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PROPERTIES OF A QUASI-UNIFORMLY MONOTONE OPERATOR AND ITS APPLICATION TO THE ELECTROMAGNETIC p-curl SYSTEMS

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Abstract. In this paper we propose a new concept of quasi-uniform monotonicity weaker than the uniform monotonicity which has been developed in the study of nonlinear operator equation $Au = b$. We prove that if A is a quasi-uniformly monotone and hemi-continuous operator, then A^{-1} is strictly monotone, bounded and continuous, and thus the Galerkin approximations converge. Also we show an application of a quasi-uniformly monotone and hemi-continuous operator to the proof of the well-posedness and convergence of Galerkin approximations to the solution of steady-state electromagnetic p-curl systems.

Keywords: well-posedness; uniform monotonicity; S-property; p-curl systems

MSC 2020: 35D30, 35A15, 47H05, 65N30 78M10, 78M30

1. Introduction

The theory of monotone operators is a powerful tool in the study of nonlinear operator equations such as nonlinear elliptic partial differential equations. It is well known that a sufficient condition for the well-posedness and convergence of Galerkin approximations to the solution of a nonlinear operator equation is the uniform monotonicity and hemi-continuity, see [5], [9], [12], [14]. However, among nonlinear operator equations which are important in the real-life applications, there are many equations without uniform monotonicity or exhibiting difficulties in the proof of coercivity and S-property, see [5].

The goal of this paper is to introduce a new concept of a quasi-uniformly monotone operator weaker than the well-known uniformly monotone one, to show some of its nice properties and then its application to the electromagnetic p-curl systems.

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For the nonlinear operator equation, if the quasi-uniform monotonicity of the operator is obtained, we can directly prove the well-posedness and convergence of Galerkin approximations without the proof of coercivity and S-property. Therefore, the method proposed in this paper is more efficient in the sense that the proof of well-posedness and convergence of Galerkin approximations is simpler than that of previous methods in the case that the proof of coercivity and S-property is difficult or much more complicated.

Firstly we propose the definition of a quasi-uniformly monotone operator, which enables us to study nonlinear operator equations without uniform monotonicity, and prove the strict monotonicity, S-property and coercivity. This leads us to obtaining the well-posedness and convergence of Galerkin approximations for nonlinear operator equations with quasi-uniform monotonicity and hemi-continuity. Moreover, we show that some nonlinear differential operators which play important role in the real-life applications such as fluid mechanics and electromagnetics possess the quasi-uniform monotonicity and hemi-continuity.

Next, in this paper we use the quasi-uniform monotonicity and hemi-continuity of the operator to prove the well-posedness and convergence of Galerkin approximations to the solution of a steady-state electromagnetic p-curl system.

Steady-state Maxwell's equations are usually stated as

$$\operatorname{curl} H = J, \quad \operatorname{curl} E = 0, \quad \operatorname{div} D = \varrho, \quad \operatorname{div} B = 0. \tag{1.1}$$

In the nonlinear media, if the relations $E = \lambda(|J|)J$, $B = \mu H$ hold provided that the permeability is constant, then from (1.1) it follows that

$$\operatorname{curl}(\lambda(|\operatorname{curl} H|)\operatorname{curl} H) = 0, \quad \operatorname{div} H = 0,$$

and if the relations $H = \lambda(|B|)B, B = \operatorname{curl} A$ hold, then

$$\operatorname{curl}(\lambda(|\operatorname{curl} A|)\operatorname{curl} A) = J, \quad \operatorname{div} A = 0.$$

Here E, H are the electric and magnetic field, B is the magnetic flux density, D is the electric displacement, J is the current density, A is the magnetic vector potential, ε is the electric permittivity, μ is the magnetic permeability, ϱ is the electric charge density, and $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a given function.

Therefore, steady-state Maxwell's equations in nonlinear media can be considered as the quasi-linear PDE

$$\operatorname{curl}(\lambda(|\operatorname{curl} u(x)|)\operatorname{curl} u(x)) = f(x), \quad \operatorname{div} u(x) = 0, \quad x \in \Omega, \tag{1.2}$$

where Ω is a bounded simply connected three-dimensional domain.

In recent years, great attention has been focused on the study of Maxwell's equations in nonlinear media, see [2], [1], [13], [4], [6], [10], [11]. In [6], the authors have established the level set method for the inverse problem to the equation (1.2) in the case of $\lambda(s) = d + cs^b/(a^b + s^b)$, $a, b, c, d > 0$. In [11], the authors studied a linear fully discrete approximation scheme for the electric field and the convergence and error estimation when the relation between the electric field and the current density is $\lambda(s) = s^n$. More recently, there have been reported theoretical results for the equation (1.2), which is called the electromagnetic p-curl system, in the case of $\lambda(s) = s^{p-2}$, $f = f(x, u)$ and its some variations including $p(x)$, $p(x, t)$, see [2], [1], [13], [4]. In [4], the existence and non-existence of solution for $p(x)$-curl systems with the source $\lambda g(x, u) + \mu f(x, u)$ have been studied in the case of $6/5 < p(x) < 3$. In [2], [1], the unique existence and finite time extinction of the solution of $p(x, t)$-curl , p-curl systems with a nonlinear source have been studied in the case of $6/5 < p(x, t), p$, respectively. In [13], the authors proved the existence and multiplicity of solution of $p(x)$-curl systems having nonlinear source and other terms.

