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INITIAL COEFFICIENTS FOR GENERALIZED SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED WITH SUBORDINATION

Gagandeep Singh[1], Gurcharanjit Singh[2], and Gurmeet Singh[3]

Abstract. This paper is concerned with certain generalized subclasses of bi-univalent functions defined with subordination in the open unit disc $E = \{z :| z |< 1\}$. The bounds for the initial coefficients for the functions in these classes are studied. The earlier known results follow as special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f having Taylor-Maclaurin series of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k\,, \tag{1}$$

defined in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Further, the class of functions $f \in \mathcal{A}$ and univalent in E, is denoted by $\mathcal{S}$. By $\mathcal{U}$, we denote the class of Schwarz functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc E and satisfy the conditions $u(0) = 0$ and $|u(z)| < 1$.

For $\delta \geq 1$ and $f \in \mathcal{A}$, Al-Oboudi [2] introduced the following differential operator:

$$D_\delta^0 f(z) = f(z)\,,$$
$$D_\delta^1 f(z) = (1-\delta)f(z) + \delta z f'(z),$$

and in general,

$$D_\delta^n f(z) = D(D_\delta^{n-1} f(z)) = (1-\delta)D_\delta^{n-1} f(z) + \delta z (D_\delta^{n-1} f(z))', n \in \mathcal{N}$$

or equivalent to

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}\,,$$

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with $D^n_\delta f(0) = 0$. For $\delta = 1$, the operator $D^n_\delta f(z)$ reduces to the Sãlãgean operator introduced in [13].

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarz function $u(z) \in \mathcal{U}$ such that $f(z) = g(u(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

It is obvious that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z(z \in E)$$

and

$$f(f^{-1}(w)) = w\Big(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\Big)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. The class of functions bi-univalent in E and given by (1) is denoted by Σ. Some examples of the functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. But, the well known Koebe function $f(z) = \frac{z}{(1-z)^2}$ is not a member of Σ.

Lewin [9] was the first, who investigated the class Σ and proved that $|a_2| < 1.51$. Subsequently, bounds for the initial coefficients of various sub-classes of bi-univalent functions were studied by various authors in [4, 5, 8, 10, 11] and more recently by Abirami et al. [1], Sivapalan et al. [18] and Singh et al. [15]–[17].

In the sequel, we lay down once and for all that $0 \leq \alpha \leq 1$, $\lambda \geq 0$, $0 < \beta \leq 1$, $0 \leq \eta < 1$, $\delta \geq 1$, $-1 \leq B < A \leq 1$, $z \in E$, $w \in E$ and $g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots$

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_\Sigma^{\lambda,\alpha,\beta}(A, B; s, t)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{(s-t)z[f'(z)]^\lambda}{f(sz) - f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^\lambda}{(f(sz) - f(tz))'} \prec \left(\frac{1+Az}{1+Bz}\right)^\beta$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^\lambda}{g(sw) - g(tw)} + \alpha\frac{(s-t)[(wg'(w))']^\lambda}{(g(sw) - g(tw))'} \prec \left(\frac{1+Aw}{1+Bw}\right)^\beta,$$

where s, $t \in \mathcal{C}$ with $s \neq t$, $|t| \leq 1$.

The following observations are obvious:

(i) $\mathcal{S}_\Sigma^{1,\alpha,\beta}(A, B; 1, -1) \equiv \mathcal{M}_\Sigma^s(\beta, \alpha; A, B)$, the class studied by Singh [14].

(ii) $\mathcal{S}_\Sigma^{\lambda,0,\beta}(1, -1; s, t) \equiv \mathcal{S}_\Sigma^{\lambda,\beta}(s, t)$, the class studied by Mazi and Opoola [12].

(iii) For $0 \leq \gamma < 1$, $\mathcal{S}_{\Sigma}^{\lambda,0,1}(1-2\gamma,-1;s,t) \equiv \mathcal{S}_{\Sigma}^{\lambda}(\gamma,s,t)$, the class studied by Mazi and Opoola [12].

(iv) $\mathcal{S}_{\Sigma}^{\lambda,0,\beta}(1,-1;1,0) \equiv \mathcal{S}_{\Sigma}^{\lambda,\beta}$, the class studied by Joshi and Pawar [7].

(v) For $0 \leq \gamma < 1$, $\mathcal{S}_{\Sigma}^{\lambda,0,1}(1-2\gamma,-1;1,0) \equiv \mathcal{S}_{\Sigma}^{\lambda}(\gamma)$, the class studied by Joshi and Pawar [7].

