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ON THE CONJUGATE TYPE VECTOR AND THE STRUCTURE OF A NORMAL SUBGROUP

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Abstract. Let N be a normal subgroup of a group G. The structure of N is given when the G-conjugacy class sizes of N is a set of a special kind. In fact, we give the structure of a normal subgroup N under the assumption that the set of G-conjugacy class sizes of N is $(p_{1n_1}^{a_{1n_1}}, \ldots, p_{11}^{a_{11}}, 1) \times \ldots \times (p_{rn_r}^{a_{rn_r}}, \ldots, p_{r1}^{a_{r1}}, 1)$, where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, n_i\}$.

Keywords: index; conjugacy class size; Baer group

MSC 2020: 20E45, 20D60

1. Introduction

All groups considered in this paper are finite. Let G be a group and x an element in G. We denote by x^G the conjugacy class of G containing x, that is, $x^G = \{g^{-1}xg\colon g \in G\}$. Then the size of x^G is $|G : C_G(x)|$, which is sometimes called the *index* of x in G. Let $cs(G) = \{|x^G|\colon x \in G\}$. Suppose that $cs(G) = \{n_1, n_2, \ldots, n_r\}$, where $n_1, n_2, \ldots, n_r$ are different numbers with $n_1 > n_2 > \ldots > n_r = 1$. In 1953, Itô in [9] called the *vector* $(n_1, n_2, \ldots, n_r)$ the conjugate type vector of G, and the group G is said to be a group with type $(n_1, n_2, \ldots, n_r)$ if G has conjugate type vector $(n_1, n_2, \ldots, n_r)$. In the same paper, Itô proved that a group G is nilpotent if G has type $(n_1, 1)$. Since then, the relationship between the conjugate type of a group G and the property of G attracts interest of many authors. Camina in [8]

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gave the structure of a group G under the assumption that the conjugate type vector of G is the product of several conjugate type vectors, see [4], [5] for more examples.

Let N be a normal subgroup of a group G and write $cs_G(N) = \{|x^G|\colon\ x \in N\}$. Since N is a union of some G-conjugacy classes contained in N, the set $cs_G(N)$ has a strong influence on the structure of N, and many interesting results are obtained, for instance, see [1], [11].

Suppose that $cs_G(N) = \{n_1, n_2, \ldots, n_r\}$, where $n_1, n_2, \ldots, n_r$ are different numbers with $n_1 > n_2 > \ldots > n_r = 1$. In this paper, we call the vector $(n_1, n_2, \ldots, n_r)$ the *G-conjugate type vector* of N. It is obvious that $cs_G(N) = \{m_1, m_2, \ldots, m_t\}$ and for each m_i there exists n_j such that m_i is a divisor of n_j. Furthermore, if $w = (a_1, a_2, \ldots, a_s)$ and $v = (b_1, b_2, \ldots, b_t)$, we define $w \times v = \{a_i b_j\colon\ i = 1, 2, \ldots, s,$ $j = 1, 2, \ldots, t\}$.

Motivated by the results in [8], in this short paper, we consider the structure of a normal subgroup N of G under the assumption that the G-conjugate type vector of N is of a particular type, and the following theorem is obtained:

Theorem 3.1. *Let G be a group and N a normal subgroup of G. Suppose that G-conjugate type vector of N is*

$$(p_{1n_1}^{a_{1n_1}}, \ldots, p_{11}^{a_{11}}, 1) \times \ldots \times (p_{rn_r}^{a_{rn_r}}, \ldots, p_{r1}^{a_{r1}}, 1),$$

where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, n_i\}$. Then $n_i = 2$ and $N = A_1 \times \ldots \times A_r$, and the G-conjugate type vector of A_i is $(p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1)$ for each $i \in \{1, \ldots, r\}$.

Furthermore, one of the following holds for A_i (up to multiplication by central Sylow subgroups):

(1) *A_i is abelian;*

(2) *A_i is a non-abelian p_{i1} or p_{i2}-group;*

(3) *A_i is a non-nilpotent $\{p_{i1}, p_{i2}\}$-group with abelian Sylow subgroups.*

Recall that a group G is called a *p-Baer group* if every p-element in G has prime power index in G, and G is called a *Baer group* if every element of the group with prime power order has prime power index in G. The structure of p-Baer groups and Baer groups are characterized in [2]. If S is a nonempty subset of G, following [4], we set $K_S = \{x \in G\colon\ xS = S\}$. Then $|K_S|$ divides $|S|$. Other notation and terminology are standard, see [10] for instance.

2. Preliminaries

In this section, we give some lemmas which are useful in the proofs of our main results.

The following lemma is a famous result as Thompson's Lemma, and the proof can be found in many books of group theory, see Theorem 8.2.8 of [10] for example.