In this paper, we prove that the p-curl operator is quasi-uniformly monotone and hemi-continuous if $1 < p < 2$. Based on it, we next show the well-posedness and convergence of Galerkin approximations to the solution of electromagnetic p-curl systems.

The paper is organized as follows. In Section 2, we introduce the definition of a quasi-uniformly monotone operator and prove some its properties. Moreover, we give some examples of the quasi-uniformly monotone operators. In Section 3, we give the weak formulation of the problem (1.2) and consider some functional spaces and their norms. In Section 4, we prove that the operator defined by electromagnetic p-curl systems is quasi-uniformly monotone if $1 < p < 2$ and locally Hölder continuous if $1 < p$. Since the locally Hölder continuous operator is hemi-continuous, we show the local Hölder continuity in this paper.

2. Properties of a quasi-uniformly monotone operator

Let us introduce the concept of a quasi-uniformly monotone operator.

Definition 2.1. Let V be the Banach space, V^* the conjugate space of V, $A\colon V \to V^*$ an operator, $\langle\cdot,\cdot\rangle$ duality pairing in $V^* \times V$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, \infty)$.

If $\mu\colon \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ is continuous, strictly increasing in the first variable, not increasing in the second variable, with

$$\forall\, t \in \mathbb{R}_{++}, \quad \mu(0, t) = 0, \quad \mu(\infty, t) = \infty, \quad \lim_{t\to 0} \mu(t, t) = 0, \quad \lim_{t\to\infty} \mu(t, t) = \infty$$

and

$$\langle Au - Av, u - v\rangle \geqslant \|u - v\|\mu(\|u - v\|, \|u\| + \|v\|) \quad \forall\, u, v \in V,$$

then the operator A is called a quasi-uniformly monotone.

From Definition 2.1 it is obvious that the following implications for an operator hold:

uniformly monotone $\Rightarrow$ quasi-uniformly monotone $\Rightarrow$ strictly monotone.

Definition 2.2 ([9], III. Definition 1.4)**.** Let V be a Banach space, V^* the conjugate space of V, $A\colon V \to V^*$ an operator. Then we say that the operator A has S-property provided the following holds: if u_n tends to u weakly and $\langle Au_n - Au, u_n - u\rangle \to 0$, then $u_n \to u$.

For another definition of S-property, we refer to [8], [14].

Now let us consider the operator equation

$$Au = b, \quad b \in V^*, \tag{2.1}$$

and the Galerkin approximation equation

$$u_n \in V_n, \quad \langle Au_n, v\rangle = \langle b, v\rangle \quad \forall\, v \in V_n. \tag{2.2}$$

Here V_n is finite dimensional subspace of the space V.

Lemma 2.1. *If the operator A is quasi-uniformly monotone, then it is coercive and has the S-property.*

P r o o f. From Definition 2.1 it follows, that if $u \neq 0$, then

$$\begin{aligned}\frac{\langle Au, u\rangle}{\|u\|} &= \frac{\langle Au - A0 + A0, u\rangle}{\|u\|} = \frac{\langle Au - A0, u - 0\rangle}{\|u\|} + \frac{\langle A0, u\rangle}{\|u\|}\\ &\geqslant \frac{\|u - 0\|\mu(\|u - 0\|, \|u\| + \|0\|)}{\|u\|} - \|A0\|\\ &= \mu(\|u\|, \|u\|) - \|A0\| \to \infty \quad \text{as } \|u\| \to \infty.\end{aligned}$$

Therefore, A is coercive.

Now let us show the S-property. Let

$$u_n \overset{w}{\to} u, \quad \langle Au_n - u, u_n - u\rangle \to 0, \quad u_n, u \in V \quad \text{as } n \to \infty.$$

It is clear that

$$\exists\, c \in R_{++}, \quad \|u\| \leqslant c, \quad \|u_n\| \leqslant c \quad \forall\, n \in \mathbb{N}.$$

This together with Definition 2.1 yields that

$$\langle Au_n - u, u_n - u\rangle \geqslant \|u-u_n\|\mu(\|u-u_n\|, \|u\|+\|u_n\|) \geqslant \|u-u_n\|\mu(\|u-u_n\|, 2c),$$

which implies that $\|u-u_n\|\mu(\|u-u_n\|, 2c) \to 0$ as $n\to\infty$.

