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# On Balancing and Lucas-balancing Quaternions

*Bijan Kumar Patel, Prasanta Kumar Ray*

**Abstract.** The aim of this article is to investigate two new classes of quaternions, namely, balancing and Lucas-balancing quaternions that are based on balancing and Lucas-balancing numbers, respectively. Further, some identities including Binet’s formulas, summation formulas, Catalan’s identity, etc. concerning these quaternions are also established.

## 1 Introduction

Quaternions were introduced by W. R. Hamilton in the middle of 19th century; they are an extension of complex numbers. A quaternion  $q$  is a hyper-complex number defined by the equation

$$q = ae_0 + be_1 + ce_2 + de_3 = (a, b, c, d)$$

where  $a, b, c, d$  are members of the set of real numbers  $\mathbb{R}$  and  $e_0, e_1, e_2, e_3$  with  $e_0 = 1$  form a standard orthonormal basis in  $\mathbb{R}^4$ . The set of quaternions is usually denoted by  $\mathbb{H}$  and constitutes a non-commutative field known as skew field that extends the complex field  $\mathbb{C}$ . The standard basis vectors  $e_0, e_1, e_2, e_3$  satisfy the quaternion multiplication as per the following multiplication table (Table ??).

If  $p$  and  $q$  are any two quaternions in  $\mathbb{H}$ , say,

$$p = (p_0, p_1, p_2, p_3) \quad \text{and} \quad q = (q_0, q_1, q_2, q_3),$$

then their addition and subtraction are defined as

$$p \pm q = (p_0 \pm q_0)e_0 + (p_1 \pm q_1)e_1 + (p_2 \pm q_2)e_2 + (p_3 \pm q_3)e_3.$$

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**Affiliation:**

Bijan Kumar Patel – Department of Mathematics, KIIT Deemed to be University, Bhubaneswar, India

*E-mail:* [iiit.bijan@gmail.com](mailto:iiit.bijan@gmail.com)

Prasanta Kumar Ray – Department of Mathematics, Sambalpur University, Jyoti Vihar, Burla, India

*E-mail:* [prasantamath@suniv.ac.in](mailto:prasantamath@suniv.ac.in)

*	1	$e_1$	$e_2$	$e_3$
1	1	$e_1$	$e_2$	$e_3$
$e_1$	$e_1$	-1	$e_3$	$-e_2$
$e_2$	$e_2$	$-e_3$	-1	$e_1$
$e_3$	$e_3$	$e_2$	$-e_1$	-1

Table 1: The multiplication table for the basis of  $\mathbb{H}$

Further, if we rewrite  $p = p_0 + P$  and  $q = q_0 + Q$  where  $P = p_1e_1 + p_2e_2 + p_3e_3$  and  $Q = q_1e_1 + q_2e_2 + q_3e_3$ , then their multiplication is defined as

$$pq = p_0q_0 - P \cdot Q + p_0Q + q_0P + P \times Q.$$

Here “ $\cdot$ ” and “ $\times$ ” are respectively the scalar and vector products of the vectors. The complex conjugate of  $q = q_0 + Q$ , denoted by  $\bar{q}$  is defined as  $\bar{q} = q_0 - Q$ , while the norm of  $q$ , denoted by  $|q|$ , is given as  $|q| = \sqrt{q\bar{q}}$ .

Fibonacci and Lucas quaternions were introduced by Horadam [??], and are defined by the equations

$$QF_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

and

$$QL_n = L_n e_0 + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3.$$

Here  $F_n$  and  $L_n$  denote the  $n^{\text{th}}$  Fibonacci and Lucas number, respectively. Some more properties including recurrence relation were studied in [??]. Iyer [??] derived some relations between the Fibonacci and Lucas quaternions. Halici [??] investigated the Fibonacci and Lucas quaternions and derived some identities of them which includes Binet’s formulas and generating functions. Subsequently, Akyigit et al. [??] generalized the Fibonacci quaternions and studied many of their properties. Recently, Çimen and Ipek [??] defined the Pell and Pell-Lucas quaternions as follows:

