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ON THE MULTIPLICITY OF LAPLACIAN EIGENVALUES FOR UNICYCLIC GRAPHS

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Abstract. Let G be a connected graph of order n and U a unicyclic graph with the same order. We firstly give a sharp bound for $m_G(\mu)$, the multiplicity of a Laplacian eigenvalue μ of G. As a straightforward result, $m_U(1) \leqslant n-2$. We then provide two graph operations (i.e., grafting and shifting) on graph G for which the value of $m_G(1)$ is nondecreasing. As applications, we get the distribution of $m_U(1)$ for unicyclic graphs on n vertices. Moreover, for the two largest possible values of $m_U(1) \in \{n-5, n-3\}$, the corresponding graphs U are completely determined.

Keywords: unicyclic graph; Laplacian eigenvalue; multiplicity; bound

MSC 2020: 05C50

1. Introduction

All graphs considered in this paper are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and $A(G)$ be the *adjacency matrix* of G. For any $v \in V(G)$ we denote by $d(v)$ the degree of v. Let $D(G)$ be the diagonal matrix of vertex degrees of G. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix*. The polynomial $\psi(G;\mu) = \det(\mu I - L(G))$, where I is the identity matrix, is the *characteristic polynomial* of G with respect to $L(G)$. Since $L(G)$ is positive semidefinite matrix, its eigenvalues can be ordered as

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1} \geqslant \mu_n = 0.$$

We denote by $m_G(\mu)$ the multiplicity of Laplacian eigenvalue μ of G. Let μ_1, $\mu_2, \ldots, \mu_t$ be t distinct Laplacian eigenvalues of G. Then the Laplacian spectrum

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of G is denoted by $\mathrm{Spec}_L(G) = \{\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_t^{m_t}\}$, where $m_G(\mu_i) = m_i$ is the multiplicity of μ_i for $i = 1, 2, \ldots, t$, and $\sum\limits_{i=1}^{t} m_i = n$. Fiedler in [10] showed that $\mu_{n-1}(G) > 0$ if and only if G is connected, so $\mu_{n-1}(G)$ is called *algebraic connectivity* of graph G, and the corresponding vector is called *Fiedler vector*.

As usual, we always write, P_n, C_n $(n \geqslant 3)$ and S_n $(n \geqslant 2)$ for the *path*, the *cycle* and the *star* on n vertices. Connected graphs in which the number of edges equals the number of vertices are called *unicyclic graphs*. The *girth* g of a graph G is the length of a shortest cycle of G. We denote by $\mathcal{U}(n, g)$ the set of all connected unicyclic graphs with girth g and order n, where $n \geqslant g \geqslant 3$. The graph $G' = G - e$ (or $G' = G + e$) is obtained from G by deleting (or adding) an edge e of G, and $G \setminus v$ is obtained from G by deleting v and its incident edges of G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any two vertices of G.

Let $G = (V, E)$ be a graph on n vertices. Then for a nonempty $V' \subseteq V$, we denote by $G[V']$ the subgraph of G induced by V'. A *pendant vertex* is a vertex of degree 1 and a *quasi-pendant vertex* is a vertex adjacent to a pendant vertex. The number of pendant and quasi-pendant vertices of G is denoted by $p(G)$ and $q(G)$ (or p and q if there is no confusion). Let $V_P(G) = \{v \in V(G)\colon\ v$ is a pendant vertex of $G\}$ and $V_Q(G) = \{v \in V(G)\colon\ v$ is a quasi-pendant vertex of $G\}$. Then all vertices of $V_R(G) = V \setminus (V_P(G) \cup V_Q(G))$ are called the *inner vertices* of G. We denote $|V_R(G)| = r$, $|V_P(G)| = p$ and $|V_Q(G)| = q$. Clearly, $r = n - p - q$. Other notions and symbols not defined here are standard, one can also see [6], for instance.

Graphs with few distinct eigenvalues form an interesting class of graphs and possess nice combinatorial properties, see [8], [13], [15], [16], [17], [18]. The number of distinct eigenvalues are closely related to their multiplicities (the less number of eigenvalues, the higher multiplicity of some eigenvalues would be). Therefore, *considering the multiplicity of eigenvalues is a related problem to investigate the graphs with few of distinct eigenvalues.* Grone, Merris and Sunder in [11] proved that for a tree T with n vertices, $m_T(\mu) = 1$ if $\mu > 1$ is a Laplacian integral eigenvalue of T. Moreover, they gave some results on the multiplicity of 1 as a Laplacian eigenvalue. Barik, Lal and Pati in [4] gave a complete characterization of trees that have 1 as the third smallest Laplacian eigenvalue. After that, Guo, Feng and Zhang in [12] presented the distribution of $m_T(1)$ for tree T of order n, and they showed that for every $\sigma \in S = \{0, 1, 2, \ldots, n-4, n-2\}$ there exists a tree T of order n such that $m_T(1) = \sigma$. Andrade et al. in [3], page 83, (2), showed that if $V_R(G) = \emptyset$, then

$$m_G(1) = p(G) - q(G)$$

for any graph G. Clearly, it always holds if T is a tree with $|V_R(T)| = 0$. Recently, Akbari, van Dam and Fakharan in [2] further considered the multiplicities of the other (non-integral) Laplacian eigenvalues of trees. For the unicyclic graph that contains a perfect matching, Akbari, Kiani and Mirzakhah in [1] determined the multiplicity of Laplacian eigenvalue 2.

Motivated by the above researches, we firstly give a sharp bound for $m_G(\mu)$ (see Theorem 3.1) in this paper, and it then follows that $m_U(1) \leqslant n-2$ for a unicyclic graph U. In addition, we provide two graph operations (i.e., *grafting* and *shifting*, see Theorems 4.1 and 4.2) of G, which preserve the non-decreasing property for the value of $m_G(1)$. As applications, we obtain the distribution of $m_U(1)$ for unicyclic graphs on n vertices, see Theorem 4.5. Moreover, for the two largest possible values of $m_U(1) \in \{n-5, n-3\}$, the corresponding graphs U are completely determined, see Theorems 4.3 and 4.6.

2. Preliminaries

In this section, we cite some lemmas as a preparation for later use.

Lemma 2.1 ([6], page 187). *If e is an edge of the graph G and $G' = G - e$, then*

$$\mu_1(G) \geqslant \mu_1(G') \geqslant \ldots \geqslant \mu_{n-1}(G) \geqslant \mu_{n-1}(G') \geqslant \mu_n(G) = \mu_n(G') = 0.$$

Lemma 2.2 ([11], Theorem 2.3). *Let μ be an eigenvalue of $L(T)$ for some tree T with $n \geqslant 2$ vertices. Then*

$$m_T(\mu) \leqslant p(T) - 1.$$

Lemma 2.3 ([14], Corollary 2.1). *Let G be a connected graph on $n \geqslant 3$ vertices. If G has a cutpoint v, then $\mu_{n-1}(G) \leqslant 1$, where equality holds if and only if v is adjacent to every vertex of G. Especially, if G is a unicyclic graph, then the equality holds if and only if $G \cong S_n^3$, see Figure* 1.

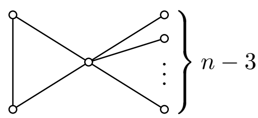


Figure 1. Graph S_n^3.

