recovered from ω^θ by the axisymmetric Biot-Savart law. And thus, tremendous efforts and interesting progresses have been made on the regularity problem, see [1], [3], [2], [4], [7], [8], [11], [12], [15], [16], [19], [20], [22], [21], [24], [17], [23] and references cited therein. The above sources are what we need (now being in a state of Serrin type for r^du^r, r^du^θ, r^du^z, $r^d\omega^\theta$, $r^d\partial_ru^r$, $r^d\partial_ru^\theta$ and $r^d\partial_zu^z$) to ensure the global regularity. Another interesting topic is to consider what we have, or equivalently, the a priori estimates. Besides (1.8), Chae and Lee [1] showed that

$$r^3\omega^\theta \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; H^1(\mathbb{R}^3)). \tag{1.9}$$

Then the weight in (1.9) was lowered down from 3 to 2 by Lei and Zhang [12] and Zhang, Ouyang and Yang [26] independently. Recently, Zhang [25] has refined the weight further as

$$r^d\omega^\theta \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; H^1(\mathbb{R}^3)), \quad \frac{3}{2} < d \leqslant 3, \tag{1.10}$$

by the use of the weighted Sobolev inequality (see [5])

$$\left.\begin{aligned} &1<\gamma,\ \beta<\infty,\ -1\leqslant a,\ b\leqslant 2,\ 0\leqslant b-a<1,\\ &\frac{1}{\gamma}=\frac{1}{\beta}-\frac{1+a-b}{2}\end{aligned}\right\} \Rightarrow \|r^au^r\|_{L^\gamma(\Omega)} \leqslant C\|r^b\omega^\theta\|_{L^\beta(\Omega)} \tag{1.11}$$

with $\Omega = \{(r,z);\ r>0,\ z\in\mathbb{R}\}$.

As far as the point-wise bound is concerned, Loftus and Zhang [14] proved

$$|\omega^\theta| \leqslant \frac{C}{r^5}, \quad |\omega^r| + |\omega^z| \leqslant \frac{C}{r^{10}}, \quad 0<r\leqslant\frac{1}{2}. \tag{1.12}$$

The first bound was improved in Lei, Navas and Zhang [10] as

$$|\omega^\theta| \leqslant \frac{C|\ln r|}{r^{7/2}}, \quad 0 < r \leqslant \frac{1}{2}, \tag{1.13}$$

by re-scaling and dimensional reduction techniques. Furthermore, Lei, Navas and Zhang in the same paper showed the bound of $\boldsymbol{b}$,

$$|\boldsymbol{b}| \leqslant \frac{C|\ln r|^{1/2}}{r^2}, \quad 0 < r \leqslant \frac{1}{2}. \tag{1.14}$$

The purpose of the present paper is to improve the second bound in (1.12). More precisely, we have

Theorem 1.1. *Suppose $\boldsymbol{u}$ is a smooth, axisymmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with the initial data $\boldsymbol{u}_0 = \boldsymbol{u}(\cdot, -T) = u_0^r \boldsymbol{e}_r + u_0^\theta \boldsymbol{e}_\theta + u_0^z \boldsymbol{e}_z \in L^2(\mathbb{R}^3)$ and $\boldsymbol{\omega} = \omega^r \boldsymbol{e}_r + \omega^\theta \boldsymbol{e}_\theta + \omega^z \boldsymbol{e}_z$ is the vorticity. Assume further $r u_0^\theta \in L^\infty(\mathbb{R}^3)$. Then there exists a constant C depending only on the initial data such that*

$$|\omega^r| + |\omega^z| \leqslant \frac{C|\ln r|}{r^{17/2}}, \quad 0 < r \leqslant \frac{1}{2}. \tag{1.15}$$

2. Proof of Theorem 1.1

In this section, we prove (1.15). The main ingredient we introduce in this paper, is analyzing ω^r, ω^z and ω^θ/r on different hollow cylinders. We follow ideas from [10] and [14].

Let (x, t) be the point in the statement of Theorem 1.1. For simplicity, we assume $t = 0$ and $z = 0$. Let $R > 0$, $S > 0$ and $0 < A < B$ be constants. Denote by

$$C_{AR,BR} = \{(r, \theta, z);\ AR \leqslant r \leqslant BR,\ 0 \leqslant \theta \leqslant 2\pi,\ |z| \leqslant BR\} \subset \mathbb{R}^3 \tag{2.1}$$

the hollowed out cylinder centered at the origin with the inner radius AR, outer radius BR and height extending up and down BR, and thus $C_{AR,BR}$ has the total height of $2BR$. If $R = 1$, we simply write $C_{A,B}$ instead of $C_{A1,B1}$.

Let

$$P_{AR,BR,SR} = C_{AR,BR} \times (-S^2R^2, 0) \tag{2.2}$$

be the parabolic region. Again, we write $P_{A,B,S}$ for $P_{A1,B1,S1}$.

Since any point (x,t) aside from that in the z axis belongs to a parabolic region $P_{2k,3k,3k/4}$ for some $k>0$ and r is comparable with k, it suffices to prove

$$
|\omega^r(x,t)| + |\omega^z(x,t)| \leqslant \frac{C|\ln k|}{k^{17/2}} \quad \forall\,(x,t)\in P_{2k,3k,3k/4}. \tag{2.3}
$$

We use the scaling property of the Navier-Stokes flow to shift the consideration from $P_{2k,3k,3k/4}$ to $P_{2,3,3/4}$. Indeed, recall that if $(\boldsymbol{u}(x,t),\Pi(x,t))$ is a solution to the system, then $(\tilde{\boldsymbol{u}}(x,t)=k\boldsymbol{u}(kx,k^2t),\widetilde{\Pi}(x,t)=k^2\Pi(kx,k^2t))$ where $k>0$ is a parameter of solutions. Consequently, if $(\boldsymbol{u}(x,t),\Pi(x,t))$ is a solution to the axisymmetric Navier-Stokes equations in $P_{k,4k,k}$, then $(\tilde{\boldsymbol{u}}(\tilde{x},\tilde{t}),\widetilde{\Pi}(\tilde{x},\tilde{t}))$ with the variables $\tilde{x}=x/k$, $\tilde{t}=t/k^2$ is the solution of the equation in $(\tilde{x},\tilde{t})\in P_{1,4,1}$.

During the proof below, we drop the "tilde" notation for all relevant quantities when computations take place on the scaled cylinders, say $P_{1,4,1}$; and at last, we scale down the quantities (on $P_{k,4k,k}$ for example) to the original solution. In $P_{1,4,1}$, we do our analysis on the governing equations of $\Gamma \overset{\text{def}}{=} \omega^r$, $\Omega \overset{\text{def}}{=} \omega^\theta/r$ and $\Phi \overset{\text{def}}{=} \omega^z$, i.e.,

$$
\partial_t\Gamma + (\boldsymbol{b}\cdot\nabla)\Gamma - \Big(\Delta - \frac{1}{r^2}\Big)\Gamma - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \tag{2.4}
$$

$$
\partial_t\Omega + (\boldsymbol{b}\cdot\nabla)\Omega - \Delta\Omega - \frac{2}{r}\partial_r\Omega - \frac{u^\theta\partial_z u^\theta}{r^2} = 0, \tag{2.5}
$$

$$
\partial_t\Phi + (\boldsymbol{b}\cdot\nabla)\Phi - \Delta\Phi - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0. \tag{2.6}
$$

We divide the following proof into three steps.

Step 1. L^{4q} estimate of $\overline{\Omega}_+$, $\overline{\Gamma}_+$, $\overline{\Phi}_+$ by a refined cut-off function. Noting that

$$
\Lambda \overset{\text{def}}{=} \|u^\theta\|_{L^\infty(P_{1,4,1})} \leqslant \|ru^\theta\|_{L^\infty(\mathbb{R}^3)} < \infty, \tag{2.7}
$$

we may put

$$
\overline{\Omega}_+ \overset{\text{def}}{=} \begin{cases} \Omega(x,t)+\Lambda+1, & \Omega(x,t)\geqslant 0,\\ \Lambda+1, & \Omega(x,t)<0, \end{cases} \tag{2.8}
$$

which is exactly $\Lambda+1$ plus the positive part of Ω. We list some properties of $\overline{\Omega}_+$, which make sense of the calculations below,

(1) $\overline{\Omega}_+ \geqslant \Lambda+1$;
(2) all derivatives of $\overline{\Omega}_+$ vanish on the set where $\Omega(x,t)<0$;
(3) $\overline{\Omega}_+$ is Lipschitz and Ω is smooth by assumption;
(4) at interfaces where $\Omega(x,t)=0$, the boundary terms upon integration by parts cancel out.