On the other hand, by Definition 2.1 the function $f(s) := s\mu(s,2c)$ is continuous and strictly increasing in R_+, and $f(0)=0, f(\infty)=\infty$. Hence, there exists the inverse function $g(t)\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(0)=0$, $g(\infty)=\infty$. Moreover, it is also continuous and strictly increasing. Let $a_n := f(\|u-u_n\|)$. Then $g(a_n) = \|u-u_n\| \to 0$ as $n\to\infty$, since $a_n \to 0$ as $n\to\infty$. Thus A has the S-property. $\square$

Lemma 2.2 ([5], [9], [12], [14])**.** *Let an operator $A\colon V\to V^*$ be strictly monotone, coercive, hemi-continuous and have the S-property. Let V be a real separable reflexive Banach space and V_n, $n=1, 2, \ldots$, be finite dimensional subspaces of V such that $V_1\subset V_2\subset \ldots$ and let $\bigcup\limits_{n=1}^{\infty} V_n$ be dense in V. Then there exists $A^{-1}\colon V^*\to V$ such that A^{-1} is strictly monotone, bounded and continuous. Moreover, a unique solution u_n to* (2.2) *converges to a unique solution u to the equation* (2.1) *as $n\to\infty$.*

From the two lemmas above, we immediately have the following theorem.

Theorem 2.1. *Let $A\colon V\to V^*$be quasi-uniformly monotone and hemi-continuous, and V, V_n satisfy the assumption of Lemma* 2.2*. Then the result of Lemma* 2.2 *continues to hold.*

We close this section with some examples of quasi-uniformly monotone and locally Hölder continuous operators. Let Ω be a bounded domain of $\mathbb{R}^d$, V a subspace of the Sobolev space $W_p^1(\Omega), 1<p<2$, $\lambda\in C(\mathbb{R}_+)$. Let the operator A be defined as

$$\langle Au,\ v\rangle = \int_\Omega \lambda(|\nabla u|)\nabla u\cdot\nabla v\,\mathrm{d}\Omega \quad \forall\, u,v\in V.$$

If $\lambda(s)$ satisfies one of the following relations, then the operator A is quasi-uniformly monotone and locally Hölder continuous.

▷ Gas flow through packed beds in the industrial furnaces, see [5].

$$\lambda(s) = \frac{1}{(a^2+bs)^{1/2}+a}, \quad a>0,\ b>0.$$

▷ Laminar flow of non-Newtonian fluid, see [5].

$$\lambda(s) = (a+bs)^{c-2}, \quad a>0,\ b>0,\ 1<c<2.$$

▷ Porous flow of fluid, see [14].

$$\lambda(s) = as^{c-2}, \quad a > 0,\ 1 < c < 2.$$

▷ Transonic flow of compressible ideal fluid gas, see [5].

$$\lambda(s) = \Bigl(1 + \frac{s^2}{2c}\Bigr)^{-c}, \quad 0 < c < 1/2.$$

3. Weak formulation of the equation (1.2)

Before we give the weak formulation of the equation (1.2), we introduce the notations to be used later. Let Ω be a bounded simply connected domain with $C^{0,1}$ boundary $\Gamma = \partial\Omega$ in $\mathbb{R}^3$, $(f)_\Omega, (f)_\Gamma$ the integrals of f over Ω and Γ. We denote the outward normal and tangential unit vector at $x \in \Gamma$ by $n(x), \tau(x)$, respectively. Let us put

$$\gamma_n u = (u\cdot n)n, \quad \gamma_t u = n \times u, \quad \gamma_\tau u = u - \gamma_n u.$$

Let $p > 1$, $1/p + 1/q = 1$. We define Sobolev type spaces and norms as

$$\begin{aligned}
W^{k,p}(\Omega) &= \{v \in L^p(\Omega); \nabla^\alpha v \in L^p(\Omega), |\alpha| \leqslant k\},\\
W^p(\operatorname{div}^0,\Omega) &= \{u \in [L^p(\Omega)]^3\colon\ \operatorname{curl} u \in [L^p(\Omega)]^3, \operatorname{div} u = 0\},\\
W^p_n(\Omega) &= \{u \in W^p(\operatorname{div}^0,\Omega)\colon\ \gamma_n u|_\Gamma = 0\},\\
W^p_t(\Omega) &= \{u \in W^p(\operatorname{div}^0,\Omega)\colon\ \gamma_t u|_\Gamma = 0\},\\
\|u\|_{\mathrm{curl},\,p} &= \|\operatorname{curl} u\|_{p,\Omega}.
\end{aligned}$$