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda}(k,\beta;A,B)$ if the following conditions are satisfied:

$$\frac{z[(D^k f(z))']^{\lambda}}{D^k f(z)} \prec \Big(\frac{1+Az}{1+Bz}\Big)^{\beta}$$

and

$$\frac{w[(D^k g(w))']^{\lambda}}{D^k g(w)} \prec \Big(\frac{1+Aw}{1+Bw}\Big)^{\beta}.$$

Specifically,

(i) $\mathcal{S}_{\Sigma}^{\lambda}(k,\beta;1,-1) \equiv \mathcal{S}_{\Sigma}^{\lambda}(k,\beta)$, the class studied by Joshi et al. [6].

(ii) For $0 \leq \gamma < 1$, $\mathcal{S}_{\Sigma}^{\lambda}(k,1;1-2\gamma,-1) \equiv \mathcal{S}_{\Sigma}^{\lambda}(k,\gamma)$, the class studied by Joshi et al. [6].

Definition 1.3. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$ if the following conditions are satisfied:

$$(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^{\lambda}}{(f(sz)-f(tz))'} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]z}{1+Bz}\Big)^{\beta}$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-g(tw)} + \alpha\frac{(s-t)[(wg'(w))']^{\lambda}}{(g(sw)-g(tw))'} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]w}{1+Bw}\Big)^{\beta},$$

where $s,t \in \mathcal{C}$ with $s \neq t$, $|t| \leq 1$.

In particular, $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,0}(A,B;s,t) \equiv \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta}(A,B;s,t)$.

Definition 1.4. A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{\lambda,\delta,\eta}(k,\beta;A,B)$ if the following conditions are satisfied:

$$\frac{z[(D_{\delta}^k f(z))']^{\lambda}}{D_{\delta}^k f(z)} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]z}{1+Bz}\Big)^{\beta}$$

and

$$\frac{w[(D_{\delta}^k g(w))']^{\lambda}}{D_{\delta}^k g(w)} \prec \Big(\frac{1+[B+(A-B)(1-\eta)]w}{1+Bw}\Big)^{\beta}.$$

Particularly, $\mathcal{S}_{\Sigma}^{\lambda,1,0}(k,\beta;A,B) \equiv \mathcal{S}_{\Sigma}^{\lambda}(k,\beta;A,B)$.

For deriving our main results, we need to the following lemma

Lemma 1.1 ([3]). *If* $p(z) = \dfrac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, $u(z) \in \mathcal{U}$, *then*

$$|p_n| \leq (A-B)(1-\eta)\,, \quad n \geq 1\,.$$

2. The class $\mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$

Theorem 2.1. *If* $f \in \mathcal{S}_{\Sigma}^{\lambda,\alpha,\beta,\eta}(A,B;s,t)$, *then*

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)(1-\eta)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}} \tag{2}$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2(1-\eta)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}. \tag{3}$$

Proof. From Definition 1.3, by principle of subordination, we have

$$(1-\alpha)\frac{(s-t)z[f'(z)]^{\lambda}}{f(sz)-f(tz)} + \alpha\frac{(s-t)[(zf'(z))']^{\lambda}}{(f(sz)-f(tz))'} = \left(\frac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)}\right)^{\beta} = [p(z)]^{\beta},\ u \in \mathcal{U} \tag{4}$$

and

$$(1-\alpha)\frac{(s-t)w[g'(w)]^{\lambda}}{g(sw)-g(tw)} + \alpha\frac{(s-t)[(wg'(w))']^{\lambda}}{(g(sw)-g(tw))'} = \left(\frac{1+[B+(A-B)(1-\eta)]v(w)}{1+Bv(w)}\right)^{\beta} = [q(w)]^{\beta},\ v \in \mathcal{U}\,, \tag{5}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \ldots$.

On expanding and equating the coefficients of z and z^2 in (4) and of w and w^2 in (5), we obtain

$$(1+\alpha)(2\lambda-s-t)a_2 = \beta p_1\,, \tag{6}$$

$$(1+3\alpha)[(s^2+2st+t^2)-2\lambda(s+t-\lambda+1)]a_2^2 + (1+2\alpha)(3\lambda-s^2-st-t^2)a_3 = \beta p_2 + \frac{\beta(\beta-1)p_1^2}{2} \tag{7}$$

and

$$-(1+\alpha)(2\lambda-s-t)a_2 = \beta q_1\,, \tag{8}$$

$$[(6\lambda-s^2-t^2)-2\lambda(s+t-\lambda+1)-\alpha(6\lambda(s+t-\lambda-1)+(s-t)^2)]a_2^2 - (1+2\alpha)(3\lambda-s^2-st-t^2)a_3 = \beta q_2 + \frac{\beta(\beta-1)q_1^2}{2}\,. \tag{9}$$