Similar considerations apply to $\overline{\Gamma}_+$ and $\overline{\Phi}_+$. For $q > 1$, direct computations yield that

$$\partial_t \overline{\Omega}_+^q + (\boldsymbol{b}\cdot\nabla)\overline{\Omega}_+^q - \Delta\overline{\Omega}_+^q - \frac{2}{r}\partial_r\overline{\Omega}_+^q - \frac{2qu^\theta\partial_z u^\theta}{r^2}\overline{\Omega}_+^{q-1} + q(q-1)\overline{\Omega}_+^{q-2}|\nabla\overline{\Omega}_+|^2 = 0. \tag{2.9}$$

Let $\frac{5}{8} \leqslant \sigma_2 < \sigma_1 \leqslant 1$. Put

$$\begin{aligned} C(\sigma_i) &= \{(r,\theta,z);\ 5-4\sigma_i < r < 4\sigma_i,\ 0 \leqslant \theta \leqslant 2\pi,\ |z| < 4\sigma_i\}, \\ P(\sigma_i) &= C(\sigma_i) \times (-\sigma_i^2, 0) \subset P_{\sigma_i, 4\sigma_i, \sigma_i} \end{aligned} \tag{2.10}$$

for $i = 1, 2$. Choose $\psi = \phi(y)\eta(s)$ to be a refined cut-off function satisfying

$$\begin{aligned} &\operatorname{supp}\phi \subset C(\sigma_1), \quad \phi|_{C(\sigma_2)} = 1, \quad 0 \leqslant \phi \leqslant 1, \quad \frac{|\nabla\phi|}{\phi^{1/2}} \leqslant \frac{c_1}{\sigma_1 - \sigma_2}, \\ &\operatorname{supp}\eta \subset (-\sigma_1^2, 0), \quad \eta|_{[-\sigma_2^2, 0]} = 1, \quad 0 \leqslant \eta \leqslant 1, \quad |\eta'| \leqslant \frac{c_2}{(\sigma_1-\sigma_2)^2}. \end{aligned}$$

Let $f = \overline{\Omega}_+^q$, test (2.9) by $f\psi^2$, and notice that

$$\begin{aligned} \int_{P(\sigma_1)} \partial_s f \cdot f\psi^2 \,\mathrm{d}y\,\mathrm{d}s &= \frac{1}{2}\int_{P(\sigma_1)} \partial_s(f^2)\psi^2 \,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2}\int_{P(\sigma_1)} [\partial_s(f^2\psi^2) - f^2\partial_s(\psi^2)] \,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2}\int_{-\sigma_1^2}^0 \int_{C(\sigma_1)} \partial_s(f^2\psi^2)\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{2}\int_{P(\sigma_1)} f^2\phi^2 \cdot 2\eta\partial_s\eta \,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\psi^2(y,0)\,\mathrm{d}y - \int_{P(\sigma_1)} f^2\phi^2 \cdot \eta\partial_s\eta \,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y,0)\,\mathrm{d}y - \int_{P(\sigma_1)} f^2\phi^2 \cdot \eta\partial_s\eta \,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

as well as

$$\begin{aligned} \int_{P(\sigma_1)} -\Delta f \cdot f\psi^2 \,\mathrm{d}y\,\mathrm{d}s &= \int_{P(\sigma_1)} \nabla f \cdot \nabla(f\psi^2)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{P(\sigma_1)} \nabla f \cdot [\nabla(f\psi)\cdot\psi + f\psi\cdot\nabla\psi]\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{P(\sigma_1)} \{\nabla(f\psi)[\nabla(f\psi) - f\nabla\psi] + f\nabla f\cdot\psi\nabla\psi\}\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{P(\sigma_1)} [|\nabla(f\psi)|^2 - (f\nabla\psi + \psi\nabla f)\cdot f\nabla\psi + f\nabla f\cdot\psi\nabla\psi]\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{P(\sigma_1)} [|\nabla(f\psi)|^2 - f^2|\nabla\psi|^2]\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

We obtain

$$(2.11)\quad \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)\,\mathrm{d}y + \int_{P(\sigma_1)} |\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s + q(q-1)\int_{P(\sigma_1)} \psi^2\overline{\Omega}_+^{2(q-1)}|\nabla\overline{\Omega}_+|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$= -\int_{P(\sigma_1)} (\boldsymbol{b}\cdot\nabla)f\cdot f\psi^2\,\mathrm{d}y\,\mathrm{d}s + \int_{P(\sigma_1)} (\phi^2\eta\partial_s\eta + |\nabla\psi|^2)f^2\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_{P(\sigma_1)} \frac{2}{r}\partial_r f\cdot f\psi^2\,\mathrm{d}y\,\mathrm{d}s + \int_{P(\sigma_1)} \frac{2qu^\theta\partial_z u^\theta}{r^2}\overline{\Omega}_+^{2q-1}\psi^2\,\mathrm{d}y\,\mathrm{d}s$$
$$\equiv F_1 + F_2 + F_3 + F_4.$$

Similarly, we have

$$(2.12)\quad \partial_t\overline{\Gamma}_+^q + (\boldsymbol{b}\cdot\nabla)\overline{\Gamma}_+^q - \Delta\overline{\Gamma}_+^q + \frac{q}{r^2}\overline{\Gamma}_+^q - q(\omega^r\partial_r + \omega^z\partial_z)u^r\cdot\overline{\Gamma}_+^{q-1} + q(q-1)\overline{\Gamma}_+^{q-2}|\nabla\overline{\Gamma}_+|^2 = 0.$$

Set $g = \overline{\Gamma}_+^q$ and test (2.12) by $g\widetilde{\psi}^2$, where $\widetilde{\psi}$ is ψ modified through replacing σ_1 by $\tilde{\sigma}_1 = (\sigma_2+\sigma_1)/2$. We find that

$$(2.13)\quad \frac{1}{2}\int_{C(\widetilde{\sigma}_1)} g^2(y,0)\widetilde{\phi}^2(y)\,\mathrm{d}y + \int_{P(\widetilde{\sigma}_1)} |\nabla(g\widetilde{\psi})|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$+ q(q-1)\int_{P(\widetilde{\sigma}_1)} \widetilde{\psi}^2\overline{\Gamma}_+^{2(q-1)}|\nabla\overline{\Gamma}_+|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$+ q\int_{P(\widetilde{\sigma}_1)} \left|\frac{g\widetilde{\psi}}{r}\right|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$= -\int_{P(\widetilde{\sigma}_1)} (\boldsymbol{b}\cdot\nabla)g\cdot g\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_{P(\widetilde{\sigma}_1)} (\widetilde{\phi}^2\widetilde{\eta}\partial_s\widetilde{\eta} + |\nabla\widetilde{\psi}|^2)g^2\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_{P(\widetilde{\sigma}_1)} q(\omega^r\partial_r + \omega^z\partial_z)u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s$$
$$\equiv G_1 + G_2 + G_3.$$

Finally,

$$(2.14)\quad \partial_t\overline{\Phi}_+^q + (\boldsymbol{b}\cdot\nabla)\overline{\Phi}_+^q - \Delta\overline{\Phi}_+^q - q(\omega^r\partial_r + \omega^z\partial_z)u^z\cdot\overline{\Phi}_+^{q-1} + q(q-1)\overline{\Phi}_+^{q-2}|\nabla\overline{\Phi}_+|^2 = 0.$$

Letting $h = \overline{\Phi}_+^q$ and testing (2.14) by $h\widetilde{\psi}^2$, we deduce that

$$
\begin{aligned}
(2.15)\quad & \frac{1}{2}\int_{C(\widetilde{\sigma}_1)} h^2(y,0)\widetilde{\phi}^2(y)\,\mathrm{d}y \\
&\quad + \int_{P(\widetilde{\sigma}_1)} |\nabla(h\widetilde{\psi})|^2\,\mathrm{d}y\,\mathrm{d}s + q(q-1)\int_{P(\widetilde{\sigma}_1)} \widetilde{\psi}^2\overline{\Phi}_+^{2(q-1)}|\nabla\overline{\Phi}_+|^2\,\mathrm{d}y\,\mathrm{d}s \\
&= -\int_{P(\widetilde{\sigma}_1)} (\boldsymbol{b}\cdot\nabla)h\cdot h\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s + \int_{P(\widetilde{\sigma}_1)} (\widetilde{\phi}^2\widetilde{\eta}\partial_s\widetilde{\eta} + |\nabla\widetilde{\psi}|^2)h^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \int_{P(\widetilde{\sigma}_1)} q(\omega^r\partial_r + \omega^z\partial_z)u^z\cdot\overline{\Phi}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s \\
&\equiv H_1 + H_2 + H_3.
\end{aligned}
$$

We will estimate F_i, G_j and H_k term by term.