Then we obtain from Corollaries 2.8 and 2.11 of [3] that for all $u \in [W^{k+1,p}(\Omega)]^3$,

$$\begin{aligned}
\|u\|_{k+1,p,\Omega} &\leqslant c(\|\operatorname{curl} u\|_{k,p,\Omega} + \|\operatorname{div} u\|_{k,p,\Omega} + \|\gamma_t u\|_{k+1/q,p,\Gamma}),\\
\|u\|_{k+1,p,\Omega} &\leqslant c(\|\operatorname{curl} u\|_{k,p,\Omega} + \|\operatorname{div} u\|_{k+1,p,\Omega} + \|\gamma_n u\|_{k+1/q,p,\Gamma}).
\end{aligned}$$

Here the constant c is independent of u. Setting $k = 0$, it holds that

$$\|u\|_{1,p,\Omega} \leqslant c\|u\|_{\mathrm{curl},\,p} \quad \forall\, u \in W^p_n(\Omega) \cup W^p_t(\Omega).$$

So the norm $\|\cdot\|_{\mathrm{curl},p}$ in subspaces $W^p_n(\Omega)$, $W^p_t(\Omega)$ of the space $W^{1,p}(\Omega)$ is equivalent to the norm of the space $W^{1,p}(\Omega)$. Using Lemma II.1.25 of [9] and Theorems I.6.12, II.3.23 of [7], we have the following lemma.

Lemma 3.1. *The spaces $W^p_n(\Omega)$, $W^p_t(\Omega)$ are separable reflexive Banach spaces with norm $\|\cdot\|_{\mathrm{curl},p}$ and this norm is equivalent to that of the space $W^{1,p}(\Omega)$.*

It is well known that for $u, v \in [C^1(\Omega)]^3$

$$\begin{aligned}
(\operatorname{div}(u\times v))_\Omega &= (n\cdot(u\times v))_\Gamma,\\
\operatorname{div}(u\times v) &= v\cdot \operatorname{curl} u - u\cdot \operatorname{curl} v,\\
n\cdot(u\times v) &= u\cdot(v\times n) = v\cdot(n\times u),\\
n\cdot(u\times v) &= -u\cdot\gamma_t v = v\cdot\gamma_t u,\\
u\cdot\gamma_t v &= (\gamma_n u + \gamma_\tau u)\cdot\gamma_t v = \gamma_\tau u\cdot\gamma_t v,
\end{aligned}$$

$$(v\cdot\operatorname{curl} u)_\Omega - (u\cdot\operatorname{curl} v)_\Omega = (v\cdot\gamma_t u)_\Gamma = -(u\cdot\gamma_t v)_\Gamma = -(\gamma_\tau u\cdot\gamma_t v)_\Gamma = (\gamma_\tau v\cdot\gamma_t u)_\Gamma.$$

Replacing u by $\lambda(|\operatorname{curl} u|)\operatorname{curl} u$ in the above relations, we get

$$\begin{aligned}
(3.1)\qquad (v\cdot\operatorname{curl}(\lambda(|\operatorname{curl} u|)\operatorname{curl} u))_\Omega &- (\lambda(|\operatorname{curl} u|)\operatorname{curl} u\cdot\operatorname{curl} v)_\Omega\\
&= -(\lambda(|\operatorname{curl} u|)\gamma_\tau\operatorname{curl} u\cdot\gamma_t v)_\Gamma\\
&= (\lambda(|\operatorname{curl} u|)\gamma_t\operatorname{curl} u\cdot\gamma_\tau v)_\Gamma \quad \forall\, u\in[C^2(\Omega)]^3, v\in[C^1(\Omega)]^3.
\end{aligned}$$

Here we use the boundary conditions

$$(3.2)\qquad \gamma_n u = 0,\quad \lambda(|\operatorname{curl} u|)\gamma_t\operatorname{curl} u = g \quad \text{on } \Gamma,$$

$$(3.3)\qquad \gamma_t u = 0 \quad \text{on } \Gamma.$$

From the equation (3.1), if u is the solution of (1.2), (3.2), then we have

$$u\in W_n^p(\Omega),\quad (\lambda(|\operatorname{curl} u|)(\operatorname{curl} u)\cdot(\operatorname{curl} v))_\Omega = (f\cdot v)_\Omega - (g\cdot\gamma_t v)_\Gamma \quad \forall\, v\in C^1(\Omega)^3\cap W_n^p(\Omega)$$

and if u is the solution of (1.2) and (3.3), then

$$u\in W_t^p(\Omega),\quad (\lambda(|\operatorname{curl} u|)(\operatorname{curl} u)\cdot(\operatorname{curl} v))_\Omega = (f\cdot v)_\Omega \quad \forall\, v\in C^1(\Omega)^3\cap W_t^p(\Omega).$$