Lemma 2.1. *Let $P \times Q$ be the direct product of a p-group P and a p'-group Q and suppose that $P \times Q$ acts on a p-group G. If $C_G(P) \subseteq C_G(Q)$, then Q acts trivially on G.*

Lemma 2.2 (Wielandt)**.** *Let G be a group and x an element of G. If both $|x|$ and $|x^G|$ are powers of a prime p, then $x \in O_p(G)$.*

Lemma 2.3. *Let G be a group and N a normal subgroup of G. Suppose that p^a is the highest power of the prime p which divides the G-conjugacy class sizes of elements in N. If there is a p-element x in N such that $|x^G| = p^a$, then N has a normal p-complement.*

Proof. Since x is a p-element and $|x^G| = p^a$, we have that $\langle x^G \rangle \leqslant O_p(G)$ by Lemma 2.2. Therefore, $\langle x^G \rangle \leqslant O_p(G) \cap N \leqslant O_p(N)$. Write $H = \langle x^G \rangle$ and $Z = C_G(H) \cap N = C_N(H)$. For every p'-element $y \in C_N(x)$, the hypothesis of this lemma implies that $(p, |C_G(x) : C_G(xy)|) = 1$. Since N is normal in G, we conclude that $(p, |C_N(x) : C_N(xy)|) = 1$. That is, $(p, |C_N(x) : C_N(x) \cap C_N(y)|) = 1$. Therefore, $H \cap C_N(x) \leqslant H \cap C_N(y)$, that is, $C_H(x) \leqslant C_H(y)$. Now by Lemma 2.1, we have that $y \in C_N(H) = Z$.

Since $|x^N|$ divides $|x^G|$, we have that $|x^N|$ is a power of p. From the above paragraph, we see that Z contains all the p'-elements in $C_N(x)$, and thus $|N : Z|$ is a power of p. Now let w be an arbitrary p'-element in Z. By the previous argument, p does not divide $|C_N(x) : C_N(w) \cap C_N(x)|$. As Z is a normal subgroup of $C_N(x)$, we have that p does not divide $|Z : C_Z(w)|$. Therefore, every p'-element in Z has index in Z prime to p, so by [6], Lemma 1, we have that $Z = K \times P$, where K is a p-complement of Z and P is a Sylow p-subgroup of Z. Therefore, K is a normal p-complement of N since $|N : Z|$ is a power of p. □

Lemma 2.4. *Let G be a group and N a normal subgroup of G such that p^a is the highest power of the prime p which divides the G-conjugacy class size of an element in N. Assume that there exists a p-element x in N such that $|x^G| = p^a$. If m is a G-conjugacy class size in N such that $(m, p) = 1$, then there exists a p'-element in N, say y, such that $|(xy)^G| = p^a m$.*

P r o o f. By Lemma 2.3, we see that N has a normal p-complement K. As $|x^N|$ divides $|x^G|$, $|x^N|$ is a power of p, and thus $K \leqslant C_N(x)$. Let u be a p'-element in $C_N(x)$. Then p does not divide $|C_G(x) : C_G(ux)|$. Since N is normal in G, we have that p does not divide $|C_N(x) : C_N(ux)|$. That is to say, p does not divide the index of u in $C_N(x)$. Therefore, by [6], Lemma 1, $C_N(x) = P_x \times K$ with P_x a Sylow p-subgroup of $C_N(x)$ and K a normal p-complement of $C_N(x)$. Let y be an element in N such that $|y^G| = m$. Then p does not divide $|y^N|$ since $|y^N|$ divides $|y^G|$, whence y centralizes a Sylow p-subgroup of N, and thus y centralizes $O_p(N)$. Since $x \in O_p(G)$ by Lemma 2.2, we have that $x \in O_p(G) \cap N \leqslant O_p(N)$. Therefore, y centralizes x. We may assume that $y \in K$, and thus $|(xy)^G| = p^a m$, as required. $\square$

Lemma 2.5 ([7], Proposition 1)**.** *Let G be a group and p a prime. Suppose that $x \in G$ such that $|x^G|$ is a power of p. Then $[x^G, x^G] \subseteq O_p(G)$.*

Lemma 2.6. *Let G be a group and p and r two primes. Suppose that there is an r-element $x \in G$ such that $|x^G|$ is a power of p. If we set $B = x^G$, then $\langle BB^{-1}\rangle \subseteq O_{p,r}(G)$.*

P r o o f. First suppose that $O_p(G) = 1$. Then by Lemma 2.5, $[B, B] = 1$. It follows that $\langle B\rangle$ is an abelian normal subgroup of G, and thus $\langle B\rangle \leqslant F(G)$. As x is an r-element, we have that $\langle B\rangle \leqslant O_r(G)$. Since $\langle BB^{-1}\rangle \leqslant \langle B\rangle$, we have that $\langle BB^{-1}\rangle \leqslant O_r(G)$.