Estimation of F_1, G_1, and H_1. Since $\boldsymbol{b}$ is divergence-free, we have

$$
\begin{aligned}
F_1 = -\frac{1}{2}\int_{P(\sigma_1)} (\boldsymbol{b}\cdot\nabla)(f^2)\cdot\psi^2\,\mathrm{d}y\,\mathrm{d}s &= \frac{1}{2}\int_{P(\sigma_1)} (\boldsymbol{b}\cdot\nabla)(\psi^2)\cdot f^2\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{P(\sigma_1)} (\boldsymbol{b}\cdot\nabla)\psi\cdot\psi f^2\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

To get a fine estimate, we need to use the dimension reduction argument explored in [10]. Put

$$
\begin{aligned}
\overline{C}(\sigma_i) &= \{(r,z); (r,\theta,z)\in C(\sigma_i)\}, \\
\overline{P}(\sigma_i) &= \{(r,z,s); (r,\theta,z,s)\in P(\sigma_i)\}, \quad i = 1,2,
\end{aligned}
$$

and if $y = (r,\theta,z)$, $\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, we set

$$
\overline{y} = (r,z),\ \mathrm{d}\overline{y} = \mathrm{d}r\,\mathrm{d}z.
$$

Since $\boldsymbol{u}$ is divergence-free, that is, $\partial_r(ru^r)+\partial_z(ru^z) = 0$, there exists an axisymmetric stream function L such that

$$
u^r = -\partial_z L, \quad u^z = \frac{1}{r}\partial_r(rL).
$$

For the later application of the Poincaré inequality, we introduce a function $a(t)$, depending only on the time. Upon integrating by parts, it follows that

$$\begin{aligned}
F_1 &= \int_{P(\sigma_1)} (\boldsymbol{b}\cdot\nabla)\psi\cdot\psi f^2\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{P(\sigma_1)} (u^r\partial_r\psi + u^z\partial_z\psi)\cdot\psi f^2\,\mathrm{d}y\,\mathrm{d}s \\
&= 2\pi\int_{\overline{P}(\sigma_1)} -\partial_z(rL-a)\partial_r\psi\cdot\psi f^2\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s \\
&\quad + 2\pi\int_{\overline{P}(\sigma_1)} \partial_r(rL-a)\partial_z\psi\cdot\psi f^2\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s \\
&= 2\pi\int_{\overline{P}(\sigma_1)} (rL-a)[\partial_z\partial_r\psi\cdot\psi f^2 + \partial_r\psi\cdot\partial_z(\psi f^2)]\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s \\
&\quad - 2\pi\int_{\overline{P}(\sigma_1)} (rL-a)[\partial_r\partial_z\psi\cdot\psi f^2 + \partial_z\psi\cdot\partial_r(\psi f^2)]\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s \\
&= 2\pi\int_{\overline{P}(\sigma_1)} (rL-a)\partial_r\psi\cdot\partial_z(\psi f^2)\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s \\
&\quad - 2\pi\int_{\overline{P}(\sigma_1)} (rL-a)\partial_z\psi\cdot\partial_r(\psi f^2)\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

Since

$$\nabla(\psi f^2) = \nabla(\psi f)\cdot f + \psi f\cdot\nabla f = \nabla(\psi f)\cdot f + f[\nabla(\psi f) - f\nabla\psi] = 2\nabla(\psi f)\cdot f - f^2\nabla\psi,$$

we can estimate F_1 as

$$\begin{aligned}
F_1 &\leqslant C\|rL-a\|_{L^\infty(\overline{C}(\sigma_1))}\|\nabla\psi\|_{L^\infty(\overline{C}(\sigma_1))} \\
&\quad\times[\|\nabla(\psi f)\|_{L^2(\overline{P}(\sigma_1))}\|f\|_{L^2(\overline{P}(\sigma_1))} + \|\nabla\psi\|_{L^\infty(\overline{C}(\sigma_1))}\|f\|^2_{L^2(\overline{C}(\sigma_1))}] \\
&\leqslant \frac{1}{4}\int_{P(\sigma_1)} |\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{C}{(\sigma_1-\sigma_2)^2}\Big[\sup_{t\in(-\sigma_1^2,0)}\|rL-a(t)\|^2_{L^\infty(\overline{C}(\sigma_1))} + 1\Big]\int_{\overline{P}(\sigma_1)} f^2\,\mathrm{d}\overline{y}\,\mathrm{d}s.
\end{aligned} \tag{2.16}$$

Here and in what follows, we employ the fact that r is comparable to 1 in $P(\sigma_1)$ and hence the two-dimensional volume element $\mathrm{d}r\,\mathrm{d}z$ is comparable to the three-dimensional one $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ for axisymmetric functions.

We are going to estimate $\|rL - a(t)\|_{L^\infty(\overline{C}(\sigma_1))}$. In order to apply the two-dimensional logarithmic Sobolev inequality (see [6], Appendix)

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leqslant C(\|\nabla f\|_{L^2(\mathbb{R}^2)}+\|f\|_{L^2(\mathbb{R}^2)}+1)\ln^{1/2}(\|\Delta f\|_{L^2(\mathbb{R}^2)}+\|f\|_{L^2(\mathbb{R}^2)}+\mathrm{e}), \tag{2.17}$$

we introduce the cut-off function $\theta = \theta(r,z) \in C_c^\infty(\mathbb{R}^2)$ such that

$$\theta|_{\overline{C}(\sigma_1)} = 1, \quad \operatorname{supp}\theta \subset \overline{C}\Big(\frac{9}{8}\sigma_1\Big), \quad 0 \leqslant \theta \leqslant 1, \quad |\overline{\nabla}\theta| + |\Delta_2\theta| \leqslant 1.$$

Here and hereafter, $\overline{\nabla}$ is the two-dimensional gradient and $\Delta_2 = \partial_r^2 + \partial_z^2$ is the two-dimensional Laplacian with respect to the variables r and z. Applying (2.17) to $rL - a(t)$ then gives

$$\begin{aligned}
&\|rL - a(t)\|_{L^\infty(\overline{C}(\sigma_1))}\\
&\quad \leqslant \|rL - a(t)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad \leqslant C[\|\overline{\nabla}((rL - a(t))\theta)\|_{L^2(\mathbb{R}^2)} + \|(rL - a(t))\theta\|_{L^2(\mathbb{R}^2)} + 1]\\
&\qquad \times \ln^{1/2}[\|\Delta_2((rL - a(t))\theta)\|_{L^2(\mathbb{R}^2)} + \|(rL - a(t))\theta\|_{L^2(\mathbb{R}^2)} + \mathrm{e}]\\
&\quad \leqslant C[\|\theta\overline{\nabla}(rL - a(t))\|_{L^2(\mathbb{R}^2)} + \|rL - a(t)\|_{L^2(\overline{C}(9\sigma_1/8))} + 1]\\
&\qquad \times \ln^{1/2}[\|\Delta_2((rL - a(t))\theta)\|_{L^2(\mathbb{R}^2)} + \|rL - a(t)\|_{L^2(\overline{C}(9\sigma_1/8))} + \mathrm{e}].
\end{aligned}$$

Choose $a(t)$ to be the average of rL on $\overline{C}(\frac{9}{8}\sigma_1)$ under the two-dimensional volume element $\mathrm{d}r\,\mathrm{d}z$, we may employ the two-dimensional Poincaré inequality to deduce

$$\begin{aligned}
&\|rL - a(t)\|_{L^\infty(\overline{C}(\sigma_1))}\\
&\quad \leqslant C[\|\overline{\nabla}(rL)\|_{L^2(\overline{C}(9\sigma_1/8))} + 1]\\
&\qquad \times \ln^{1/2}[\|\Delta_2((rL - a(t))\theta)\|_{L^2(\overline{C}(9\sigma_1/8))} + \|\overline{\nabla}(rL)\|_{L^2(\overline{C}(9\sigma_1/8))} + \mathrm{e}].
\end{aligned} \tag{2.18}$$