Now, if we define the operator A as

$$\langle Au, v\rangle = (\lambda(|\operatorname{curl} u|)(\operatorname{curl} u)\cdot(\operatorname{curl} v))_\Omega,$$

then the solution of the problem (1.2), (3.2) or (1.2), (3.3) satisfies the variation equation

$$(3.4)\qquad u\in W_n^p(\Omega),\quad \langle Au, v\rangle = (f\cdot v)_\Omega - (g\cdot\gamma_t v)_\Gamma \quad \forall\, v\in W_n^p(\Omega),$$

or

$$(3.5)\qquad u\in W_t^p(\Omega),\quad \langle Au, v\rangle = (f\cdot v)_\Omega \quad \forall\, v\in W_t^p(\Omega).$$

In (3.4), (3.5), it is assumed that the given functions f, g are as smooth as the bounds $(f\cdot v)_\Omega$, $(g\cdot\gamma_t v)_\Gamma$.

4. Quasi-uniform monotonicity and local Hölder continuity of A

In the end of Section 2 we gave some examples of quasi-uniformly monotone and hemi-continuous operators. Here we want to show that the operator A defined by (4.1) below is quasi-uniformly monotone and locally Hölder continuous.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with $C^{0,1}$ boundary. Let us define the operator A as*

$$\langle Au, v\rangle = (|\operatorname{curl} u|^{p-2}(\operatorname{curl} u)\cdot(\operatorname{curl} v))_\Omega, \quad 1 < p < \infty, \tag{4.1}$$

for $u, v \in W$ $(:= W_n^p(\Omega)$ or $W_t^p(\Omega))$. Then the operator A is locally Hölder continuous. Moreover, it is quasi-uniformly monotone if $1 < p < 2$, strongly monotone if $p = 2$ and uniformly monotone if $p > 2$.

Since the local Hölder continuity implies the hemi-continuity, from Theorems 2.1 and 4.1 we have immediately:

Corollary 4.1. *The weak formulation* (3.4) (*or* (3.5)) *of the steady-state electromagnetic p-*curl *systems* (1.2), (3.2) (*or* (1.2), (3.3)) *is well-posed and the Galerkin approximations converge in the norm of the space $W_n^p(\Omega)$ (or $W_t^p(\Omega)$).*

Remark 4.1. If $f = f(x, u)$ is a nonlinear function in the equation (1.2), our result also holds under certain assumptions, for example, the monotonicity and hemi-continuity of the operator $Fu = -f(x, u)$.

Proof of Theorem 4.1. Now let $u, v \in \mathbb{R}^3$, $u \neq 0$, $v \neq 0$, $p > 1$. The following inequality is well known, see [1]: for all $u, v \in \mathbb{R}^3$

$$\exists\, c_p > 0,\ (|u|^{p-2}u - |v|^{p-2}v)\cdot(u - v) \geqslant \begin{cases} c_p|u - v|^2(|u| + |v|)^{p-2}, & 1 < p < 2,\\ c_p|u - v|^p, & p \geqslant 2. \end{cases} \tag{4.2}$$

In this paper, we explain how the constant c_p is related to p in more detail. To this end we use the following inequalities without proof: for $1 < p < \infty$, $r > 0$, $s > 0$, $t > 0$,

$$(s^{p-1} - t^{p-1})(s - t) \geqslant \eta_p(s - t)^2 \times \begin{cases} (s + t)^{p-2} & \text{if } 1 < p < 2,\\ (s^{p-2} + t^{p-2}) & \text{if } 2 \leqslant p, \end{cases} \tag{4.3}$$
$$|s^{p-1} - t^{p-1}| \leqslant \xi_p(s + t)|s - t|^{m_p},$$

where

$$
(4.4)\qquad m_p := \begin{cases} p-1 & \text{if } 1 < p \leqslant 2, \\ 1 & \text{if } 2 \leqslant p, \end{cases} \qquad \xi_p(r) := \begin{cases} 1 & \text{if } 1 < p \leqslant 2, \\ \widetilde{\xi}_p r^{p-2} & \text{if } 2 \leqslant p, \end{cases}
$$

$$
1 \leqslant \widetilde{\xi}_p \leqslant p, \quad 0 < \eta_p \leqslant \begin{cases} p-1 & \text{if } 1 < p < 2, \\ 1/2 & \text{if } 2 \leqslant p, \end{cases}
$$

while

$$
(4.5)\qquad \tilde{e}_r(s+t)^r \leqslant s^r + t^r \leqslant e_r(s+t)^r,
$$

$$
e_r := \begin{cases} 1 & \text{if } 1 \leqslant r, \\ 2^{1-r} & \text{if } 0 < r \leqslant 1, \end{cases} \qquad \tilde{e}_r := \begin{cases} 2^{1-r} & \text{if } 1 \leqslant r, \\ 1 & \text{if } 0 < r \leqslant 1. \end{cases}
$$