$$QP_n = P_n e_0 + P_{n+1} e_1 + P_{n+2} e_2 + P_{n+3} e_3$$

and

$$QPL_n = Q_n e_0 + Q_{n+1} e_1 + Q_{n+2} e_2 + Q_{n+3} e_3,$$

where  $P_n$  and  $Q_n$  are the  $n^{\text{th}}$  Pell and Pell-Lucas numbers respectively. As usual, Pell and Pell-Lucas numbers are defined recursively by

$$P_n = 2P_{n-1} + P_{n-2}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}$$

for  $n \geq 2$  with their respective initials

$$(P_0, P_1) = (0, 1) \quad \text{and} \quad (Q_0, Q_1) = (1, 1).$$

Consequently, Szynal-Liana and Włoch [??] obtained several identities concerning  $QP_n$  and  $QPL_n$  using matrix methods. Motivated by the work of Szynal-Liana and Włoch, Catarino [??] introduced the Modified Pell and the Modified  $k$ -Pell quaternions and established some of their properties. Motivated by these works, in this paper we introduce the balancing and Lucas-balancing quaternions and establish some identities.

It is worth defining balancing and Lucas-balancing numbers. A balancing number  $B$  is a solution of the Diophantine equation

$$1 + 2 + 3 + \dots + (B - 1) = (B + 1) + (B + 2) + \dots + (B + R)$$

with  $R$  as a balancer corresponding to  $B$  [??]. For each balancing number  $B$ , the square root of  $8B^2 + 1$  is called a Lucas-balancing number [??]. The  $n^{\text{th}}$  balancing number  $B_n$  and the  $n^{\text{th}}$  Lucas-balancing number  $C_n$  are defined recursively by

$$B_n = 6B_{n-1} - B_{n-2}$$

with  $(B_0, B_1) = (0, 1)$  and

$$C_n = 6C_{n-1} - C_{n-2}$$

with  $(C_0, C_1) = (1, 3)$  respectively for  $n \geq 2$ . The Binet formulas for  $B_n$  and  $C_n$  are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

where  $\lambda_1 = 3 + \sqrt{8}$  and  $\lambda_2 = \lambda_1^{-1}$ .

In this article we introduce two new classes of quaternions, namely, balancing and Lucas-balancing quaternions and then derive some of their properties. Further, we also study various results of these classes of quaternions including recurrence relations, Binet's formulas, summation formulas, Catalan's identity etc.

## 2 Balancing and Lucas-balancing quaternions

In this section we define balancing and Lucas-balancing quaternions and derive some properties of these quaternions.

**Definition 1.** Let  $B_n$  and  $C_n$  denote the  $n^{\text{th}}$  balancing and the  $n^{\text{th}}$  Lucas-balancing numbers respectively. Then balancing and Lucas-balancing quaternions are respectively defined as

$$QB_n = B_n e_0 + B_{n+1} e_1 + B_{n+2} e_2 + B_{n+3} e_3 = \sum_{r=0}^3 B_{n+r} e_r,$$

and

$$QC_n = C_n e_0 + C_{n+1} e_1 + C_{n+2} e_2 + C_{n+3} e_3 = \sum_{r=0}^3 C_{n+r} e_r,$$

where  $e_0, e_1, e_2$  and  $e_3$  are the standard orthonormal basis vectors in  $\mathbb{R}^4$ .

We can observe from the above definition that addition and subtraction of these quaternions can be obtained as follows:

$$QB_n \pm QC_n = \sum_{r=0}^3 (B_r \pm C_r)e_r.$$

Balancing and Lucas-balancing quaternions satisfy similar recurrence relations as those of balancing and Lucas-balancing numbers. The following propositions demonstrate this fact.

**Proposition 1.** *The recurrence relations for balancing and Lucas-balancing quaternions are respectively*

$$QB_n = 6QB_{n-1} - QB_{n-2} \quad \text{and} \quad QC_n = 6QC_{n-1} - QC_{n-2}$$

for  $n \geq 2$ .

*Proof.* Using the recurrence relation of  $\{B_n\}_{n \geq 2}$ , we have

$$\begin{aligned} QB_n &= \sum_{r=0}^3 B_{n+r}e_r \\ &= \sum_{r=0}^3 (6B_{n-1+r} - B_{n-2+r})e_r \\ &= 6QB_{n-1} - QB_{n-2}, \end{aligned}$$

which completes the proof. The proof is similar for Lucas-balancing quaternions.  $\square$

The following lemma is useful while deriving the Binet formulas for both  $QB_n$  and  $QC_n$ .