To proceed further, we calculate

$$\begin{aligned}
|\overline{\nabla}(rL)|^2 &= |\partial_r(rL)|^2 + |\partial_z(rL)|^2 = r^2 \cdot \Big|\frac{1}{r}\partial_r(rL)\Big|^2 + r^2|-\partial_z L|^2\\
&= r^2|u^z|^2 + r^2|u^r|^2 = r^2|\boldsymbol{b}|^2,\\
\Delta_2((rL - a(t))\theta) &= \Delta_2(rL)\theta + 2\overline{\nabla}(rL)\cdot\nabla\theta + (rL - a(t))\Delta_2\theta\\
&= \theta(\partial_r^2 + \partial_z^2)(rL) + 2\overline{\nabla}(rL)\cdot\nabla\theta + (rL - a(t))\Delta_2\theta\\
&= \theta r\Big(\partial_r^2 + \partial_z^2 + \frac{2}{r}\partial_r\Big)L + 2\overline{\nabla}(rL)\cdot\nabla\theta + (rL - a(t))\Delta_2\theta\\
&= \theta r\Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)L + \theta\Big(\partial_r + \frac{1}{r}\Big)L\\
&\qquad + 2\overline{\nabla}(rL)\cdot\nabla\theta + (rL - a(t))\Delta_2\theta\\
&= -\theta r\omega^\theta + \theta u^z + 2\overline{\nabla}(rL)\cdot\nabla\theta + (rL - a(t))\Delta_2\theta,\\
|\Delta_2((rL - a(t))\theta)| &\leqslant |r\omega^\theta - u^z| + 2r|\boldsymbol{b}| + C|rL - a(t)|.
\end{aligned}$$

Applying the two-dimensional Poincaré inequality again, we get

$$\begin{aligned}
&\|\Delta_2((rL-a(t))\theta)\|_{L^2(\overline{C}(9\sigma_1/8))}\\
&\qquad \leqslant C\|\omega^\theta\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|\boldsymbol{b}\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|rL-a(t)\|_{L^2(\overline{C}(9\sigma_1/8))}\\
&\qquad \leqslant C\|\omega^\theta\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|\boldsymbol{b}\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|\nabla(rL)\|_{L^2(\overline{C}(9\sigma_1/8))}\\
&\qquad \leqslant C\|\omega^\theta\|_{L^2(\overline{C}(9\sigma_1/8))}+2C\|\boldsymbol{b}\|_{L^2(\overline{C}(9\sigma_1/8))}.
\end{aligned}$$

With the above estimates, (2.18) becomes

$$\begin{aligned}
\text{(2.19)}\qquad &\|rL-a(t)\|_{L^\infty(\overline{C}(\sigma_1))}\\
&\quad\leqslant C[\|\boldsymbol{u}(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+1]\\
&\qquad\times\ln^{1/2}[C\|\omega^\theta(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|\boldsymbol{u}(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+\mathrm{e}].
\end{aligned}$$

Consequently, (2.16) reduces to

$$\text{(2.20)}\qquad F_1\leqslant\frac14\int_{P(\sigma_1)}|\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s+C\frac{\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)}f^2\,\mathrm{d}\overline{y}\,\mathrm{d}s,$$

where

$$\begin{aligned}
\text{(2.21)}\qquad &\overline{K}=\overline{K}(\boldsymbol{u},\boldsymbol{\omega})\equiv\sup_{t\in(-\sigma_1^2,0)}\{[\|\boldsymbol{u}(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+1]\\
&\qquad\times\ln^{1/2}[C\|\omega^\theta(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+C\|\boldsymbol{u}(\cdot,t)\|_{L^2(\overline{C}(9\sigma_1/8))}+\mathrm{e}]\}.
\end{aligned}$$

Similarly,

$$\text{(2.22)}\qquad G_1\leqslant\frac14\int_{P(\widetilde{\sigma}_1)}|\nabla(g\psi)|^2\,\mathrm{d}y\,\mathrm{d}s+C\frac{\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\tilde{\sigma}_1)}g^2\,\mathrm{d}\overline{y}\,\mathrm{d}s,$$

$$\text{(2.23)}\qquad H_1\leqslant\frac14\int_{P(\widetilde{\sigma}_1)}|\nabla(h\psi)|^2\,\mathrm{d}y\,\mathrm{d}s+C\frac{\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\tilde{\sigma}_1)}h^2\,\mathrm{d}\overline{y}\,\mathrm{d}s.$$

Estimation of F_2, G_2, and H_2. Employing the routine treatment, we obtain

$$\begin{aligned}
\text{(2.24)}\qquad F_2&\leqslant\int_{P(\sigma_1)}(|\partial_s\eta|+|\nabla\phi|^2)f^2\,\mathrm{d}y\,\mathrm{d}s=\int_{P(\sigma_1)}\Big(|\partial_s\eta|+\phi\Big|\frac{\nabla\phi}{\phi^{1/2}}\Big|^2\Big)f^2\,\mathrm{d}y\,\mathrm{d}s\\
&\leqslant\frac{C}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)}f^2\,\mathrm{d}y\,\mathrm{d}s\leqslant\frac{C}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)}f^2\,\mathrm{d}\overline{y}\,\mathrm{d}s.
\end{aligned}$$

Similarly,

$$\text{(2.25)}\qquad G_2\leqslant\frac{C}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\widetilde{\sigma}_1)}g^2\,\mathrm{d}\overline{y}\,\mathrm{d}s,$$

$$\text{(2.26)}\qquad H_2\leqslant\frac{C}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\widetilde{\sigma}_1)}h^2\,\mathrm{d}\overline{y}\,\mathrm{d}s.$$

Estimation of F_3. Integrating by parts provides

$$F_3 = \int_{-\sigma_1^2}^{0} \mathrm{d}s \int_{-4\sigma_1}^{4\sigma_1} \mathrm{d}z \int_{5-4\sigma_1}^{4\sigma_1} \frac{2}{r}\partial_r f \cdot f\psi^2 \cdot 2\pi r \,\mathrm{d}r \tag{2.27}$$

$$= 2\pi \int_{-\sigma_1^2}^{0} \mathrm{d}s \int_{-4\sigma_1}^{4\sigma_1} \mathrm{d}z \int_{5-4\sigma_1}^{4\sigma_1} \partial_r(f^2)\psi^2 \,\mathrm{d}r$$

$$= -2\pi \int_{-\sigma_1^2}^{0} \mathrm{d}s \int_{-4\sigma_1}^{4\sigma_1} \mathrm{d}z \int_{5-4\sigma_1}^{4\sigma_1} f^2 \cdot 2\psi\partial_r\psi \,\mathrm{d}r$$

$$= -\int_{P(\sigma_1)} \frac{2}{r}\partial_r\psi \cdot \psi f^2 \,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant \int_{P(\sigma_1)} \frac{2}{r}\eta^2 \frac{\partial_r\phi}{\phi^{1/2}} \cdot \phi^{3/2} f^2 \,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant \frac{C}{\sigma_1-\sigma_2}\int_{P(\sigma_1)} f^2 \,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant \frac{C}{\sigma_1-\sigma_2}\int_{\overline{P}(\sigma_1)} f^2 \,\mathrm{d}\overline{y}\,\mathrm{d}s.$$

Estimation of F_4. Integrating by parts gives

$$F_4 = q\int_{P(\sigma_1)} \partial_z\Big|\frac{u^\theta}{r}\Big|^2 \frac{\overline{\Omega}_+^{2q}\psi^2}{\overline{\Omega}_+}\,\mathrm{d}y\,\mathrm{d}s \tag{2.28}$$

$$= -q\int_{P(\sigma_1)}\Big|\frac{u^\theta}{r}\Big|^2\Big[\partial_z(|f\psi|^2)\frac{1}{\overline{\Omega}_+} + \overline{\Omega}_+^{2q}\psi^2\cdot\Big(-\frac{\partial_z\overline{\Omega}_+}{\overline{\Omega}_+^2}\Big)\Big]\,\mathrm{d}y\,\mathrm{d}s$$

$$= -q\int_{P(\sigma_1)}\Big|\frac{u^\theta}{r}\Big|^2\Big[\partial_z(|f\psi|^2)\frac{1}{\overline{\Omega}_+} - \frac{1}{2q}\psi^2\partial_z\overline{\Omega}_+^{2q}\frac{1}{\overline{\Omega}_+}\Big]\,\mathrm{d}y\,\mathrm{d}s$$

$$= -q\int_{P(\sigma_1)}\Big|\frac{u^\theta}{r}\Big|^2\frac{1}{\overline{\Omega}_+}\Big\{\partial_z(|f\psi|^2) - \frac{1}{2q}[\partial_z(|f\psi|^2) - f^2\partial_z(\psi^2)]\Big\}\,\mathrm{d}y\,\mathrm{d}s$$

$$= -q\int_{P(\sigma_1)}\Big|\frac{u^\theta}{r}\Big|^2\frac{1}{\overline{\Omega}_+}\Big[\frac{2q-1}{2q}\partial_z(|f\psi|^2) + \frac{1}{q}f^2\psi\partial_z\psi\Big]\,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant Cq\int_{P(\sigma_1)}\Lambda^2\cdot\frac{1}{\Lambda}\Big[|f\psi|\cdot|\partial_z(f\psi)| + \frac{1}{q}f^2\psi|\partial_z\psi|\Big]\,\mathrm{d}y\,\mathrm{d}s$$

$$= Cq\int_{P(\sigma_1)}\Lambda^2\cdot\frac{1}{\Lambda}\Big[|f\psi|\cdot|\partial_z(f\psi)| + \frac{1}{q}f^2\eta^2\phi^{3/2}\Big|\frac{\partial_z\psi}{\phi^{1/2}}\Big|\Big]\,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant C\Big[\Lambda^2q^2 + \frac{\Lambda}{\sigma_1-\sigma_2}\Big]\int_{P(\sigma_1)} f^2\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{4}\int_{P(\sigma_1)}|\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant C\Big[\Lambda^2q^2 + \frac{\Lambda}{\sigma_1-\sigma_2}\Big]\int_{P(\sigma_1)} f^2\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{4}\int_{P(\sigma_1)}|\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s,$$

where, in the last three lines, we have used that

$$\overline{\Omega}_+ \geqslant \Lambda > 0, \quad \Big|\frac{u^\theta}{r}\Big| = \frac{|ru^\theta|}{r^2} \leqslant \frac{\Lambda}{\sigma_1^2} \leqslant \Lambda\Big(\frac{8}{5}\Big)^2.$$