From the inequality (4.5), it follows that

$$
s^{p-2} + t^{p-2} \geqslant \tilde{e}_{p-2}(s+t)^{p-2} \quad \text{if } 2 \leqslant p.
$$

Combining the previous inequality with (4.3), we get

$$
(4.6)\qquad (s^{p-1} - t^{p-1})(s-t) \geqslant \widetilde{\eta}_p (s-t)^2 (s+t)^{p-2},
$$

$$
\widetilde{\eta}_p := \eta_p \times \begin{cases} 1 & \text{if } 1 < p \leqslant 3, \\ 2^{3-p} & \text{if } 3 > p. \end{cases}
$$

Since the function $f(x) = x^{-r}$, $r > 0$, is convex, we obtain

$$
\Big(\frac{s+t}{2}\Big)^{-r} \leqslant \frac{1}{2}(s^{-r} + t^{-r}), \quad s, t > 0,
$$

and therefore,

$$
(4.7)\qquad s^{-r} + t^{-r} \geqslant 2^{1+r}(s+t)^{-r}, \quad r, s, t > 0.
$$

We use (4.6) to get

$$
\begin{aligned}
(4.8)\qquad & (|u|^{p-2}u - |v|^{p-2}v) \cdot (u - v) \\
&= |u|^{p-2}|u|^2 + |v|^{p-2}|v|^2 - u \cdot v(|u|^{p-2} + |v|^{p-2}) \\
&= (|u|^{p-1} - |v|^{p-1})(|u| - |v|) + (|u|^{p-2} + |v|^{p-2})(|u||v| - u \cdot v) \\
&\geqslant \widetilde{\eta}_p(|u| - |v|)^2(|u| + |v|)^{p-2} + (|u|^{p-2} + |v|^{p-2})(|u||v| - u \cdot v).
\end{aligned}
$$

Considering (4.5) and (4.7) yields that

$$
\begin{aligned}
|u|^{p-2}+|v|^{p-2} &\geqslant \tilde{e}_{p-2}(|u|+|v|)^{p-2} \quad \text{if } p-2\geqslant 0,\\
|u|^{p-2}+|v|^{p-2} &\geqslant 2^{p-1}(|u|+|v|)^{p-2} \quad \text{if } -1<p-2<0,
\end{aligned}
$$

which can be rewritten as

$$
|u|^{p-2}+|v|^{p-2} \geqslant \widetilde{\widetilde{\eta}}_p(|u|+|v|)^{p-2}, \quad \widetilde{\widetilde{\eta}}_p = \begin{cases} 2^{p-1} & \text{if } 1<p<2,\\ 1 & \text{if } 2\leqslant p\leqslant 3,\\ 2^{3-p} & \text{if } 3<p. \end{cases} \tag{4.9}
$$

Connecting (4.8) and (4.9), we obtain that

$$
\begin{aligned}
&(|u|^{p-2}u-|v|^{p-2}v)\cdot(u-v)\\
&\quad\geqslant [\widetilde{\eta}_p(|u|-|v|)^2+\widetilde{\widetilde{\eta}}_p(|u||v|-u\cdot v)](|u|+|v|)^{p-2}\\
&\quad\geqslant \min\Big\{\widetilde{\eta}_p, \frac{1}{2}\widetilde{\widetilde{\eta}}_p\Big\}[(|u|-|v|)^2+2(|u||v|-u\cdot v)](|u|+|v|)^{p-2}\\
&\quad= \min\Big\{\widetilde{\eta}_p, \frac{1}{2}\widetilde{\widetilde{\eta}}_p\Big\}|u-v|^2(|u|+|v|)^{p-2}.
\end{aligned}
$$

As a result, we have that for all $p>1$, and $u,v\in\mathbb{R}^3$, $u\neq 0$, $v\neq 0$,

$$
(|u|^{p-2}u-|v|^{p-2}v)\cdot(u-v)\geqslant \widetilde{\widetilde{\widetilde{\eta}}}_p|u-v|^2(|u|+|v|)^{p-2}, \tag{4.10}
$$

where

$$
\widetilde{\widetilde{\widetilde{\eta}}}_p := \min\Big\{\widetilde{\eta}_p, \frac{1}{2}\widetilde{\widetilde{\eta}}_p\Big\} > 0.
$$

On the other hand, from (4.3) and (4.4), it follows that for $p\geqslant 2$

$$
\begin{aligned}
||u|^{p-2}u-|v|^{p-2}v| &= ||u|^{p-2}(u-v)+(|u|^{p-2}-|v|^{p-2})v|\\
&\leqslant |u|^{p-2}|u-v|+||u|^{p-2}-|v|^{p-2}|\,|v|,\\
||u|^{p-2}-|v|^{p-2}|\,|v| &\leqslant |u|^{p-2}||u|-|v||+||u|^{p-1}-|v|^{p-1}|\\
&\leqslant |u|^{p-2}||u|-|v||+\widetilde{\xi}_p(|u|+|v|)^{p-2}||u|-|v||.
\end{aligned}
$$