(6) and (8) together gives

$$p_1 = -q_1 \tag{10}$$

and

$$2(1+\alpha)^2(2\lambda - s - t)^2 a_2^2 = \beta^2(p_1^2 + q_1^2)\,. \tag{11}$$

Adding (7) and (9) and using (11), it yields

$$\begin{aligned}&[(2\lambda - 4\lambda(s+t-\lambda) + 2st) + 2\alpha((s^2+4st+t^2) - 6\lambda(s+t-\lambda))]a_2^2\\ &\qquad = \beta(p_2+q_2) + \frac{(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2 a_2^2}{\beta}\,.\end{aligned} \tag{12}$$

(12) gives

$$a_2^2 = \frac{\beta^2(p_2+q_2)}{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}. \tag{13}$$

On applying Lemma 1.1 to the coefficients p_2 and q_2, we can easily obtain (2). Now subtracting (9) from (7), we get

$$-2(1+2\alpha)(3\lambda-s^2-st-t^2)a_2^2+2(1+2\alpha)(3\lambda-s^2-t^2-st)a_3 = \beta(p_2-q_2)\,. \tag{14}$$

Using (10) and (11) in (14), using Lemma 1.1 and on applying triangle inequality, (3) can be easily obtained.

□

On putting $\eta = 0$, Theorem 2.1 gives the following result:

Corollary 2.1. *If $f \in \mathcal{S}_\Sigma^{\lambda,\alpha,\beta}(A,B;s,t)$, then*

$$|a_2| \le \frac{\beta\sqrt{2(A-B)}}{\sqrt{\beta[(2\lambda-4\lambda(s+t-\lambda)+2st)+2\alpha((s^2+4st+t^2)-6\lambda(s+t-\lambda))]-(\beta-1)(1+\alpha)^2(2\lambda-s-t)^2}}$$

and

$$|a_3| \le \frac{\beta(A-B)}{(1+2\alpha)(3\lambda-s^2-st-t^2)} + \frac{(A-B)^2\beta^2}{(1+\alpha)^2(2\lambda-s-t)^2}\,.$$

For $\eta = 0, \lambda = 1, s = 1, t = -1$, Theorem 2.1 gives the following result due to Singh [14]:

Corollary 2.2. *If $f \in \mathcal{M}_\Sigma^s(\beta,\alpha;A,B)$, then*

$$|a_2| \le \frac{\beta\sqrt{A-B}}{\sqrt{2((1+\alpha)^2-\beta\alpha^2)}}$$

and

$$|a_3| \leq \frac{\beta^2(A-B)^2}{4(1+\alpha)^2} + \frac{\beta(A-B)}{2(1+2\alpha)}\,.$$

3. The class $\mathcal{S}_{\Sigma}^{\lambda,\delta,\eta}(k,\beta;A,B)$

Theorem 3.1. *If $f \in \mathcal{S}_{\Sigma}^{\lambda,\delta,\eta}(k,\beta;A,B)$, then*

$$|a_2| \leq \frac{\beta\sqrt{2(A-B)(1-\eta)}}{\sqrt{4\beta(3\lambda-1)(1+2\delta)^k + [4\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2(1+\delta)](1+\delta)^{2k}}} \tag{15}$$

and

$$|a_3| \leq \frac{\beta(A-B)(1-\eta)}{(3\lambda-1)(1+2\delta)^k} + \frac{2\beta^2(A-B)^2(1-\eta)^2}{(2\lambda-1)^2(1+\delta)^{2k+1}}\,. \tag{16}$$

Proof. From Definition 1.4, by principle of subordination, we have

$$\frac{z[(D_\delta^k f(z))']^\lambda}{D_\delta^k f(z)} = \left(\frac{1+[B+(A-B)(1-\eta)]u(z)}{1+Bu(z)}\right)^\beta = [p(z)]^\beta,\ u \in \mathcal{U} \tag{17}$$

and

$$\frac{w[(D_\delta^k g(w))']^\lambda}{D_\delta^k g(w)} = \left(\frac{1+[B+(A-B)(1-\eta)]v(w)}{1+Bv(w)}\right)^\beta = [q(w)]^\beta,\ v \in \mathcal{U}, \tag{18}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \ldots$.