The *nullity* of a matrix M, denoted by $\nu(M)$, is the dimension of the *null space* of M. Let G be a graph with $V(G) = V_P \cup V_Q \cup V_R$, and let $L_R(G)$ be the principal submatrix of $L(G) - I_n$ that corresponds to the inner vertices V_R, where I_n is an identity matrix of order n.

Lemma 2.4 ([11], page 226). *Let G be a connected graph on n vertices. Then*

$$m_G(1) = p - q + \nu(L_R(G)). \tag{1}$$

Remark 2.1. Grone et al. in [11] gave a proof of (1) but not in the form of theorem. We here cite it as a lemma to be used later. In addition, Andrade et al. in [3] also redefined the equation in terms of $m_{L_R(G)}(1)$, the multiplicity of eigenvalue 1 in $L_R(G)$.

Let $Gu : vH$ be the graph obtained from G and H by joining a vertex u of G to a vertex v of H. Especially, if $H = P_2$ $(= vw)$, we denote it by $Gu : vw$ for short.

Lemma 2.5 ([12], Corollary 2.3). *Let H be a graph and S_n be a star on $n \geqslant 3$ vertices. Set $G = Hu : vS_n$.*

(a) *If v is a pendant vertex of S_n, then*

$$m_G(1) = m_H(1) + n - 3.$$

(b) *If v is the center of S_n, then*

$$m_G(1) = m_{Hu:vw}(1) + n - 2.$$

Lemma 2.6 ([6], Proposition 7.5.6). *If G is a connected graph with s distinct Laplacian eigenvalues, then* $\operatorname{diam}(G) \leqslant s - 1$.

A graph G is *P_5-free* if it does not contain induced subgraph P_5. For a connected P_5-free graph we list the following lemma.

Lemma 2.7 ([15], Lemma 3.4). *Let G be a connected P_5-free graph on $n \geqslant 5$ vertices with* $\operatorname{diam}(G) = 3$. *Then at least one of f_i for $i = 1, 2, 3, 4, 5$ (shown in Figure 2) is an induced subgraph of G.*

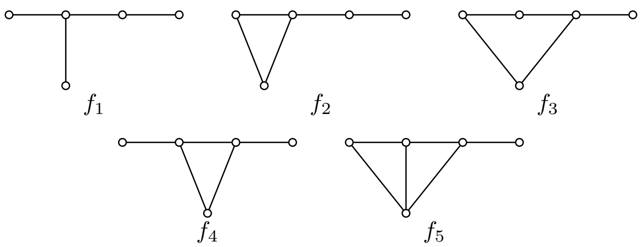


Figure 2. Graphs f_1, f_2, f_3, f_4 and f_5.

3. A bound of the multiplicity of Laplacian eigenvalue 1 of a unicyclic graph

In this section, we mainly consider the bound of the multiplicity of Laplacian eigenvalue 1 of a unicyclic graph. A connected graph with n vertices and ε edges is called a *k-cyclic graph* if $k = \varepsilon - n + 1$. Such a k is the so called *cyclomatic number* of G and is denoted by $c(G)$.

Theorem 3.1. *Let G be a connected graph with $n \geqslant 3$ vertices, size ε and $p(G)$ pendant vertices. Suppose that μ is a Laplacian eigenvalue of G. If $G \cong C_n$, then $\max\{m_G(\mu)\} = 2$, where $\mu = 2 - 2\cos(2\pi j/n)$ for $j = 1, 2, \ldots, \lceil \frac{1}{2}n \rceil - 1$, and otherwise,*

$$\max\{m_G(\mu)\colon \mu \text{ is a eigenvalue of } L(G)\} \leqslant p(G) + 2\varepsilon - 2n + 1, \tag{2}$$

where the equality holds if $G \cong S_n$.

Proof. Let μ be a Laplacian eigenvalue of G with the multiplicity $m_G(\mu)$. If $G \cong C_n$, then $p(G) = 0$. It is well-known that $\mathrm{Spec}_L(C_n) = \{2 - 2\cos(2\pi j/n)\colon j = 0, 1, \ldots, n-1\}$, see also [5], page 9. So we have $m_{C_n}(\mu) \leqslant 2$ for any Laplacian eigenvalue μ of C_n, and $m_{C_n}(\mu) = 2$ if $\mu = 2 - 2\cos(2\pi j/n)$ for $j = 1, 2, \ldots, \lceil \frac{1}{2}n \rceil - 1$.

In what follows, we always assume that G is a connected graph and $G \ncong C_n$. Let $k = \varepsilon - n + 1$. Then G is also a k-cyclic graph. In order to prove the conclusion, we only need to show

$$\max\{m_G(\mu)\colon \mu \text{ is a eigenvalue of } L(G)\} \leqslant p(G) + 2k - 1. \tag{3}$$

To promote the proof, we give a fact that can be verified by simple observations.

Fact 3.1. Let G be a $k(\geqslant 1)$-cyclic graph. If there exists an edge $uv \in E(G)$ that lies on some cycle of G such that either $d(u) = 2$ and $d(v) > 2$ or $d(u) > 2$ and $d(v) = 2$, then $p(G - uv) = p(G) + 1$ and $c(G - uv) = c(G) - 1 = k - 1$, otherwise there exists edge uv on some cycle with $d(u) > 2$ and $d(v) > 2$ such that $p(G - uv) = p(G)$ and $c(G - uv) = c(G) - 1 = k - 1$.

We distinct two cases to show inequality (3) as follows.

If G has at least one pendent vertex, we prove it by induction on k. When $k = 0$, G is a tree. Then by Lemma 2.2 we have $m_G(\mu) \leqslant p(G) - 1 = p(G) + 2k - 1$. Clearly, the conclusion holds. Now assume that inequality (3) holds for such graphs with cyclomatic number less than $k \geqslant 1$, and let G be a k-cyclic graph. Note that $p(G) \geqslant 1$ and $k \geqslant 1$. If μ is a simple eigenvalue, then $m_G(\mu) = 1 \leqslant p(G) + 2k - 1$.

Otherwise, μ is a multiple Laplacian eigenvalue of G. Since $G \ncong C_n$ is a k-cyclic graph, by Fact 3.1 there exists an edge $uv \in E(G)$ such that $G' = G - uv$ (say) is a spanning subgraph of G, and G' is a $(k-1)$-cyclic graph with $p(G') \leqslant p(G) + 1$. Thus, it follows from Lemma 2.1 that μ is also a Laplacian eigenvalue of G'. By our assumption, $m_{G'}(\mu) \leqslant p(G') + 2(k-1) - 1 \leqslant p(G) + 1 + 2k - 3 = p(G) + 2k - 2$. It therefore follows from Lemma 2.1 that μ is also a Laplacian eigenvalue of G and $m_G(\mu) \leqslant p(G) + 2k - 1$.

If G has no pendent vertex, then $p(G) = 0$, meanwhile, it implies that $k \geqslant 2$ since $G \ncong C_n$. Here we need to prove $m_G(\mu) \leqslant 2k - 1$ for any μ. Since k is finite, according to Fact 3.1, there exists the smallest integer $1 \leqslant l < k$ such that $G' = G - \{e_1, e_2, \ldots, e_l\}$ is a $(k-l)$-cyclic graph with just one pendent vertex, where e_i is an edge of G for $i = 1, 2, \ldots, l$. Then by the conclusion above we have $m_{G'}(\mu) \leqslant 1 + 2(k-l) - 1 = 2k - 2l$. Note that G can be obtained from G' by adding e_i $(i = 1, 2, \ldots, l)$ in proper order. Therefore, by Lemma 2.1 it follows that $m_G(\mu) \leqslant 2k - 2l + l = 2k - l \leqslant 2k - 1$, as desired.