Estimation of G_3 and H_3. Applying integration by parts to G_3 could provide some cancellation,

$$\begin{aligned}
G_3 &= q\int_{P(\tilde\sigma_1)} (\omega^r\partial_r + \omega^z\partial_z)u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s\\
&= q\int_{P(\tilde\sigma_1)} \Big[-\partial_z u^\theta\partial_r u^r + \frac{1}{r}\partial_r(ru^\theta)\partial_z u^r\Big]\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s\\
&= q\int_{P(\tilde\sigma_1)} -\partial_z u^\theta\partial_r u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s\\
&\quad + 2\pi q\int_{-\tilde\sigma_1^2}^0 \mathrm{d}s\int_{-4\tilde\sigma_1}^{4\tilde\sigma_1}\mathrm{d}z\int_{5-4\tilde\sigma_1}^{\tilde\sigma_1}\partial_r(ru^\theta)\partial_z u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2\,\mathrm{d}y\,\mathrm{d}s\\
&= q\int_{P(\tilde\sigma_1)} u^\theta[\partial_z\partial_r u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2 + \partial_r u^r\cdot\partial_z(\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2)]\,\mathrm{d}y\,\mathrm{d}s\\
&\quad - 2\pi q\int_{-\tilde\sigma_1^2}^0 \mathrm{d}s\int_{-4\tilde\sigma_1}^{4\tilde\sigma_1}\mathrm{d}z\\
&\quad \times\int_{5-4\tilde\sigma_1}^{4\tilde\sigma_1} ru^\theta[\partial_r\partial_z u^r\cdot\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2 + \partial_z u^r\cdot\partial_r(\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2)]\,\mathrm{d}y\,\mathrm{d}s\\
&= q\int_{P(\tilde\sigma_1)} u^\theta\cdot\partial_r u^r\cdot\partial_z(\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2)\,\mathrm{d}y\,\mathrm{d}s\\
&\quad - q\int_{P(\tilde\sigma_1)} u^\theta\cdot\partial_z u^r\cdot\partial_r(\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Since

$$\begin{aligned}
\nabla(\overline{\Gamma}_+^{2q-1}\widetilde{\psi}^2) &= (2q-1)\overline{\Gamma}_+^{2q-2}\nabla\overline{\Gamma}_+\widetilde{\psi}^2 + 2\overline{\Gamma}_+^{2q-1}\widetilde{\psi}\nabla\widetilde{\psi}\\
&= \frac{2q-1}{q}\overline{\Gamma}_+^{q-1}\nabla(\overline{\Gamma}_+^q)\widetilde{\psi}^2 + 2\overline{\Gamma}_+^{2q-1}\widetilde{\psi}\nabla\widetilde{\psi}\\
&= \frac{2q-1}{q}\overline{\Gamma}_+^{q-1}\psi[\nabla(\overline{\Gamma}_+^q\widetilde{\psi}) - \overline{\Gamma}_+^q\nabla\widetilde{\psi}] + 2\overline{\Gamma}_+^{2q-1}\widetilde{\psi}\nabla\widetilde{\psi}\\
&= \frac{2q-1}{q}\overline{\Gamma}_+^{q-1}\psi\nabla(\overline{\Gamma}_+^q\widetilde{\psi}) + \frac{1}{q}\overline{\Gamma}_+^{2q-1}\widetilde{\psi}\nabla\widetilde{\psi},
\end{aligned}$$

we have

$$\begin{aligned}
G_3 &\leqslant q\int_{P(\tilde\sigma_1)}\frac{1}{r}|ru^\theta|\cdot|\nabla u^r|\cdot\Big[\overline{\Gamma}_+^{q-1}\widetilde{\psi}|\nabla(g\widetilde{\psi})| + \frac{1}{q}\tilde\eta^2\overline{\Gamma}_+^{2q-1}\tilde\phi^{3/2}\Big|\frac{\nabla\tilde\phi}{\tilde\phi^{1/2}}\Big|\Big]\,\mathrm{d}y\,\mathrm{d}s\\
&\leqslant Cq\Lambda\|\nabla u^r\|_{L^{2q}(P(\tilde\sigma_1))}\\
&\quad\times\Big[\|\overline{\Gamma}_+\|_{L^{2q}(P(\tilde\sigma_1))}^{q-1}\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))} + \frac{1}{q(\sigma_1-\sigma_2)}\|\overline{\Gamma}_+\|_{L^{2q}(P(\tilde\sigma_1))}^{2q-1}\Big].
\end{aligned}$$

Applying Lemma 3.3,

$$G_3 \leqslant Cq\Lambda\Big[\|\omega^\theta\|_{L^{2q}(P(\sigma_1))} + \frac{1}{\sigma_1-\sigma_2}\|\boldsymbol{b}\|_{L^{2q}(P(\sigma_1))}\Big] \times \Big[\|\overline{\Gamma}_+\|^{q-1}_{L^{2q}(P(\tilde\sigma_1))}\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))} + \frac{1}{q(\sigma_1-\sigma_2)}\|\overline{\Gamma}_+\|^{2q-1}_{L^{2q}(P(\tilde\sigma_1))}\Big]. \tag{2.29}$$

Put

$$\overline{\Omega}_- = \begin{cases} -\Omega+\Lambda+1, & \Omega\leqslant 0,\\ \Lambda+1, & \Omega>0,\end{cases}$$

and, similarly, Γ_- and Φ_-. Also, set $f_1 = \overline{\Omega}^q_-$, $g_1 = \overline{\Gamma}^q_-$, $h_1 = \overline{\Phi}^q_-$. Then

$$|\omega^\theta|^q \leqslant f + f_1 \quad \text{on } P(\sigma_1).$$

Moreover, $0<\sigma_1<1$ implies

$$\|\boldsymbol{b}\|_{L^{2q}(P(\sigma_1))} \leqslant \|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}\Big[\int_{P(\sigma_1)} 1\,\mathrm{d}y\,\mathrm{d}s\Big]^{1/(2q)} \leqslant C\|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}.$$

With the above two observations, (2.29) becomes

$$\begin{aligned} G_3 &\leqslant \frac{Cq\Lambda}{\sigma_1-\sigma_2}[\|\omega^\theta\|_{L^{2q}(P(\sigma_1))} + \|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}]\\ &\quad\times\Big[\|g\|^{(q-1)/q}_{L^2(P(\tilde\sigma_1))}\cdot\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))} + \frac{1}{q(\sigma_1-\sigma_2)}\|g\|^{(2q-1)/q}_{L^2(P(\tilde\sigma_1))}\Big]\\ &\leqslant \frac{Cq\Lambda}{\sigma_1-\sigma_2}[\|f+f_1\|^{1/q}_{L^2(P(\sigma_1))} + \|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}]\\ &\quad\times\Big[\|g\|^{(q-1)/q}_{L^2(P(\tilde\sigma_1))}\cdot\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))} + \frac{1}{q(\sigma_1-\sigma_2)}\|g\|^{(2q-1)/q}_{L^2(P(\tilde\sigma_1))}\Big]\\ &\leqslant C\Big[\Big(\frac{q\Lambda}{\sigma_1-\sigma_2}\Big)^{1/q}\|f\|^{1/q}_{L^2(P(\sigma_1))} + \Big(\frac{q\Lambda}{\sigma_1-\sigma_2}\Big)^{1/q}\|f_1\|^{1/q}_{L^2(P(\sigma_1))}\Big]\\ &\quad\times\Big(\frac{q\Lambda}{\sigma_1-\sigma_2}\Big)^{(q-1)/q}\|g\|^{(q-1)/q}_{L^2(P(\tilde\sigma_1))}\cdot\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))}\\ &\quad + C\Big[\frac{\Lambda}{(\sigma_1-\sigma_2)^2}\Big]^{1/(2q)}\|f\|^{1/q}_{L^2(P(\sigma_1))}\cdot\Big[\frac{\Lambda}{(\sigma_1-\sigma_2)^2}\Big]^{(2q-1)/(2q)}\|g\|^{(2q-1)/q}_{L^2(P(\tilde\sigma_1))}\\ &\quad + \frac{Cq\Lambda}{\sigma_1-\sigma_2}\|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}\|g\|^{(q-1)/q}_{L^2(P(\tilde\sigma_1))}\|\nabla(g\widetilde{\psi})\|_{L^2(P(\tilde\sigma_1))}\\ &\quad + \frac{C\Lambda}{(\sigma_1-\sigma_2)^2}\|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}\|g\|^{(2q-1)/q}_{L^2(P(\tilde\sigma_1))}. \end{aligned}$$