Now suppose that $O_p(G) \neq 1$, we can set $\overline{G} = G/O_p(G)$. Then $O_p(\overline{G}) = 1$. Since $|\bar{x}^{\overline{G}}|$ divides $|x^G|$, we have that $|\bar{x}^{\overline{G}}|$ is a power of p. By the above paragraph, we have that $\langle \overline{B}\overline{B}^{-1}\rangle \leqslant O_r(\overline{G})$. Therefore, $\langle BB^{-1}\rangle \subseteq O_{p,r}(G)$. $\square$

Lemma 2.7. *Let G be a group and N a normal subgroup of G. Suppose that $x, y \in N$ such that $|x^G| = p^a$ and $|y^G| = q^b$, where p and q are distinct primes with $p^a < q^b$. If there is no element in N with G-conjugacy class size divisible by pq, then x is a q-element (up to multiplication by central elements).*

P r o o f. Write $x = x_1x_2\ldots x_s$ such that each x_i is an element of a prime power order, $x_ix_j = x_jx_i$ for all i and j and $(|x_i|, |x_j|) = 1$ for $i \neq j$. Since x is not central in G, we may assume that $x_1 \notin Z(G)$ and that x_1 is an r-element for a prime r.

Write $B = x_1^G$, $C = y^G$ and $D = CB$. Since $|B|$ divides $|x^G|$, we have that $(|B|, |y^G|) = 1$. Therefore, similarly as in [3], Lemma 1(b), we see that D is a G-conjugacy class contained in N and $|D|$ divides $|C||B|$. In fact, since $(|y^G|, |x_1^G|) = 1$, we have that $G = C_G(y)C_G(x_1)$. For every $y^g x_1^h \in CB$, we have that $gh^{-1} \in G = C_G(y)C_G(x_1)$. Then there exist $a \in C_G(y)$ and $b \in C_G(x_1)$ such that $gh^{-1} = a^{-1}b$. So $ag = bh$. Furthermore, $y^g x_1^h = y^{ag} x_1^{bh} = (yx_1)^{ag} \in (yx_1)^G$. Therefore, $CB \subseteq (yx_1)^G$. Conversely, it is obvious that $(yx_1)^G \subseteq y^G x_1^G = CB$. Therefore, $CB = y^G x_1^G = (yx_1)^G$ is a conjugacy class. Now it is clear that $|D| \geqslant |C|$. So by the hypothesis of this lemma, $|D| = |C|$. Repeating the argument we see that DB^{-1} is a G-conjugacy class contained in N, and thus $C = CBB^{-1}$ since $C \subseteq CBB^{-1}$. Therefore, $H = \langle BB^{-1} \rangle \leqslant K_C$. Since $|K_c|$ divides $|C|$, we have that $|H|$ divides $|C|$, whence $|H|$ is a power of q. According to Lemma 2.6, we have that $\langle BB^{-1} \rangle \subseteq O_{p,r}(G)$, which forces $r = q$. Therefore, $x_2 x_3 \ldots x_s$ is central in G and by replacing x with x_1 we can assume that x is a q-element. $\square$

Lemma 2.8 ([11], Lemma 2.2). *Let G be a group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p-subgroup.*

3. Proof of the Main Result

In this section, we give the proof of the main result.

Theorem 3.1. *Let G be a group and N a normal subgroup of G. Suppose that the G-conjugate type vector of N is*

$$(p_{1n_1}^{a_{1n_1}}, \ldots, p_{11}^{a_{11}}, 1) \times \ldots \times (p_{rn_r}^{a_{rn_r}}, \ldots, p_{r1}^{a_{r1}}, 1),$$

where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, n_i\}$. Then $n_i = 2$ and $N = A_1 \times \ldots \times A_r$, and the G-conjugate type vector of A_i is $(p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1)$ for each $i \in \{1, \ldots, r\}$.

Furthermore, one of the following holds for A_i (up to multiplication by central Sylow subgroups):

(1) *A_i is abelian;*

(2) *A_i is a non-abelian p_{i1} or p_{i2}-group;*

(3) *A_i is a non-nilpotent $\{p_{i1}, p_{i2}\}$-group with abelian Sylow subgroups.*