Moreover, if $G \cong S_n$, then $p(G) = n - 1$. Note that $\mathrm{Spec}_L(S_n) = \{n, 1^{n-2}, 0\}$. Hence, $\max\{m_G(\mu)\} = m_G(1) = n - 2 = p(G) + 2\varepsilon - 2n + 1$, and thus the equality of (2) holds.

Summing up the above, the proof is completed. □

Let U be a unicyclic graph. Then $k = 1$. From Theorem 3.1 we have the following corollary.

Corollary 3.1. *Let U be a unicyclic graph with $p(U)$ pendant vertices, and μ be an eigenvalue of $L(U)$. Then*

$$m_U(\mu) \leqslant p(U) + 2,$$

where the equality holds if and only if $U \cong C_n$ and $\mu = 2 - 2\cos(2\pi j/n)$ for $j = 1, 2, \ldots, \lceil \frac{1}{2}n \rceil - 1$.

Proof. From Theorem 3.1, the inequality is obvious. For the equality, we only prove the sufficiency. Let U be a unicyclic graph with $m_U(\mu) = p(U) + 2$. Assume that $U \ncong C_n$, then there exists an edge such that $U - e$ is a tree (say T) with $p(T) \leqslant p(U) + 1$. It follows from Lemma 2.1 that $m_T(\mu) \geqslant p(U) + 1$, which leads to $m_T(\mu) \geqslant p(T)$, it contradicts Lemma 2.2, and thus $U \cong C_n$. We notice that $\mathrm{Spec}_L(C_n) = \{2 - 2\cos(2\pi j/n)\colon\ j = 0, 1, \ldots, n-1\}$, it therefore follows that the sufficiency holds. □

From Corollary 3.1, we give a sharp bound of the multiplicity of 1 for unicyclic graph below.

Corollary 3.2. *Let U be a unicyclic graph with $p(U)$ pendants and $q(U)$ quasi-pendents. Then*

$$p(U) - q(U) \leqslant m_U(1) \leqslant p(U) + 2, \tag{4}$$

where the right-hand side of the equality holds if and only if $U \cong C_{6t}$ $(t \geqslant 1)$.

Proof. From Lemma 2.4 and Corollary 3.1, inequality (4) always holds.

We then prove the right-hand side of equality of (4). Let U be a unicyclic graph with $m_U(1) = p+2$. Then by Corollary 3.1 we know that U is just a cycle. Note that $\mathrm{Spec}_L(C_n) = \{2 - 2\cos(2\pi j/n)\colon\ j = 0, 1, \ldots, n-1\}$. Therefore, $2 - 2\cos(2\pi j/n) = 1$ for some j if and only if $n = 6t$ for $t \geqslant 1$ since $\cos(2\pi j/n) = \frac{1}{2}$, which implies that $j = \frac{1}{6}n$ or $j = \frac{5}{6}n$, i.e., $6 \mid n$. Thus, the right equality of (4) holds if and only if $U \cong C_{6t}$ for $t \geqslant 1$. □

Remark 3.1. In fact, Grone and Merris obtained $m_G(1) = p - q + \nu(L_R(G))$ for any graph G (see [11] and Lemma 2.4), which is very useful for a graph if $\nu(L_R(G))$ can be determined. In Corollary 3.2, we give a sharp bound of $m_U(1)$ for a unicyclic graph from the perspective of its structure. It is worth to mention that the lower bound is obvious by Grone and Merris' result, which was also obtained by Faria earlier, see [9], page 260. In addition, Andrade et al. in [3], page 83, have proved that $m_G(1) = p - q$ if G has no inner vertex, and thus, the left-hand side of the equality of (4) always holds for unicylic graphs.

Let U be a unicyclic graph on n vertices, and $p(U)$ the number of pendant vertices. Then $p(U) \leqslant n-3$ since $g \geqslant 3$. Thus, by Theorem 3.1 we have the following corollary.

Corollary 3.3. *Let U be a unicyclic graph on n vertices. Then $m_U(1) \leqslant n - 2$.*

In fact, the upper bound given in Corollary 3.3 cannot be achieved. In the next section, we will further give the distribution of $m_U(1)$.

4. Distribution of $m_U(1)$ and the characterization for the extremal unicyclic graphs

Grone et al. in [11] and Andrade et al. in [3], respectively, presented a pretty formula to count the multiplicity of the Laplacian eigenvalue 1, i.e., $m_G(1) = p - q + \nu(L_R(G))$ and $m_G(1) = p - q + m_{L_R(G)}(1)$ from which, however, we still cannot compute $m_G(1)$ in terms of the structure of G. Let us alone find which value $m_G(1)$ will take. In this section, we first provide two graph operations (i.e., *grafting* and *shifting*) of a connnected graph G that preserve the non-decreasingness of $m_G(1)$.

By using those tools we will determine the distribution of $m_U(1)$ among all unicylic graphs $\mathcal{U}$ and then completely characterize extremal graphs U with respect to the first and second large values of $m_U(1)$.

A subgraph of G induced by a quasi-pendant and its pendant vertices is called a *star composition* of G. For example, let u be a quasi-pendant with pendant vertices $u_1, u_2, \ldots, u_{n_1}$ in G, and denote $V' = \{u, u_1, \ldots, u_{n_1}\} \subseteq V(G)$. Then $G[V'] \cong S_{n_1+1}$. For convenience, we say that G has a S_{n_1+1}-*composition* of G, see Figure 3 for instance.

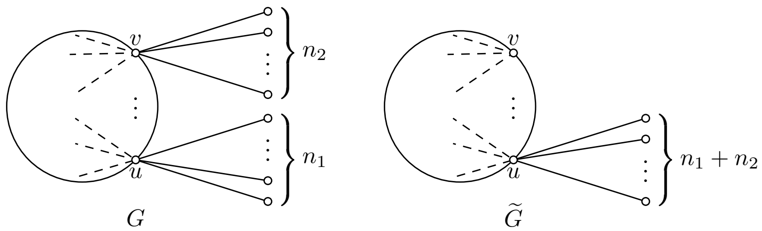


Figure 3. Graphs G and $\widetilde{G}$.

From Lemma 2.4 we find that the *star compositions* of graphs play an important role in the multiplicity of Laplacian eigenvalue 1, which was confirmed by Lemma 3.1 of [7]. It is worth to mention that $\mu_{n-1}(G)$ is the algebraic connectivity of G. Although $\mu_{n-1}(G)$ can contribute the multiplicity to $m_G(1)$ if $\mu_{n-1}(G) = 1$, it follows from Lemma 2.3 that $\mu_{n-1}(G) = 1$ depends on whether there exists a cut-vertex v adjacent to every vertex of G or not.