On expanding and equating the coefficients of z and z^2 in (17) and of w and w^2 in (18), we obtain

$$(2\lambda-1)(1+\delta)^k a_2 = \beta p_1\,, \tag{19}$$

$$(3\lambda-1)(1+2\delta)^k a_3 + (2\lambda^2-4\lambda+1)(1+\delta)^{2k} a_2^2 = \beta p_2 + \frac{\beta(\beta-1)p_1^2}{2} \tag{20}$$

and

$$-(2\lambda-1)(1+\delta)^k a_2 = \beta q_1\,, \tag{21}$$

$$[2(3\lambda-1)(1+2\delta))^k + (2\lambda^2-4\lambda+1)(1+\delta)^{2k}]a_2^2 - (3\lambda-1)(1+2\delta)^k a_3 = \beta q_2 + \frac{\beta(\beta-1)q_1^2}{2}\,. \tag{22}$$

(19) and (21) together give

$$p_1 = -q_1 \tag{23}$$

and

$$(2\lambda-1)^2(1+\delta)^{2k+1} a_2^2 = \beta^2(p_1^2+q_1^2)\,. \tag{24}$$

Adding (20) and (22) and using (24), it yields

$$[2\beta(3\lambda-1)(1+2\delta)^k+\{2\beta(2\lambda^2-4\lambda+1)-\frac{(\beta-1)}{2}(2\lambda-1)^2(1+\delta)\}(1+\delta)^{2k}]a_2^2$$
$$=\beta^2(p_2+q_2)\,. \tag{25}$$

(25) gives

$$a_2^2= \tag{26}$$
$$\frac{2\beta^2(p_2+q_2)}{4\beta(3\lambda-1)(1+2\delta)^k+\{4\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2(1+\delta)\}(1+\delta)^{2k}}\,.$$

On applying Lemma 1.1 to the coefficients p_2 and q_2 in (26), we can easily obtain (15).

Now subtracting (22) from (20), we get

$$2(3\lambda-1)(1+2\delta)^k a_3-2(3\lambda-1)(1+2\delta)^k a_2^2=\beta(p_2-q_2)\,. \tag{27}$$

Using (e24), (e27) yields

$$a_3=\frac{\beta^2(p_1^2+q_1^2)}{(2\lambda-1)^2(1+\delta)^{2k+1}}+\frac{\beta(p_2-q_2)}{2(3\lambda-1)(1+2\delta)^k}\,. \tag{28}$$

Applying Lemma 1.1 to the coefficients p_2, q_2 and p_1 in (28), (16) is obvious. □

For $\delta=1$, $\eta=0$, the following result can be easily obtained from Theorem 3.1:

Corollary 3.1. *If $f\in\mathcal{S}_\Sigma^\lambda(k,\beta;A,B)$, then*

$$|a_2|\le\frac{\beta\sqrt{2(A-B)}}{\sqrt{2\beta(3\lambda-1)3^k+[2\beta(2\lambda^2-4\lambda+1)-(\beta-1)(2\lambda-1)^2]2^{2k}}}$$

and

$$|a_3|\le\frac{\beta(A-B)}{(3\lambda-1)3^k}+\frac{\beta^2(A-B)^2}{(2\lambda-1)^2 2^{2k}}\,.$$

For $\delta=1$, $\eta=0$, $A=1$, $B=-1$, Theorem 3.1 gives the following result due to Joshi et al. [6]:

Corollary 3.2. *If $f\in\mathcal{S}_\Sigma^\lambda(k,\beta;A,B)$, then*

$$|a_2|\le\frac{2\beta}{\sqrt{2\beta(3\lambda-1)3^k+\{2\beta(2\lambda^2-4\lambda-1)-(\beta-1)(2\lambda-1)^2\}2^{2k}}}$$

and

$$|a_3|\le\frac{2\beta}{(3\lambda-1)3^k}+\frac{4\beta^2}{(2\lambda-1)^2 2^{2k}}\,.$$

Putting $\delta=1$, $\eta=0$, $A=1-2\gamma$, $B=-1$ and $\beta=1$ in Theorem 3.1, we obtain the following result due to Joshi et al. [6]:

Corollary 3.3. *If* $f \in \mathcal{S}_{\Sigma}^{\lambda}(k,\gamma)$, *then*

$$|a_2| \leq \frac{2\sqrt{1-\gamma}}{\sqrt{2(3\lambda-1)3^k + [(2\lambda-1)^2 - (4\lambda-1)]2^{2k}}}$$

and

$$|a_3| \leq \frac{4(1-\gamma)^2}{(2\lambda-1)^2 2^{2k}} + \frac{2(1-\gamma)}{(3\lambda-1)3^k}\,.$$

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