Proof. We first consider the case $r = 2$. Let $x, y_i \in N$ such that $|x^G| = p_{11}^{a_{11}}$ and $|y_i^G| = p_{1i}^{a_{1i}}$ for $2 \leqslant i \leqslant s$. Then by Lemma 2.7, x is a p_{1i}-element for each i. Thus, $n_1 = 2$ and x is a p_2-element. For every p_{12}'-element $y \in C_N(x)$, the hypothesis implies that p_{12} does not divide $|C_N(x) : C_N(xy)|$, whence $C_N(x) = P_2 \times L$ by [6], Lemma 1, where P_2 is a Sylow p_{12}-subgroup of N. Therefore, p_{12} does not divide the index of any p_{2j}-element in N for $j = 1, \ldots, t$. Similarly, we have $n_2 = 2$, and if z is an element in N such that $|z^G| = p_{21}^{a_{21}}$, then z is a p_{22}-element. Furthermore, we have that $C_N(z) = Q_2 \times K$, where Q_2 is a Sylow p_{22}-subgroup of N, and p_{22} does not divide the index of any p_{1i}-element in N for $i = 1, 2$.

Now assume that w is an element in N such that $|w^G| = p_{12}^{a_{12}}$. By the above paragraph we see that w is neither a p_{21}-element nor a p_{22}-element. If w is a p_{12}-element, we may assume that $w \in P_2$. It follows that $L \leqslant C_G(w)$. Then $p_{12}^{a_{12}} = |w^G|$ divides $|G : L| = |G : C_G(x)||C_G(x) : C_N(x)||C_N(x) : K|$, which is a contradiction. Therefore, w must be a p_{11}-element. Let v be an arbitrary p'_{11}-element in $C_N(w)$. Since $p_{11}p_{12}$ does not divide any G-conjugacy class size of element in N, p_{11} does not divide $|C_G(w) : C_G(wv)|$ and thus, p_{11} does not divide $|C_N(w) : C_N(wv)|$ since N is normal in G. Therefore, $C_N(w) = P_1 \times M$, where P_1 is a Sylow p_{11}-subgroup of N. Recall that $|N : C_N(w)|$ is a p_{12}-number. If u is a p_{21}- or p_{22}-element in N, then u is contained in a conjugation of M and thus, p_{11} does not divide $|u^N|$. Combining this with the above paragraph, we see that $|u^N|$ is a power of p_{21} or p_{22}. Similarly, if h is a p_{11}- or p_{12}-element in N, then $|h^N|$ is a power of p_{11} or p_{12}.

In the following, we suppose that $r > 2$. Let x be an element of N such that $|x^G| = p_{11}^{a_{11}}$. Then as in the first paragraph of the proof we have that $n_1 = 2$ and that x is a p_{12}-element. For every p'_{12}-element $y \in C_N(x)$ we have that

$$|C_G(x) : C_G(x) \cap C_G(y)| = |C_G(x) : C_G(xy)|$$

is prime to p_{12}. Since $C_N(x) \trianglelefteq C_G(x)$, we have that $|y^{C_N(x)}|$ is a p'_{12}-number. Therefore, we have that $C_N(x) = P_{12} \times K$, where P_{12} is a Sylow p_{12}-subgroup of N. It is easy to see that N is a p_{12}-Baer group. Furthermore, all $\{p_{11}, p_{12}\}'$-elements have index coprime to p_{12}. On the other hand, it follows from Lemma 2.4 that all p_{12}-elements have index $p_{11}^{a_{11}}$ or are central. So an element of index $p_{12}^{a_{12}}$ must be a p_{11}-element, we can assume that w is such an element. Then by arguing similarly as for the element x, we have that N is a p_{11}-Baer group. Thus, by [7], Theorem A, we see that $P_{11}P_{12}$ is a normal subgroup of N. Notice that every element of order prime to p_{11} and p_{12} have index prime to p_{11} and p_{12}. Therefore, $P_{11}P_{12}$ is centralized by all $\{p_{11}, p_{12}\}'$-elements of N. If we set $A_1 = P_{11}P_{12}$, then A_1 satisfies the theorem. Similarly, we can find all A_i for $2 \leqslant i \leqslant r$.

Let $i \in \{1, \ldots, r\}$. Suppose that A_i is not abelian. For every element $x \in A_i \setminus Z(A_i)$, since A_i is normal in G, we have that $|x^{A_i}|$ divides $|x^G|$. Since $cs_G(A_i) = \{p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1\}$, we have that $|x^{A_i}|$ is a power of p_{i1} or p_{i2}. Then according to Lemma 2.8, the $\{p_{i1}, p_{i2}\}$-complement of A_i is central in A_i. Up to multiplication by central Sylow subgroups, we can assume that A_i is a $\{p_{i1}, p_{i2}\}$-group. Recall that $p_{i1}p_{i2}$ does not divide $|x^{A_i}|$ for any $x \in A_i$. If A_i is nilpotent, then A_i is a p_{i1} or a p_{i2}-group. If A_i is not nilpotent, since A_i is a Baer group, we have that every Sylow subgroup of A_i is abelian by Theorem of [2]. □

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