These two inequalities imply that

$$
\begin{aligned}
||u|^{p-2}u-|v|^{p-2}v| &\leqslant |u|^{p-2}|u-v|+|u|^{p-2}||u|-|v||+\xi_p(|u|+|v|)^{p-2}|u-v|\\
&\leqslant 2(|u|+|v|)^{p-2}|u-v|+\widetilde{\xi}_p(|u|+|v|)^{p-2}|u-v|.
\end{aligned}
$$

Hence,

$$
||u|^{p-2}u-|v|^{p-2}v|\leqslant (2+\widetilde{\xi}_p)(|u|+|v|)^{p-2}|u-v| \quad \text{if } p\geqslant 2. \tag{4.11}
$$

If $1 < p < 2$, then by (4.3), (4.4), (4.5), we get

$$\begin{aligned}
||u|^{p-2}u - |v|^{p-2}v|^2 &= (|u|^{p-1} - |v|^{p-1})^2 + 2|u|^{p-2}|v|^{p-2}(|u||v| - u\cdot v) \\
&\leqslant (||u| - |v||^{p-1})^2 + 2(|u||v|)^{p-2}(|u||v| - u\cdot v) \\
&\leqslant [(|u| - |v|)^2]^{p-1} + 2(|u||v|)^{p-2}(|u||v| - u\cdot v)^{p-1}(2|u||v|)^{2-p} \\
&= [(|u| - |v|)^2]^{p-1} + 2^{4-2p}[2(|u||v| - u\cdot v)]^{p-1} \\
&\leqslant 2^{4-2p}\{[(|u| - |v|)^2]^{p-1} + [2(|u||v| - u\cdot v)]^{p-1}\} \\
&\leqslant 2^{4-2p}2^{2-p}[(|u| - |v|)^2 + 2(|u||v| - u\cdot v)]^{p-1} \\
&= 2^{6-3p}|u - v|^{2(p-1)}.
\end{aligned}$$

Therefore,

$$||u|^{p-2}u - |v|^{p-2}v| \leqslant 2^{3-3p/2}|u - v|^{p-1} \quad \text{if } 1 < p < 2. \tag{4.12}$$

From (4.11), (4.12), we conclude that for all $p > 1$ and all $u, v \in \mathbb{R}^3$, $u \neq 0$, $v \neq 0$,

$$||u|^{p-2}u - |v|^{p-2}v| \leqslant \tilde{\tilde{\xi}}_p|u - v|^{m_p}(|u| + |v|)^{p-1-m_p}, \tag{4.13}$$

where m_p is from (4.4) and

$$\tilde{\tilde{\xi}}_p := \begin{cases} 2^{3-3p/2} & \text{if } 1 < p < 2, \\ 2 + \tilde{\xi}_p & \text{if } 2 \leqslant p. \end{cases}$$

Now let us prove the quasi-uniform monotonicity and local Hölder continuity of A defined by (4.1).

Since from (4.10)

$$\tilde{\tilde{\tilde{\eta}}}_p|u - v|^2 \leqslant (|u|^{p-2}u - |v|^{p-2}v)\cdot(u - v)(|u| + |v|)^{2-p},$$

we can easily get that for $1 < p < 2$

$$\tilde{\tilde{\tilde{\eta}}}_p^{p/2}\int_\Omega |u - v|^p \, \mathrm{d}\Omega \leqslant \int_\Omega [(|u|^{p-2}u - |v|^{p-2}v)\cdot(u - v)]^{p/2}(|u| + |v|)^{p(2-p)/2} \, \mathrm{d}\Omega,$$

and furthermore by the Hölder inequality with the pair $(p/2, (2-p)/2)$

$$\begin{aligned}
&\tilde{\tilde{\tilde{\eta}}}_p^{p/2}\int_\Omega |u - v|^p \, \mathrm{d}\Omega \\
&\quad\leqslant \left[\int_\Omega (|u|^{p-2}u - |v|^{p-2}v)\cdot(u - v) \, \mathrm{d}\Omega\right]^{p/2}\left(\int_\Omega (|u| + |v|)^p \, \mathrm{d}\Omega\right)^{p(2-p)/(2p)} \\
&\quad\leqslant \left[\int_\Omega (|u|^{p-2}u - |v|^{p-2}v)\cdot(u - v) \, \mathrm{d}\Omega\right]^{p/2}(\|u\|_{p,\Omega} + \|v\|_{p,\Omega})^{p(2-p)/2}.
\end{aligned}$$