Noticing that $q > 1$, $g \geqslant \Lambda + 1 \geqslant 1$, $\overline{K} \geqslant 1$, and applying the Young inequality, we get

$$
\begin{aligned}
(2.30)\quad G_3 &\leqslant \frac{1}{4}\int_{P(\tilde{\sigma}_1)} |\nabla(g\widetilde{\psi})|^2 \,\mathrm{d}y\,\mathrm{d}s + C\frac{q^2\Lambda^2+\Lambda}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)} (f^2+f_1^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s \\
&\quad + C\Big[\frac{q^2\Lambda^2+\Lambda}{(\sigma_1-\sigma_2)^2} + \frac{q^2\Lambda^2}{(\sigma_1-\sigma_2)^2}\|\boldsymbol{b}\|^2_{L^\infty(P(\sigma_1))} + \frac{\Lambda}{(\sigma_1-\sigma_2)^2}\|\boldsymbol{b}\|_{L^\infty(P(\sigma_1))}\Big] \\
&\quad \times \int_{\overline{P}(\tilde{\sigma}_1)} g^2\,\mathrm{d}\overline{y}\,\mathrm{d}s \\
&\leqslant \frac{1}{4}\int_{P(\tilde{\sigma}_1)} |\nabla(g\widetilde{\psi})|^2 \,\mathrm{d}y\,\mathrm{d}s + \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}\int_{\overline{P}(\sigma_1)} (f^2+f_1^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s \\
&\quad + \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[1+\|\boldsymbol{b}\|^2_{L^\infty(P(\sigma_1))}]\int_{\overline{P}(\tilde{\sigma}_1)} g^2\,\mathrm{d}\overline{y}\,\mathrm{d}s \\
&\leqslant \frac{1}{4}\int_{P(\tilde{\sigma}_1)} |\nabla(g\widetilde{\psi})|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[1+\|\boldsymbol{b}\|^2_{L^\infty(P(\sigma_1))}]\int_{\overline{P}(\tilde{\sigma}_1)} (f^2+f_1^2+g^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(2.31)\qquad H_3 &\leqslant \frac{1}{4}\int_{P(\tilde{\sigma}_1)} |\nabla(h\widetilde{\psi})|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[1+\|\boldsymbol{b}\|^2_{L^\infty(P(\sigma_1))}]\int_{\overline{P}(\tilde{\sigma}_1)} (f^2+f_1^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s.
\end{aligned}
$$

Collecting (2.20), (2.22), (2.23), (2.24), (2.25), (2.26), (2.27), (2.28), (2.30), and (2.31) into (2.11) + (2.13) + (2.15), and observing that $\overline{K} \geqslant 1$, $0 < \sigma_1 - \sigma_2 < 1$, we find

$$
\begin{aligned}
(2.32)\qquad \int_{C(\sigma_1)} f^2(y,0)\phi^2(y)\,\mathrm{d}y &+ \int_{P(\sigma_1)} |\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&+ \int_{C(\tilde{\sigma}_1)} [g^2(y,0)+h^2(y,0)]\widetilde{\phi}^2(y)\,\mathrm{d}y \\
&+ \int_{P(\tilde{\sigma}_1)} |\nabla(g\widetilde{\psi})|^2 + |\nabla(h\widetilde{\psi})|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega}) + \|\boldsymbol{b}\|^2_{L^\infty(P(\sigma_1))}] \\
&\quad \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s,
\end{aligned}
$$

where $\overline{K} = \overline{K}(\boldsymbol{u},\boldsymbol{\omega})$ is defined by (2.21).

Since $\eta|_{[-\sigma_2^2,0]} = 1$, we may now replace the upper time limit 0 of the time cut-off function η by any $s \in [-\sigma_2^2, 0]$. Consequently,

$$
\begin{aligned}
(2.33)\qquad &\sup_{-\sigma_2^2<s<0}\Big[\int_{C(\sigma_1)} f^2(y,s)\phi^2(y)\,\mathrm{d}y + \int_{C(\widetilde{\sigma}_1)} [g^2(y,0)+h^2(y,0)]\widetilde{\phi}^2(y)\,\mathrm{d}y\Big]\\
&\quad + \int_{P(\sigma_1)} |\nabla(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s + \int_{P(\widetilde{\sigma}_1)} (|\nabla(g\widetilde{\psi})|^2 + |\nabla(h\widetilde{\psi})|^2)\,\mathrm{d}y\,\mathrm{d}s\\
&\leqslant \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]\\
&\quad \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s,
\end{aligned}
$$

where we have used the facts that $q > 1$, $0 < \sigma_1 - \sigma_2 < 1$, and

$$0 < \sigma_1 \leqslant 1 \Rightarrow C\Big(\frac{9\sigma_1}{8}\Big) \subset C_{0.5,5}.$$

Step 2. L^2-L^∞ estimate on solutions to (2.4)–(2.6) via Moser's iteration.

We are in a state to apply the dimension reduction argument from [10]. Since r is comparable with σ_1 or σ_2 in $C(\sigma_1)$ or $C(\tilde{\sigma}_1)$, respectively, (2.33) actually can be reformulated as

$$
\begin{aligned}
(2.34)\qquad &\sup_{-\sigma_2^2<s<0}\Big[\int_{\overline{C}(\sigma_1)} f^2(\overline{y},s)\phi^2(\overline{y})\,\mathrm{d}\overline{y} + \int_{\overline{C}(\widetilde{\sigma}_1)} [g^2(\overline{y},0)+h^2(\overline{y},0)]\widetilde{\phi}^2(\overline{y})\,\mathrm{d}\overline{y}\Big]\\
&\quad + \int_{P(\sigma_1)} |\nabla(f\psi)|^2\,\mathrm{d}\overline{y}\,\mathrm{d}s + \int_{P(\widetilde{\sigma}_1)} (|\nabla(g\widetilde{\psi})|^2 + |\nabla(h\widetilde{\psi})|^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s\\
&\leqslant \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2}[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]\\
&\quad \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s.
\end{aligned}
$$

By the two-dimensional Gagliardo-Nirenberg inequality,

$$\|f\psi\|_{L^4(\overline{C}(\sigma_1))} \leqslant C\|f\psi\|^{1/2}_{L^2(\overline{C}(\sigma_1))}\|\overline{\nabla}(f\psi)\|^{1/2}_{L^2(\overline{C}(\sigma_1))}.$$

The inequality (2.34) implies that

$$
\begin{aligned}
(2.35)\qquad \int_{\overline{P}(\sigma_2)} f^4\,\mathrm{d}\overline{y}\,\mathrm{d}s &= \int_{-\sigma_2^2}^0 \|f\psi\|^4_{L^4(\overline{C}(\sigma_1))}\,\mathrm{d}s\\
&\leqslant C\sup_{-\sigma_2^2<s<0}\|f\psi\|^2_{L^2(\overline{C}(\sigma_1))}\int_{-\sigma_2^2}^0 \|\overline{\nabla}(f\psi)\|^2_{L^2(\overline{C}(\sigma_1))}\,\mathrm{d}s\\
&\leqslant C\sup_{-\sigma_1^2<s<0}\|f\psi\|^2_{L^2(\overline{C}(\sigma_1))}\int_{P(\sigma_1)} |\overline{\nabla}(f\psi)|^2\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \left\{ \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2} [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s \right\}^2.$$