**Lemma 1.** *For any natural number  $n$ ,*

$$QC_n + \sqrt{8}QB_n = A\lambda_1^n \quad \text{and} \quad QC_n - \sqrt{8}QB_n = B\lambda_2^n,$$

where

$$A = \sum_{r=0}^3 \lambda_1^r e_r \quad \text{and} \quad B = \sum_{r=0}^3 \lambda_2^r e_r.$$

*Proof.* Using the identity  $C_n + \sqrt{8}B_n = \lambda_1^n$ , we have

$$\begin{aligned} QC_n + \sqrt{8}QB_n &= \sum_{r=0}^3 C_{n+r}e_r + \sqrt{8} \sum_{r=0}^3 B_{n+r}e_r \\ &= \sum_{r=0}^3 (C_{n+r} + \sqrt{8}B_{n+r})e_r \\ &= \sum_{r=0}^3 \lambda_1^{r+n} e_r \\ &= A\lambda_1^n, \end{aligned}$$

where  $A = \sum_{r=0}^3 \lambda_1^r e_r$ . Similarly, using the identity  $C_n - \sqrt{8}B_n = \lambda_2^n$ , the second result can be obtained. □

**Theorem 1.** *The Binet formulas for  $QB_n$  and  $QC_n$  are respectively given by*

$$QB_n = \frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2} \quad \text{and} \quad QC_n = \frac{A\lambda_1^n + B\lambda_2^n}{2},$$

where  $A = \sum_{r=0}^3 \lambda_1^r e_r$  and  $B = \sum_{r=0}^3 \lambda_2^r e_r$  with  $n \geq 0$ .

*Proof.* By virtue of Lemma ??,

$$A\lambda_1^n - B\lambda_2^n = (\lambda_1 - \lambda_2)QB_n \quad \text{and} \quad A\lambda_1^n + B\lambda_2^n = 2QC_n,$$

and the results follow. □

By using the Binet form of balancing and Lucas-balancing quaternions, we derive some identities concerning  $QB_n$  and  $QC_n$ . Before that we first define conjugates and norms of these quaternions.

**Definition 2.** The conjugates of  $QB_n$  and  $QC_n$  are respectively defined as

$$\begin{aligned} \overline{QB_n} &= B_n e_0 - B_{n+1} e_1 - B_{n+2} e_2 - B_{n+3} e_3 = B_n - \sum_{r=1}^3 B_{n+r} e_r, \\ \overline{QC_n} &= C_n e_0 - C_{n+1} e_1 - C_{n+2} e_2 - C_{n+3} e_3 = C_n - \sum_{r=1}^3 C_{n+r} e_r, \end{aligned}$$

and the norms of  $QB_n$  and  $QC_n$  are respectively defined as

$$\begin{aligned} N_{QB_n} &= \overline{QB_n}QB_n = B_n^2 + B_{n+1}^2 + B_{n+2}^2 + B_{n+3}^2 = \sum_{r=0}^3 B_{n+r}^2, \\ N_{QC_n} &= \overline{QC_n}QC_n = C_n^2 + C_{n+1}^2 + C_{n+2}^2 + C_{n+3}^2 = \sum_{r=0}^3 C_{n+r}^2. \end{aligned}$$

**Proposition 2.** *If  $n \geq 2$ , then*

- (i)  $QB_n + \overline{QB_n} = 2B_n$ ,
- (ii)  $QB_n^2 + QB_n\overline{QB_n} = 2B_nQB_n$ ,
- (iii)  $QB_n\overline{QB_n} = \frac{1}{32}(B_{2n+7} - B_{2n-1} - 8)$ .