Let G be a graph with S_{n_1+1}-composition and S_{n_2+1}-composition, and let u and v be the center of S_{n_1+1} and S_{n_2+1}, respectively. Let $\widetilde{G}$ be the graph obtained from G by deleting the pendant edges of v and then joining those isolated vertices to the center u of S_{n_1+1}. At this point, $\widetilde{G}$ has a $S_{n_1+n_2+1}$-composition, and we say $\widetilde{G}$ is obtained from G by *grafting* S_{n_2+1}-composition to S_{n_1+1}-composition, see Figure 3.

Theorem 4.1. *Let G be a connected graph with S_{n_1+1}-composition and S_{n_2+1}-composition, where u and v are the centers of S_{n_1+1} and S_{n_2+1}, respectively. Let $\widetilde{G}$ be obtained from G by grafting S_{n_2+1}-composition to S_{n_1+1}-composition. Then $m_{\widetilde{G}}(1) \geqslant m_G(1)$.*

Proof. Let $l(L_R(G))$ and $l(L_R(\widetilde{G}))$ be the rank of matrices $L_R(G)$ (of order r_1) and $L_R(\widetilde{G})$ (of order r_2), respectively. By the well-known *Rank-Nullity Theorem*, we have

$$\nu(L_R(G)) = r_1 - l(L_R(G)) \quad \text{and} \quad \nu(L_R(\widetilde{G})) = r_2 - l(L_R(\widetilde{G})).$$

Thus, it follows that

$$\nu(L_R(G)) - \nu(L_R(\widetilde{G})) = (r_1 - r_2) + (l(L_R(\widetilde{G})) - l(L_R(G))). \tag{5}$$

Suppose that $I(G)$ is the set of the inner vertices in G. We here consider two cases on vertex v after the grafting of G.

Case 1: When v becomes a pendant vertex of $\widetilde{G}$, then there exists one vertex v' (say) as its quasi-pendant vertex in $\widetilde{G}$.

Subcase 1.1: If v' has other pendant vertices as its neighbors in G, then $I(G)$ is still the set of the inner vertices of $\widetilde{G}$. So, it implies that $r_2 = r_1$ and $l(L_R(\widetilde{G})) = l(L_R(G))$. Thus, by (5) we have $\nu(L_R(G)) = \nu(L_R(\widetilde{G}))$.

Without loss of generality, we may assume that v' has n_3 pendant vertices in G. From the grafting we know that the S_{n_3+1}-composition on v' of G becomes a new S_{n_3+2}-composition in $\widetilde{G}$. Besides, the S_{n_1+1}-composition and S_{n_2+1}-composition of G become another new $S_{n_1+n_2+1}$-composition of $\widetilde{G}$. Hence, $p(\widetilde{G}) = p(G) + 1$ and $q(\widetilde{G}) = q(G) - 1$. Furthermore, it follows from Lemma 2.4 that

$$\begin{aligned} m_{\widetilde{G}}(1) &= p(\widetilde{G}) - q(\widetilde{G}) + \nu(L_R(\widetilde{G})) = p(G) + 1 - q(G) + 1 + \nu(L_R(\widetilde{G})) \\ &= p(G) - q(G) + \nu(L_R(G)) + 2 = m_G(1) + 2 > m_G(1), \end{aligned}$$

i.e., $m_{\widetilde{G}}(1) > m_G(1)$.

Subcase 1.2: If v' has just one pendant vertex v in $\widetilde{G}$, that is, v' is inner vertex of G, then $I(G) \setminus \{v'\}$ is the set of inner vertices of $\widetilde{G}$, which implies that $L_R(\widetilde{G})$ is a principal submatrix of $L_R(G)$. Hence,

$$r_2 = r_1 - 1 \tag{6}$$

and

$$l(L_R(\widetilde{G})) \leqslant l(L_R(G)). \tag{7}$$

Combining (5), (6) and (7) we get

$$\nu(L_R(G)) \leqslant \nu(L_R(\widetilde{G})) + 1. \tag{8}$$

According to the grafting we see that S_{n_1+1}-composition and S_{n_2+1}-composition of G become a new $S_{n_1+n_2+1}$-composition of $\widetilde{G}$. Besides, it produces another new S_2-composition $v'v$ in $\widetilde{G}$. Except for the above, the remainder remains unchangeable during the operation, and thus, $p(\widetilde{G}) = p(G) + 1$ and $q(\widetilde{G}) = q(G)$. Therefore, it follows from Lemma 2.4 that

$$\begin{aligned} m_{\widetilde{G}}(1) &= p(\widetilde{G}) - q(\widetilde{G}) + \nu(L_R(\widetilde{G})) = p(G) + 1 - q(G) + \nu(L_R(\widetilde{G})) \\ &\geqslant p(G) - q(G) + \nu(L_R(G)) \quad \text{(by (8))} \\ &= m_G(1), \end{aligned}$$

i.e., $m_{\widetilde{G}}(1) \geqslant m_G(1)$.

Case 2: When v becomes an inner vertex of $\widetilde{G}$, then from Figure 3 we see that $I(G)\cup\{v\}$ are the inner vertices of $\widetilde{G}$. It implies that $L_R(G)$ is a principal submatrix of $L_R(\widetilde{G})$ and thus,

$$r_2 = r_1 + 1 \quad \text{and} \quad l(L_R(\widetilde{G})) \leqslant l(L_R(G)) + 2. \tag{9}$$

Combining (5) and (9) we also get the same inequality of (8).

By the operation of $\widetilde{G}$, just S_{n_1+1}-composition and S_{n_2+1}-composition of G become a new $S_{n_1+n_2+1}$-composition of $\widetilde{G}$, and the other star compositions remain unchangeable during the operation. So we have $p(\widetilde{G}) = p(G)$, $q(\widetilde{G}) = q(G) - 1$ and

$$p(\widetilde{G}) - q(\widetilde{G}) = p(G) - (q(G) - 1) = p(G) - q(G) + 1.$$

Furthermore, it follows from Lemma 2.4 that

$$m_{\widetilde{G}}(1) = p(\widetilde{G}) - q(\widetilde{G}) + \nu(L_R(\widetilde{G})) = p(G) - q(G) + 1 + \nu(L_R(\widetilde{G})). \tag{10}$$

Hence, from (8) and (10) we get

$$\begin{aligned} m_{\widetilde{G}}(1) &= p(G) - q(G) + 1 + \nu(L_R(\widetilde{G})) \geqslant p(G) - q(G) + \nu(L_R(G)) \\ &= m_G(1) \quad \text{(by Lemma 2.4)}, \end{aligned}$$

i.e., $m_{\widetilde{G}}(1) \geqslant m_G(1)$, as required. □

In what follows, we always denote by $U(n, g; n_1, n_2, \ldots, n_q) \in \mathcal{U}(n, g)$ the unicyclic graph with S_{n_i+1}-composition attached at some vertex v_i of the exact cycle C_g, where $1 \leqslant i \leqslant q \leqslant g$ and $p = n_1 + n_2 + \ldots + n_q < n$. Especially, $U(n, g; n')$ denotes the unicyclic graph having a unique $S_{n'+1}$-composition. So, from Theorem 4.1 we have the following corollary.

Corollary 4.1. *Let $U(n, g; n_1, n_2, \ldots, n_q) \in \mathcal{U}(n, g)$ be a unicyclic graph with S_{n_i+1}-compositions for $1 \leqslant i \leqslant q$. Then*

$$\begin{aligned} m_{U(n,g;n')}(1) &= m_{U(n,g;n_1+\ldots+n_q)}(1) \geqslant \ldots \geqslant m_{U(n,g;n_1+n_q,n_2,\ldots,n_{q-1})}(1) \\ &\geqslant m_{U(n,g;n_1,\ldots,n_q)}(1), \end{aligned}$$

where $n' = n_1 + \ldots + n_q$.