Similarly,

$$\begin{aligned}
\int_{\overline{P}(\sigma_2)} g^4\,\mathrm{d}\overline{y}\,\mathrm{d}s &= \int_{-\sigma_2^2}^0 \|g\psi\|^4_{L^4(\overline{C}(\widetilde{\sigma}_1))}\,\mathrm{d}s \\
&\leqslant C \sup_{-\sigma_2^2<s<0} \|g\psi\|^2_{L^2(\overline{C}(\widetilde{\sigma}_1))} \int_{-\sigma_2^2}^0 \|\overline{\nabla}(g\psi)\|^2_{L^2(\overline{C}(\widetilde{\sigma}_1))}\,\mathrm{d}s \\
&\leqslant C \sup_{-\widetilde{\sigma}_1^2<s<0} \|g\psi\|^2_{L^2(\overline{C}(\widetilde{\sigma}_1))} \int_{P(\sigma_1)} |\overline{\nabla}(g\psi)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant \left\{ \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2} [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \right. \\
&\qquad \left. \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s \right\}^2
\end{aligned} \tag{2.36}$$

as well as

$$\int_{\overline{P}(\sigma_2)} h^4\,\mathrm{d}\overline{y}\,\mathrm{d}s \leqslant \left\{ \frac{Cq^2(\Lambda+1)^2}{(\sigma_1-\sigma_2)^2} [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \times \int_{\overline{P}(\sigma_1)} (f^2+f_1^2+g^2+h^2)\,\mathrm{d}\overline{y}\,\mathrm{d}s \right\}^2. \tag{2.37}$$

Gathering (2.35), (2.36) and (2.37) together, and setting $J_+ = \overline{\Omega}_+ + \overline{\Gamma}_+ + \overline{\Phi}_+$, $J_- = \overline{\Omega}_- + \overline{\Gamma}_- + \overline{\Phi}_-$, and $J = J_+ + J_-$, we obtain

$$\begin{aligned}
\left[\int_{\overline{P}(\sigma_2)} J_+^{2^2 q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^\Upsilon \leqslant{} & \frac{C^{1/(2q)} q^{1/q} (\Lambda+1)^{1/q}}{(\sigma_1-\sigma_2)^{1/q}} \\
& \times [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{1/(2q)} \\
& \times \left[\int_{\overline{P}(\sigma_1)} J^{2q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{1/(2q)}, \quad \text{where } \Upsilon = \frac{1}{2^2 q}.
\end{aligned}$$

Repeating the above argument to J_- gives

$$\begin{aligned}
\left[\int_{\overline{P}(\sigma_2)} J_-^{2^2 q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^\Upsilon \leqslant{} & \frac{C^{1/(2q)} q^{1/q} (\Lambda+1)^{1/q}}{(\sigma_1-\sigma_2)^{1/q}} \\
& \times [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{1/(2q)} \\
& \times \left[\int_{\overline{P}(\sigma_1)} J^{2q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{1/(2q)}.
\end{aligned}$$

Summing up the above two estimates yields

$$
\left[\int_{\overline{P}(\sigma_2)} J^{2^2 q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{\Upsilon} \leqslant 2\frac{C^{1/(2q)}q^{1/q}(\Lambda+1)^{1/q}}{(\sigma_1-\sigma_2)^{1/q}} \times [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{1/(2q)} \times \left[\int_{\overline{P}(\sigma_1)} J^{2q}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{1/(2q)}. \tag{2.38}
$$

For $i = 0, 1, 2, \ldots$, we take $q = 2^i$ in (2.38), and replace σ_1 and σ_2 by

$$
\sigma_i = 1 - \sum_{j=1}^{i} 2^{-j-2} \quad \text{and} \quad \sigma_2 = 1 - \sum_{j=1}^{i+1} 2^{-j-2},
$$

respectively. Here, we set $\sum\limits_{j=1}^{i} 2^{-j-2} = 0$ for $i = 0$. Then

$$
\begin{aligned}
\left[\int_{\overline{P}(\sigma_{i+1})} J^{2^{i+2}}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{2^{-i-2}} &\leqslant 2\frac{C^{2^{-i-1}}(2i)^{2^{-i}}(\Lambda+1)^{2^{-i}}}{(2^{-(i+1)-2})^{2^{-i}}} \\
&\quad \times [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{2^{-i-1}} \\
&\quad \times \left[\int_{\overline{P}(\sigma_i)} J^{2^{i+1}}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{2^{-i-1}} \\
&\leqslant C(\Lambda+1)^{2^{-i}}[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{2^{-i-1}} \\
&\quad \times \left[\int_{\overline{P}(\sigma_i)} J^{2^{i+1}}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{2^{-i-1}},
\end{aligned}
$$

where we have used the boundedness of the convergent sequences

$$
\lim_{i\to\infty} C^{2^{-i-1}} = 1, \quad \lim_{i\to\infty} (2i)^{2^{-i}} = 1, \quad \lim_{i\to\infty} 2^{(i+3)2^{-i}} = 1.
$$

After iteration, we deduce

$$
\begin{aligned}
\left[\int_{\overline{P}(\sigma_{i+1})} J^{2^{i+2}}\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{2^{-i-2}} &\leqslant [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]^{\sum_{i=0}^\infty 2^{-i-1}} \\
&\quad \times (\Lambda+1)^{\sum_{i=0}^\infty 2^{-i}} \left[\int_{\overline{P}_{1,4,1}} J^2\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{1/2} \\
&= [\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \\
&\quad \times C(\Lambda+1)^2 \left[\int_{\overline{P}_{1,4,1}} J^2\,\mathrm{d}\overline{y}\,\mathrm{d}s\right]^{1/2}.
\end{aligned}
$$

Letting $i \to \infty$ and noticing that

$$\sigma \overset{\text{def}}{=} 1 - \lim_{i\to\infty} \sum_{j=1}^{i} 2^{-j-2} = \frac{3}{4}, \quad 5 - 4\sigma = 2, \quad 4\sigma = 3,$$

we get

$$\begin{aligned}(2.39)\qquad \sup_{\overline{P}_{2,3,3/4}} J &\leqslant C(\Lambda+1)^2[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}]\Big[\int_{\overline{P}_{1,4,1}} J^2 \,\mathrm{d}\overline{y}\,\mathrm{d}s\Big]^{1/2} \\ &\leqslant C(\Lambda+1)^2[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \\ &\quad \times \Big\{\int_{\overline{P}_{1,4,1}} [\Omega^2 + \Gamma^2 + \Phi^2 + (\Lambda+1)^2] \,\mathrm{d}\overline{y}\,\mathrm{d}s\Big\}^{1/2}.\end{aligned}$$

Observing that $\Omega = \omega^\theta/r$ and ω^θ are equivalent on $P_{1,4,1}$, we have

$$\begin{aligned}(2.40)\qquad \sup_{\overline{P}_{2,3,3/4}} |\boldsymbol{\omega}| &\leqslant C(\Lambda+1)^2[\overline{K}^2(\boldsymbol{u},\boldsymbol{\omega})(C_{0.5,5}) + \|\boldsymbol{b}\|^2_{L^\infty(P_{1,4,1})}] \\ &\quad \times \Big\{\int_{\overline{P}_{1,4,1}} [|\boldsymbol{\omega}|^2 + (\Lambda+1)^2] \,\mathrm{d}\overline{y}\,\mathrm{d}s\Big\}^{1/2}.\end{aligned}$$

Step 3. A priori bound of $|\boldsymbol{\omega}|$ by re-scaling.

Recall that we have omitted the "tilde" in the above two steps. So what actually has been proven thus far is

$$\begin{aligned}(2.41)\qquad \sup_{(\tilde{x},\tilde{t})\in\overline{P}_{2,3,3/4}} |\widetilde{\boldsymbol{\omega}}(\tilde{x},\tilde{t})| &\leqslant C(\tilde{\Lambda}+1)^2[\overline{K}^2(\tilde{\boldsymbol{u}},\widetilde{\boldsymbol{\omega}})(C_{0.5,5}) + \|\tilde{\boldsymbol{b}}\|^2_{L^\infty(P_{1,4,1})}] \\ &\quad \times \Big\{\int_{\tilde{P}_{1,4,1}} [|\widetilde{\boldsymbol{\omega}}|^2 + (\tilde{\Lambda}+1)^2] \,\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{s}\Big\}^{1/2},\end{aligned}$$

where $\tilde{x} = x/k$, $\tilde{t} = t/k^2$. Notice that

$$\begin{aligned}&\overline{K}(\tilde{\boldsymbol{u}},\widetilde{\boldsymbol{\omega}})(C_{0.5,5}) \\ &\quad = \sup_{\tilde{t}\in(-1,0)} \{[\|\tilde{\boldsymbol{u}}(\cdot,\tilde{t})\|_{L^2(C_{0.5,5})} + 1] \\ &\qquad \times \ln^{1/2}[C\|\tilde{\omega}^\theta(\cdot,\tilde{t})\|_{L^2(C_{0.5,5})} + \|\tilde{\boldsymbol{u}}(\cdot,\tilde{t})\|_{L^2(C_{0.5,5})} + \mathrm{e}]\} \\ &\quad = \sup_{t\in[-k^2,0]} \Big\{\Big[\frac{1}{k^{1/2}}\|\boldsymbol{u}(\cdot,t)\|_{L^2(C_{0.5k,5k})} + 1\Big] \\ &\qquad \times \ln^{1/2}\Big[Ck^{1/2}\|\omega^\theta(\cdot,t)\|_{L^2(C_{0.5k,5k})} + \frac{1}{k^{1/2}}\|\boldsymbol{u}(\cdot,t)\|_{L^2(C_{0.5k,5k})} + \mathrm{e}\Big]\Big\}.\end{aligned}$$