Now replacing u, v by $\operatorname{curl} u, \operatorname{curl} v$, respectively, and using the fact that the norm $\|\cdot\|_{\mathrm{curl},p}$ is equivalent to the norm of $W^{1,p}(\Omega)$ in the spaces $W_n^p(\Omega), W_t^p(\Omega)$, we have

$$\tilde{\tilde{\tilde{\eta}}}_p^{p/2}\|u-v\|_{\mathrm{curl},p}^p \leqslant (\langle Au - Av, u - v\rangle)^{p/2}(\|u\|_{\mathrm{curl},p} + \|v\|_{\mathrm{curl},p})^{p(2-p)/2}.$$

The previous inequality implies immediately that for all $1 < p < 2$,

$$\begin{aligned}\langle Au - Av, u - v\rangle &\geqslant \tilde{\tilde{\tilde{\eta}}}_p\|u-v\|_{\mathrm{curl},p}^2(\|u\|_{\mathrm{curl},p} + \|v\|_{\mathrm{curl},p})^{p-2} \\ &= \|u-v\|_{\mathrm{curl},p}\mu_p(\|u-v\|_{\mathrm{curl},p}, \|u\|_{\mathrm{curl},p} + \|v\|_{\mathrm{curl},p}) \quad \forall\, u, v \in W,\end{aligned}$$

where

$$\mu_p(s,t) = \tilde{c}_p s t^{p-2}, \quad \tilde{c}_p = \tilde{\tilde{\tilde{\eta}}}_p > 0 \quad \text{for } 1 < p < 2.$$

Hence the operator A is quasi-uniformly monotone if $1 < p < 2$.

On the other hand, the strong monotonicity for $p = 2$ and the uniform one for $p > 2$ follow directly from the fact that by (4.2)

$$\langle Au - Av, u - v\rangle \geqslant c_p\|u-v\|_{\mathrm{curl},p}^p \quad \text{if} \quad p \geqslant 2.$$

Now it remains to show the local Hölder continuity of A. From (4.13) it follows that for $1/p + 1/q = 1$

$$\begin{aligned}&\left|\int_\Omega (|u|^{p-2}u - |v|^{p-2}v)w\,\mathrm{d}\Omega\right| \\ &\quad\leqslant \tilde{\tilde{\xi}}_p\int_\Omega |u-v|^{m_p}(|u|+|v|)^{p-1-m_p}|w|\,\mathrm{d}\Omega \\ &\quad\leqslant \tilde{\tilde{\xi}}_p\|w\|_{p,\Omega}\left[\int_\Omega |u-v|^{qm_p}(|u|+|v|)^{q(p-1-m_p)}\,\mathrm{d}\Omega\right]^{1/q}.\end{aligned} \tag{4.14}$$

For $p > 2$, setting

$$p_1 := \frac{p}{m_p q}, \quad p_2 := \frac{p}{(p-1-m_p)q}$$

and using the Hölder inequality with a pair (p_1, p_2), we get

$$\begin{aligned}&\left[\int_\Omega |u-v|^{qm_p}(|u|+|v|)^{q(p-1-m_p)}\,\mathrm{d}\Omega\right]^{1/q} \\ &\quad\leqslant \left(\int_\Omega |u-v|^p\,\mathrm{d}\Omega\right)^{m_p/p}\left(\int_\Omega (|u|+|v|)^p\,\mathrm{d}\Omega\right)^{(p-1-m_p)/p} \\ &\quad\leqslant \|u-v\|_{p,\Omega}^{m_p}(\|u\|_{p,\Omega} + \|v\|_{p,\Omega})^{p-1-m_p}.\end{aligned} \tag{4.15}$$

Since $p - m_p - 1 = 0$ for $1 < p \leqslant 2$ and $q(p-1) = p$, we have

$$\left[\int_\Omega |u-v|^{qm_p}(|u|+|v|)^{q(p-1-m_p)}\,\mathrm{d}\Omega\right]^{1/q} = \left(\int_\Omega |u-v|^p\,\mathrm{d}\Omega\right)^{1/q} = \|u-v\|_{p,\Omega}^{p/q} = \|u-v\|_{p,\Omega}^{m_p}. \tag{4.16}$$

Inserting (4.15), (4.16) into (4.14), replacing u, v by $\operatorname{curl} u, \operatorname{curl} v$ and applying the dual argument, we obtain that for $p > 1$

$$\|Au - Av\|_* \leqslant \tilde{\tilde{\xi}}_p \|u-v\|_{\mathrm{curl},p}^{m_p}(\|u\|_{\mathrm{curl},p} + \|v\|_{\mathrm{curl},p})^{p-1-m_p} \quad \forall\, u, v \in W,$$

which implies the local Hölder continuity of A. Note that $\|\cdot\|_*$ is the norm of W^* which is the conjugate space of W. $\square$

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