Let G be a connected graph with $S_{n'+1}$-composition on the center vertex v. Let v_1 be an inner vertex which is adjacent to v in G, and $\widehat{G}$ be obtained from G by deleting the edge vv_1 and identifying v and v_1 as a new vertex v', and also adding a pendant vertex on the center of $S_{n'+1}$-decomposition to obtain a new $S_{n'+2}$-composition, see Figure 4. We say that $\widehat{G}$ is obtained from G by *shifting* an inner vertex. Clearly, $\widehat{G}$ has a $S_{n'+2}$-composition.

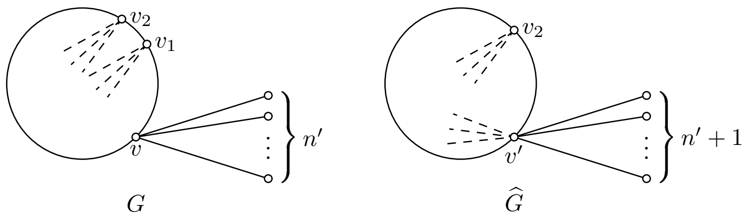


Figure 4. Graphs G and $\widehat{G}$.

Remark 4.1. In the operation of *shifting*, if v and v_1 have a common neighbor w, then $v'w$ must produce a parallel edge, but it does not affect the proof of Theorem 4.1. In addition, we mainly consider the unicyclic graphs of girth greater than 3, and thus, the above case will not appear.

Theorem 4.2. *Let G be a connected graph with $S_{n'+1}$-composition on the center vertex v. Let v_1 be an inner vertex which is adjacent to v in G, and $\widehat{G}$ be obtained from G by shifting the vertex v_1. Then $\widehat{G}$ has a $S_{n'+2}$-composition and $m_{\widehat{G}}(1) \geqslant m_G(1)$.*

Proof. Let $l(L_R(G))$ and $l(L_R(\widehat{G}))$ be the rank of matrices $L_R(G)$ (of order r_1) and $L_R(\widehat{G})$ (of order r_2), respectively. For convenience, we denote by $I(G) = \{v_1, v_2, \ldots, v_{r_1}\}$ the inner vertices of G. Then $I(\widehat{G}) = \{v_2, \ldots, v_{r_1}\}$.

According to the definition of $\widehat{G}$, $L_R(\widehat{G})$ is obtained from $L_R(G)$ by deleting the row and column corresponding to v_1. Clearly, $r_2 = r_1 - 1$. Let $L_R(\widehat{G})$ be the principal submatrix of $L_R(G)$ corresponding to $I(G) \setminus \{v_1\}$. Then

$$l(L_R(\widehat{G})) \leqslant l(L_R(G)).$$

Similarly to the proof of Theorem 4.1, we have

$$\nu(L_R(\widehat{G})) + 1 \geqslant \nu(L_R(G)). \tag{11}$$

Note that $p(\widehat{G}) = p(G) + 1$ and $q(\widehat{G}) = q(G)$. Then by Lemma 2.4 we get

$$m_G(1) = p(G) - q(G) + \nu(L_R(G)) \tag{12}$$

and

$$m_{\widehat{G}}(1) = p(\widehat{G}) - q(\widehat{G}) + \nu(L_R(\widehat{G})) = (p(G)+1) - q(G) + \nu(L_R(\widehat{G})). \tag{13}$$

Therefore, it follows from (11), (12) and (13) that

$$m_{\widehat{G}}(1) = p(G) - q(G) + \nu(L_R(\widehat{G})) + 1 \geqslant p(G) - q(G) + \nu(L_R(G)) = m_G(1),$$

that is, $m_{\widehat{G}}(1) \geqslant m_G(1)$.

The proof is completed. □

According to Theorem 4.2, we have the following corollary.

Corollary 4.2. *Let $U(n,g;n') \in \mathcal{U}(n,g)$ be a unicyclic graph with the exact $S_{n'+1}$-composition. Then*

$$m_{S_n^3}(1) = m_{U(n,3;n'+g-3)}(1) \geqslant \ldots \geqslant m_{U(n,g-1;n'+1)}(1) \geqslant m_{U(n,g;n')}(1).$$

From Corollaries 4.1 and 4.2 we have the following theorem.

Theorem 4.3. *Let U be a unicyclic graph on $n \geqslant 4$ vertices. Then*

$$m_U(1) \leqslant n-3,$$

where the equality holds if and only if $U \cong S_n^3$.

P r o o f. Let U be a unicyclic graph of $\mathcal{U}(n,g)$. For convenience, we denote $U = U(n,g;T_1,T_2,\ldots,T_g)$, where T_i is a tree-component attaching at ith vertex v_i of the exact cycle $C_g = v_1v_2\ldots v_gv_1$ in U. Applying the grafting and shifting operations (see Theorems 4.1 and 4.2) on each T_i repeatedly and appropriately, one can get the resulting graph $U(n,g;n_1,n_2,\ldots,n_g)$ (as defined before) such that

$$m_{U(n,g;n_1,n_2,\ldots,n_g)}(1) \geqslant m_U(1). \tag{14}$$

By Corollary 4.1 we have

$$m_{U(n,g;n-g)}(1) \geqslant m_{U(n,g;n_1,n_2,\ldots,n_g)}(1). \tag{15}$$

Furthermore, it follows from Corollary 4.2 that

$$m_{S_n^3}(1) = m_{U(n,3;n-3)}(1) \geqslant m_{U(n,g;n-g)}(1). \tag{16}$$

Together with (14), (15) and (16) we have $m_U(1) \leqslant m_{S_n^3}(1)$. By simple calculation, one can get $m_{S_n^3}(1) = n-3$. Thus, $m_U(1) \leqslant n-3$.

We now prove the equality. If $U \cong S_n^3$, the equality is clear. Conversely, let U be a unicyclic graph with $m_U(1) = n-3$. If U is a cycle, then by Corollary 3.2 we have $m_U(1) = n-3 \leqslant 2$ and deduce that $n \leqslant 5$, clearly, $6 \nmid n$, which is a contradiction. Thus, $U \ncong C_n$, which means that U has at least one cut-point. Then by Lemma 2.3 we have $\mu_{n-1}(U) \leqslant 1$. We claim that $\mu_{n-1}(U) = 1$ since if not, then $\mu_{n-1}(U) < 1$, which together with $m_U(1) = n-3$, means that we have $\mu_{n-2}(U) = \ldots = \mu_2(U) = 1$. Furthermore, from $\sum\limits_{i=1}^{n-1} \mu_i = 2n$ one can get $\mu_1(U) > n+2$, however, it is easy to deduce that $\mu_1(U) \leqslant n$ (see [6], Proposition 7.3.3), a contradiction. Consequently, by Lemma 2.3 we have $U \cong S_n^3$.

The proof follows. □

Remark 4.2. Applying Theorems 4.1 and 4.2 we see that $m_T(1) \leqslant n-2$ if T is a tree on $n \geqslant 6$ vertices, and equality holds if and only if $T \cong S_n$, which was also proved by Guo et al. in [12].