Whence, by (1.12) and the energy inequality, we find

$$\overline{K}(\tilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\omega}})(C_{0.5,5}) \leqslant \frac{C}{k^{1/2}} \ln^{1/2}\Big(Ck^{1/2} \cdot \frac{1}{k^5} + \frac{C}{k^{1/2}}\Big) \leqslant \frac{C|\ln k|^{1/2}}{k^{1/2}}, \quad 0 < k < \frac{1}{2}.$$

Also, $\tilde{\Lambda}$ is a scaling-invariant quantity,

$$\tilde{\Lambda} = \sup_{P_{1,4,1}} |\tilde{u}^\theta(\tilde{x}, \tilde{t})| \leqslant \|\tilde{r}\tilde{u}^\theta(\cdot, -T)\|_{L^\infty(\mathbb{R}^3)} = \|ru^\theta(x, -T)\|_{L^\infty(\mathbb{R}^3)} = \|ru_0^\theta\|_{L^\infty(\mathbb{R}^3)}.$$

Moreover, by (1.14),

$$\|\tilde{\boldsymbol{b}}\|_{L^\infty(P_{1,4,1})} = \frac{1}{k}\|\boldsymbol{b}\|_{L^\infty(P_{k,4k,k})} \leqslant \frac{C|\ln k|^{1/2}}{k^3}.$$

We may now scale down (2.41) on the original solution as in [10] and deduce for $0 < k < 1$ that

$$\begin{aligned} k^2\|\boldsymbol{\omega}\|_{L^\infty(P_{2k,3k,3k/4})} &\leqslant C\Big[\frac{|\ln k|}{k} + \frac{|\ln k|}{k^6}\Big]\Big\{\int_{\overline{P}_{k,4k,k}} \Big[\frac{1}{k}|\boldsymbol{\omega}|^2 + (\Lambda+1)^2\Big]\,\mathrm{d}y\,\mathrm{d}s\Big\}^{1/2} \\ &\leqslant \frac{C|\ln k|}{k^{13/2}}. \end{aligned}$$

This is exactly our goal (2.3). This finishes the proof of Theorem 1.1.

3. Appendix

In this appendix, we state and prove variants of Lemmas 4.1, 4.2 of [14], which are needed in (2.29). First, we have:

Lemma 3.1. *Let $\boldsymbol{u}$ be a smooth vector field on $\mathbb{R}^3$ with a sufficient decay at infinity. Then for any $q \in (1, \infty)$ there exists a constant $c(q) > 0$ such that*

$$\|\nabla \boldsymbol{u}\|_{L^q} \leqslant c(q)[\|\operatorname{curl} \boldsymbol{u}\|_{L^q} + \|\operatorname{div} \boldsymbol{u}\|_{L^q}]. \tag{3.1}$$

Proof. This is a classical result. Since the proof is short, we provide it. Put $\boldsymbol{u} = (u^1, u^2, u^3)$ and $\operatorname{curl} \boldsymbol{u} = \boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$. Then direct computations show

$$\begin{aligned} -\Delta u^1 &= -(\partial_1^2 + \partial_2^2 + \partial_3^2)u^1 \\ &= -\partial_1(\partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3) + \partial_2(\partial_1 u^2 - \partial_2 u^1) - \partial_3(\partial_3 u^1 - \partial_1 u^3) \\ &= -\partial_1 \operatorname{div} \boldsymbol{u} + \partial_2 \omega^3 - \partial_3 \omega^2. \end{aligned}$$

Thus,

$$\begin{aligned}\nabla u^1 = & -\partial_1(-\Delta)^{-1/2}\nabla(-\Delta)^{-1/2}\operatorname{div}\boldsymbol{u} + \partial_2(-\Delta)^{-1/2}\nabla(-\Delta)^{-1/2}\omega^3 \\ & -\partial_3(-\Delta)^{-1/2}\nabla(-\Delta)^{-1/2}\omega^2.\end{aligned}$$

The L^p boundedness of the Riesz transform then yields

$$\|\nabla v^1\|_{L^q} \leqslant c(q)[\|\operatorname{curl}\boldsymbol{u}\|_{L^q} + \|\operatorname{div}\boldsymbol{u}\|_{L^q}].$$

Similar considerations apply to ∇u^i, $i = 1, 2$, and this finishes the proof of Lemma 3.1. □

Lemma 3.1 has its local version, which reads

Lemma 3.2. *Let $\boldsymbol{u} \in C^\infty(P(\sigma_1))$ be a vector field. Then for $q \in (0, \infty)$ there exists a constant $c(q) > 0$ such that*

$$\begin{aligned}(3.2)\qquad & \|\nabla \boldsymbol{u}\|_{L^q(P(\sigma_2))} \\ & \leqslant c(q)\Big[\|\operatorname{curl}\boldsymbol{u}\|_{L^q(P(\sigma_1))} + \|\operatorname{div}\boldsymbol{u}\|_{L^q(P(\sigma_1))} + \frac{1}{\sigma_1-\sigma_2}\|\boldsymbol{u}\|_{L^q(P(\sigma_1))}\Big].\end{aligned}$$

Proof. Choose a cut-off function θ such that

$$\theta|_{P(\sigma_1)} = 1, \quad \operatorname{supp}\theta \subset P(\sigma_1), \quad 0 \leqslant \theta \leqslant 1, \quad |\nabla\theta| \leqslant \frac{C}{\theta_1-\theta_2}.$$

Then $\boldsymbol{u}\theta$ is compactly supported. By Lemma 3.1,

$$\begin{aligned}\|\nabla\boldsymbol{u}\|_{L^q(P(\sigma_2))} & \leqslant \|\nabla(\boldsymbol{u}\theta)\|_{L^q(P(\sigma_1))} \\ & \leqslant C[\|\operatorname{curl}(\boldsymbol{u}\theta)\|_{L^q(P(\sigma_1))} + \|\operatorname{div}(\boldsymbol{u}\theta)\|_{L^q(P(\sigma_1))}].\end{aligned}$$

Since

$$\operatorname{curl}(\boldsymbol{u}\theta) = \nabla\theta \times \boldsymbol{u} + \theta\operatorname{curl}\boldsymbol{u}, \quad \operatorname{div}(\boldsymbol{u}\theta) = (\boldsymbol{u}\cdot\nabla)\theta + \theta\operatorname{div}\boldsymbol{u},$$

we are led to (3.2), as desired. □

For axisymmetric smooth vector field, we have the following version of Lemma 3.2.

Lemma 3.3. *Let $\boldsymbol{u}$ be a divergence-free, axisymmetric, smooth vector field in $P(\sigma_1)$. Then for all $q \in (0, \infty)$ there exists a constant $c(q) > 0$ such that*

$$\begin{aligned}(3.3)\qquad & \|\nabla u^r\|_{L^q(P(\sigma_2))} + \Big\|\frac{u^r}{r}\Big\|_{rL^q(P(\sigma_2))} + \|\nabla u^z\|_{L^q(P(\sigma_2))} \\ & \leqslant c(q)\Big[\|\omega^\theta\|_{L^q(P(\sigma_1))} + \frac{1}{\sigma_1-\sigma_2}\|\boldsymbol{b}\|_{L^q(P(\sigma_1))}\Big],\end{aligned}$$

where we recall $\boldsymbol{b} = u^r\boldsymbol{e}_r + u^z\boldsymbol{e}_z$.

P r o o f. It is well-known (see [21], page 186, for a direct calculation) that

$$\operatorname{div} \boldsymbol{b} = 0, \quad \operatorname{curl} \boldsymbol{b} = \omega^\theta \boldsymbol{e}_\theta.$$

Moreover, by [21], Lemma 2.1,

$$\|\nabla u^r\|_{L^q(P(\sigma_2))} + \left\|\frac{u^r}{r}\right\|_{L^q(P(\sigma_2))} + \|\nabla u^z\|_{L^q(P(\sigma_2))} \leqslant C\|\nabla \boldsymbol{b}\|_{L^q(\sigma_2)}.$$

Then we may apply Lemma 3.2 to conclude the proof. □

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