Remark 4.3. Using the same method as in the proof of Theorem 4.3 and combining it with Lemma 2.4, one can deduce that $m_{B_n}(1) \leqslant n-4$ if B_n is a bicyclic graph on $n \geqslant 5$ vertices, and equality holds if B_n is one of B_n^1 and B_n^2, shown in Figure 5.

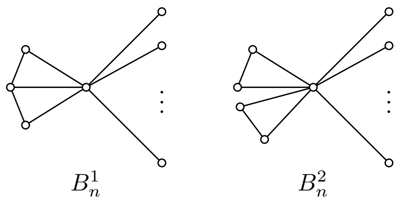


Figure 5. Graphs B_n^1 and B_n^2.

For convenience, we also use $m_U(a,b)$ (or $m_U[a,b)$) to denote the sum of its multiplicity of all Laplacian eigenvalues μ that belong to (a,b) (or $[a,b)$), where a and b are two nonnegative integers. From Theorem 4.3 we see that $m_U(1) \notin \{n-2, n-1, n\}$ for any unicyclic graph U on n vertices. Besides those, there is another exception of $m_U(1)$ below.

Theorem 4.4. *Let U be a unicyclic graph of order $n \geqslant 7$. Then $m_U(1) \neq n-4$.*

P r o o f. Assume that U is a unicyclic graph with $m_U(1) = n-4$. Then U has at most 5 distinct Laplacian eigenvalues. By Lemma 2.6 one can get $\operatorname{diam}(U) \leqslant 4$. If $\operatorname{diam}(U) \leqslant 2$, then $U \in \{S_n^3 (n \geqslant 4), C_3, C_4, C_5\}$. By Lemma 2.4, one can find that $m_{S_n^3}(1) = n-3$ and $m_{C_3}(1) = m_{C_4}(1) = m_{C_5}(1) = 0$, a contradiction. So we may further assume that $3 \leqslant \operatorname{diam}(U) \leqslant 4$ according to the above arguments. Notice that $U \ncong S_n^3$ due to $m_{S_n^3}(1) = n-3$ and we have $\mu_{n-1}(U) < 1$ by Lemma 2.3. If

$\operatorname{diam}(U) = 4$, then U contains P_5 as its induced subgraph. By a direct calculation we get

$$\operatorname{Spec}_L(P_5) = \{0, 0.3820, 1.3820, 2.6180, 3.6180\}.$$

It is easy to see that $H = P_5 \cup (n-5)K_1$ is a spanning subgraph of U, and U can be obtained from H by adding $n-4$ edges $e_1, e_2, \ldots, e_{n-4}$, where $e_i \in E(U) \setminus E(P_5)$ for $i = 1, 2, \ldots, n-4$. Let $U_i = H + \{e_1, e_2, \ldots, e_i\}$. Clearly, $U = U_{n-4}$. By Lemma 2.1 we have

$$\begin{aligned}
\mu_1(U) &= \mu_1(U_{n-4}) \geqslant \mu_1(U_{n-5}) \geqslant \ldots \geqslant \mu_1(U_1) \geqslant \mu_1(P_5) = 3.6180,\\
\mu_2(U) &= \mu_2(U_{n-4}) \geqslant \mu_2(U_{n-5}) \geqslant \ldots \geqslant \mu_2(U_1) \geqslant \mu_2(P_5) = 2.6180,\\
\mu_3(U) &= \mu_3(U_{n-4}) \geqslant \mu_3(U_{n-5}) \geqslant \ldots \geqslant \mu_3(U_1) \geqslant \mu_3(P_5) = 1.3820.
\end{aligned}$$

Therefore, $m_U(1, n] \geqslant 3$. Since $\mu_{n-1}(U) < 1$, we have $m_U[0,1) = m_U(0) + m_U(0,1) \geqslant 2$. So we get

$$m_U(1) = n - m_U[0,1) - m_U(1,n] \leqslant n - 5,$$

which contradicts $m_U(1) = n - 4$.

f_1	0	0.5188	1.0000	2.3111	4.1701
f_2	0	0.5188	2.3111	3.0000	4.1701
f_3	0	0.8299	2.0000	2.6889	4.4812
f_4	0	0.6972	1.3820	3.6180	4.3028

Table 1. The Laplacian spectra of f_1, f_2, f_3 and f_4.

So if $m_U(1) = n-4$, then $\operatorname{diam}(U) = 3$ and U contains no P_5 as its subgraph. Note that U is a unicyclic graph. Therefore, it follows from Lemma 2.7 that U has at least one of f_1, f_2, f_3 and f_4 (see Figure 2) as its subgraph. Furthermore, if U contains at least one of f_2, f_3 and f_4 as its subgraphs, then from simple calculation we get Table 1. By regarding f_i ($i = 2, 3, 4$) as P_5 as above, we can prove that $m_U(1) \leqslant n-5$, which is a contradiction. Otherwise U only contains f_1 as its subgraph. Since U is a P_5-free unicyclic graph, it follows that U must contain $C_3^2(1,1)$ (see Figure 8) as its another subgraph. By direct calculation we have

$$\operatorname{Spec}_L(C_3^2(1,1)) = \{0, 0.4859, 1.0000, 2.4280, 3.0000, 5.0861\}.$$

Thus, we can similarly deduce that $m_U(1) \leqslant n-5$, which is also a contradiction. $\square$

Guo et al. in [12] first introduced the notion of *Laplacian* 1*-realizable* for tree, we here introduce the concept on unicyclic graphs: a subset $\mathbb{N}$ of $\{0,1,\dots,n\}$ is said to be *Laplacian* 1*-realizable* for unicyclic graphs with n vertices provided that for any $\sigma\in\mathbb{N}$, if there exists at least one unicyclic graph U on n vertices such that $m_U(1)=\sigma$. It is clear that the largest $\mathbb{N}$ such that it is Laplacian 1-realizable for unicyclic graphs is just the distribution of $m_U(1)$.

Theorem 4.5. *The set* $\mathbb{N}=\{0,1,\dots,n-5,n-3\}$ *is Laplacian* 1*-realizable for unicyclic graphs with* $n\geqslant 7$ *vertices.*

Proof. For any positive integer $\sigma\in\mathbb{N}$, let U be a unicyclic graph on n vertices with $m_U(1)=\sigma$. We here distinguish the following situations.

When $\sigma=n-3$, it is clear by Theorem 4.3.

When $\sigma=n-k$ for $5\leqslant k\leqslant n-1$, we consider two subcases as follows:

(i) If k is odd, then we take $U=G_1$ (see Figure 6) with $n=\frac12(k+3)+\sum\limits_{i=1}^{(k-1)/2} r_i$, where G_1 is obtained from $C_{(k+3)/2}$ by attaching $r_i\geqslant 1$ pendent vertices on each vertex of $C_{(k+3)/2}$ except for a pair of adjacent vertices, say u and v. Clearly, $p(G_1)=n-\frac12(k+3)$ and $q(G_1)=\frac12(k-1)$, moreover, u and v are just two inner vertices of G_1. Let $L_R(G_1)$ be the principal submatrix of $L(G_1)-I_n$ that corresponds to the inner vertices u and v of G_1. Then we have

$$L_R(G_1)=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

So, one can see that $\nu(L_R(G_1))=1$. Thus, it follows from Lemma 2.4 that

$$m_{G_1}(1)=p(G_1)-q(G_1)+\nu(L_R(G_1))=n-\frac{k+3}{2}-\frac{k-1}{2}+1=n-k,$$

as desired.

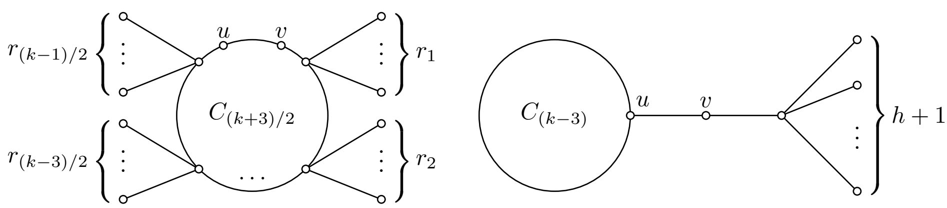
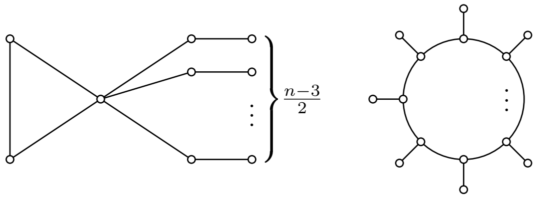


Figure 6. Graphs G_1 and G_2.

(ii) If k is even, then we take $U=G_2=C_{k-3}u:vS_{h+3}$ (see Figure 6), where $k\geqslant 6$, $h\geqslant 1$ and $n=k+h$. Since k is even, $k-3$ is odd, we further have $6\nmid(k-3)$.

Note that $\operatorname{Spec}_L(C_n) = \{2 - 2\cos(2\pi j/n)\colon j = 0, 1, \ldots, n-1\}$. Then from the proof of Corollary 3.2 we get $m_{C_{k-3}}(1) = 0$. Therefore, by Lemma 2.5 (a), $m_{G_2}(1) = m_{C_{k-3}}(1) + (h+3) - 3 = m_{C_{k-3}}(1) + h = h = n-k$, where k is even for $6 \leqslant k \leqslant n-1$.



$\check{S}_n^3$ $\qquad$ $\operatorname{Sun}(\frac{1}{2}n, n)$

Figure 7. Graphs $\check{S}_n^3$ and $\operatorname{Sun}(\frac{1}{2}n, n)$.

When $\sigma = 0$, we take $U = \check{S}_n^3$ (see Figure 7) when $n \geqslant 7$ is odd, where $\check{S}_n^3$ is obtained from C_3 by joining $\frac{1}{2}(n-3)$ rays (legs) on one of its vertices. Since $L_R(\check{S}_n^3)$ is the principal submatrix of $L(\check{S}_n^3) - I_n$ that corresponds to the inner vertices of $\check{S}_n^3$, we have

$$L_R(\check{S}_n^3) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & \frac{1}{2}(n-1) \end{pmatrix}.$$

Since $\det(L_R(\check{S}_n^3)) = -4$ for $n \geqslant 7$ and $p(\check{S}_n^3) = q(\check{S}_n^3)$, by Lemma 2.4 we have $m_{\check{S}_n^3}(1) = p(\check{S}_n^3) - q(\check{S}_n^3) + \nu(L_R(\check{S}_n^3)) = 0$. On the other hand, when n is even, we take $U = \operatorname{Sun}(\frac{1}{2}n, n)$, where $\operatorname{Sun}(\frac{1}{2}n, n)$ is the sun graph (see Figure 7) obtained from $C_{n/2}$ ($n \geqslant 8$) by appending one pendant vertex on each vertex of $C_{n/2}$. Then it follows from Lemma 2.4 that $m_{\operatorname{Sun}(n/2,n)}(1) = p(\operatorname{Sun}(\frac{1}{2}n, n)) - q(\operatorname{Sun}(\frac{1}{2}n, n)) = 0$, as required.

Sum up the above, it would meet our requirements whenever σ is taken in $\mathbb{N}$. So the proof is completed. $\square$

Remark 4.4. Let $G = C_6$ be the cycle on 6 vertices. Then $\operatorname{Spec}_L(C_6) = \{0, 1^2, 3^2, 4\}$. Clearly, 1 is a multiple Laplacian eigenvalue of C_6. If $n = 6$, then $m_{C_6}(1) = n-4 = 2$, but such a graph C_6 is excluded in Theorem 4.4, and thus $n \geqslant 7$ is necessary.

Theorem 4.3 determines the extremal graph for the largest $m_U(1) = n-3$. According to Theorem 4.5, $m_U(1) = n-5$ is the second largest multiplicity in $\mathcal{U}(n,g)$. Then we will completely characterize the unicyclic graphs U of order $n \geqslant 7$ with $m_U(1) = n-5$ in the following result.

Theorem 4.6. *Let $U \in \mathcal{U}(n,g)$ be a unicyclic graph on $n \geqslant 7$ vertices. Then $m_U(1) = n-5$ if and only if U is one of $C_3^1(t,h)$ ($t \geqslant 1$, $h \geqslant 1$), $C_3^2(t,h)$ ($t \geqslant 0$, $h \geqslant 1$) and $C_4(t,h)$ ($t \geqslant 0$, $h \geqslant 0$), and all those graphs are shown in Figure 8.*

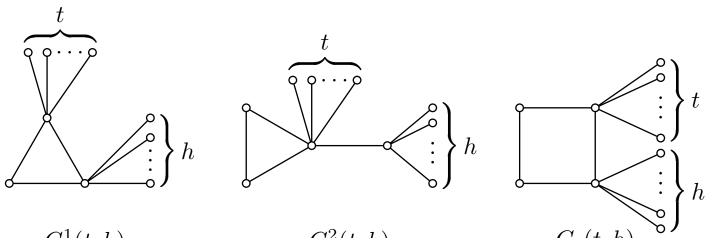


Figure 8. Graphs $C_3^1(t,h)$, $C_3^2(t,h)$ and $C_4(t,h)$.

P r o o f. Let $U \in \mathcal{U}(n,g)$ be a unicyclic graph with $m_U(1) = n-5$, where $n \geqslant 7$. If U is a cycle, then by Corollary 3.2 one can get $m_U(1) = 2$ if and only if $6 \mid n$. Thus, if $m_U(1) = n-5 = 2$, it leads to $n = 7$, a contradiction. So we always suppose that U is not a cycle in the remainder of the proof.

Assume that the girth g of U is no less than 5, that is, $g \geqslant 5$. Then by the same method as in the proof of Theorem 4.3 we have

$$\begin{aligned} m_{C_5^*(n,5;n-5)}(1) &\geqslant \ldots \geqslant m_{U(n,g;n_1+n_2+\ldots+n_g)}(1) \\ &\geqslant \ldots \geqslant m_{U(n,g;n_1,n_2,\ldots,n_g)}(1) \geqslant m_U(1), \end{aligned}$$

where $C_5^*(n,5;n-5)$ is the unicyclic graph on n vertices obtained from C_5 by attaching $n-5$ pendant vertices at any vertex of C_5.

Let $L_R(C_5^*(n,5;n-5))$ be the principal submatrix of $L(C_5^*(n,5;n-5)) - I_n$ corresponding to the inner vertices of $C_5^*(n,5;n-5)$. Then we have

$$L_R(C_5^*(n,5;n-5)) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

by simple computation, $\nu(C_5^*(n,5;n-5)) = 0$. Therefore, it follows from Lemma 2.4 that $m_{C_5^*(n,5;n-5)}(1) = n-6$. Consequently, $m_U(1) \leqslant n-6$, a contradiction. Thus, we have $3 \leqslant g \leqslant 4$. So, there are two cases to be considered:

Case 1: $g = 3$. If U has the form of $C_3(s,t,h)$ (see Figure 9) on $n = s+t+h+3$ vertices, where $s,t,h \geqslant 0$, then it is not possible to have $s,t,h \geqslant 1$. Because if $s \geqslant 1$, $t \geqslant 1$ and $h \geqslant 1$, one can see that $C_3(s,t,h)$ has no inner vertex. Then it follows from Lemma 2.4 that $m_{C_3(s,t,h)}(1) = s+t+h-3 = n-6$, a contradiction.

Subcase 1.1: Suppose $\operatorname{diam}(U) = 3$. Then U has one of the forms $C_3(s,t,h)$ on $n = s+t+h+3$ vertices or $C_3^2(t,h)$ on $n = t+h+3$ vertices. Suppose U has the form $C_3(s,t,h)$, then according to the above argument, it is not possible to have $s,t,h \geqslant 1$. If two of s, t, h in $C_3(s,t,h)$ are equal to zero, then $U \cong S_n^3$, it is also impossible.

Otherwise, we may, without loss of generality, assume that s is only equal to 0, that is, $s=0$ and $t,h \geqslant 1$, then U has the form $C_3^1(t,h)$, see Figure 8. Let $L_R(C_3^1(t,h))$ be the principal submatrix of $L(C_3^1(t,h)) - I_n$ that corresponds to the unique vertex with degree 2, and $L_R(C_3^1(t,h)) = 1$. So we have $\nu(L_R(C_3^1(t,h))) = 0$. Therefore, $m_{C_3^1(t,h)}(1) = t+h-2 = n-5$ by Lemma 2.4.

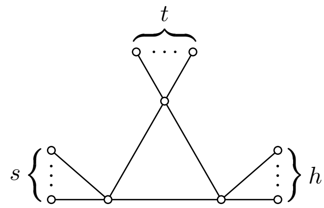


Figure 9. Graph $C_3(s,t,h)$.

If U has the form of $C_3^2(t,h)$ on $n = t+h+4$ vertices, clearly, $h=0$ implies that $U = C_3^2(t,0) \cong S_n^3$, which is impossible, and so $h \geqslant 1$. If $t=0$ and $h=1$, then by Lemma 2.4 we can obtain $m_{C_3^2(0,1)}(1) = p(C_3^2(0,1)) - q(C_3^2(0,1)) + \nu(L_R(C_3^2(0,1))) = 0$. Further, if $t=0$ and $h \geqslant 2$, one can get $m_{C_3^2(0,h)}(1) = h-1 = n-5$ by Lemma 2.5 (b). If $t \geqslant 1$ and $h \geqslant 1$, similarly to $C_3^1(t,h)$ above, we have $\nu(L_R(C_3^2(t,h)) = 1$. Thus, it follows from Lemma 2.4 that $m_{C_3^2(t,h)}(1) = t+h-2+1 = n-5$.

Subcase 1.2: Suppose that $\operatorname{diam}(U) \geqslant 4$. Then U contains at least one of J_1, J_2 and J_3 (see Figure 10) as its subgraphs. If U contains J_1 as its subgraph, then from Table 2 we know that

$$\operatorname{Spec}_L(J_1) = \{0, 0.3249, 1.4608, 3, 3, 4.2143\}.$$

From Theorem 4.3 we have $U \ncong S_n^3$. It follows from Lemma 2.3 that $m_U(0,1) \geqslant 1$, and $m_U(1,n] \geqslant 4$ by Lemma 2.1. Therefore, we get

$$m_U(1) = n - m_U(0) - m_U(0,1) - m_U(1,n] \leqslant n-6,$$

which contradicts $m_U(1) = n-5$. Using the same arguments as for J_1 we get that U does not contain J_2 and J_3 as its subgraphs.

J_1	0	0.3249	1.4608	3.0000	3.0000	4.2143	
J_2	0	0.4131	1.1369	2.3595	3.6977	4.3928	
J_3	0	0.3820	0.6086	2.2271	2.6180	3.0000	5.1642
J_4	0	0.4384	2.0000	2.0000	3.0000	4.5616	
J_5	0	0.5858	1.2679	2.0000	3.4142	4.7321	

Table 2. The Laplacian spectra of J_1, J_2, J_3, J_4 and J_5.

Case 2: $g = 4$. By similar reasoning as in the case of J_1, U does not contain J_4 and J_5 (see Figure 10) as its subgraphs. Thus, U must have the form of $C_4(t, h)$ (shown in Figure 8), where $t \geqslant 0$, $h \geqslant 0$ and $n = t + h + 4$.

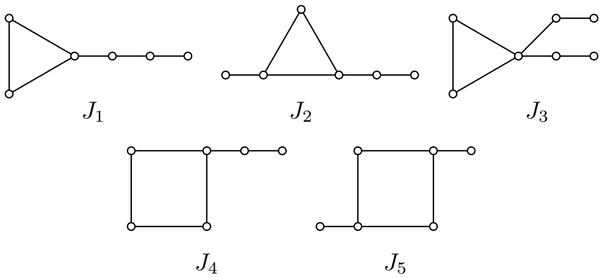


Figure 10. Graphs J_1, J_2, J_3, J_4 and J_5.

If $t \geqslant 1$ and $h \geqslant 1$, let $L_R(C_4(t, h))$ be the principal submatrix of $L(C_4(t, h)) - I_n$ corresponding to the two vertices of degree 2 in $C_4(t, h)$. Then

$$L_R(C_4(t,h)) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and $\nu(L_R(C_4(t, h))) = 1$. It follows from Lemma 2.4 that $m_{C_4(t,h)}(1) = t + h - 1 = n - 5$.

If $t = 0$ or $h = 0$, without loss of generality, we may assume that $h = 0$, then $U = C_4(t, 0)$, where $n = t + 4$. By a similar proof as above, we have

$$L_R(C_4(t,0)) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

and $\nu(L_R(C_4(t, 0))) = 0$. Therefore, $m_{C_4(t,0)}(1) = t - 1 = n - 5$.

Conversely, by Lemma 2.4 it follows that the sufficiency holds. So the proof is completed. $\square$

Unicyclic graphs with $m_U(1) \in \{n - 3, n - 5\}$ are completely characterized in Theorems 4.3 and 4.6, respectively. It means that the large multiplicity of Laplacian eigenvalue 1 can determine the unicyclic graph itself. Applying the same method as in Theorem 4.6, one can also characterize the unicyclic graphs with $m_U(1) = n-6, n-7$ and so on. However, if one keeps doing it the way we did, one has to encounter more graphs with the same value of $m_U(1)$ to be characterized, and this seems to be complex tasks. Rather, one can consider its opposite. So we propose to *characterize the unicyclic graphs on n vertices with property $m_U(1) = 0$.*

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