*Proof.* Using Definition ??, we have

$$QB_n + \overline{QB_n} = \sum_{r=0}^3 B_{n+r} e_r + B_n - \sum_{r=1}^3 B_{n+r} e_r = 2B_n,$$

which ends the proof of (i). Since

$$QB_n^2 = QB_nQB_n = QB_n(2B_n - \overline{QB_n}) = 2B_nQB_n - QB_n\overline{QB_n}$$

and so

$$QB_n^2 + QB_n\overline{QB_n} = 2B_nQB_n.$$

In order to prove (iii) we use the following identity for all positive integers  $n$  and  $m$ ,

$$\sum_{r=0}^m B_{n+r}^2 = \frac{1}{32}(B_{2m+2n+1} - B_{2n-1} - 2(m+1)) \quad ([??, \text{Theorem 2.2}]).$$

Since  $QB_n\overline{QB_n} = \sum_{r=0}^3 B_{n+r}^2$ , the identity follows by letting  $m = 3$ . □

**Proposition 3.** *If  $m$  and  $n$  are positive integers, then*

$$QB_{m+n} = B_mQC_n + C_mQB_n$$

and

$$QC_{m+n} = C_mQC_n + 8B_mQB_n.$$

*Proof.* Using the identity  $B_{m+n} = B_mC_n + C_mB_n$ , we have

$$\begin{aligned} QB_{m+n} &= \sum_{r=0}^3 B_{m+n+r}e_r \\ &= B_m \sum_{r=0}^3 C_{n+r}e_r + C_m \sum_{r=0}^3 B_{n+r}e_r \\ &= B_mQC_n + C_mQB_n. \end{aligned}$$

Similarly,

$$QC_{m+n} = \sum_{r=0}^3 C_{m+n+r}e_r.$$

Further simplification leads the right side expression to

$$\sum_{r=0}^3 (C_mC_{n+r} + 8B_mB_{n+r})e_r.$$

It follows that

$$QC_{m+n} = C_m \sum_{r=0}^3 C_{n+r}e_r + 8B_m \sum_{r=0}^3 B_{n+r}e_r,$$

and the result follows. □

The following result can also be shown analogously.

**Proposition 4.** *If  $m$  and  $n$  are positive integers, then*

$$QB_{m-n} = C_nQB_m - B_nQC_m \quad \text{and} \quad QC_{m-n} = C_nQC_m - 8B_nQB_m.$$

Replacing  $n$  by  $n+r$  in the identities  $B_n = 3B_{n-1} + C_{n-1}$  and  $C_{n+1} = 8B_n + 3C_n$  [??], we have the following formulas that are useful while proving the subsequent results.

For any natural numbers  $n$  and  $r$ ,

$$B_{n+r} = 3B_{n-1+r} + C_{n-1+r}, \tag{1}$$

$$C_{n+r} = 8B_{n-1+r} + 3C_{n-1+r}. \tag{2}$$

Using (??) and the recurrence relation for Lucas-balancing numbers, we have

$$B_{n+1+r} - B_{n-1+r} = 2C_{n+r}. \tag{3}$$

The following result demonstrates some relations between the balancing and Lucas-balancing quaternions.

**Proposition 5.** *For  $n \geq 2$  we have the following identities,*

- (i)  $QB_n = 3QB_{n-1} + QC_{n-1}$ ,
- (ii)  $QC_n = 8QB_{n-1} + 3QC_{n-1}$ ,
- (iii)  $2QC_n = QB_{n+1} - QB_{n-1}$ ,
- (iv)  $QC_n - QC_{n-1} = 2(QB_{n-1} + QB_n)$ .

*Proof.* From (??), we have

$$\begin{aligned} QB_n &= \sum_{r=0}^3 B_{n+r}e_r \\ &= \sum_{r=0}^3 (3B_{n-1+r} + C_{n-1+r})e_r \\ &= 3QB_{n-1} + QC_{n-1}, \end{aligned}$$

which implies the first identity. Similarly, applying (??) and (??), (ii) and (iii) can be derived. (iv) immediately follows from (ii) and (iii). This completes the proof.  $\square$

**Theorem 2 (Catalan’s identity).** *If  $n, r \in \mathbb{N}$ , then*

$$\begin{aligned} QB_n^2 - QB_{n+r}QB_{n-r} &= \frac{-1}{8}[(C_{2r} - C_0) + (C_{2r-1} - C_1)e_1 + (C_{2r+2} - C_2)e_2] \\ &\quad + \frac{1}{16}[(C_{2r+3} + C_{2r-3} + C_{2r-1} - C_{2r+1} - 2C_3)]e_3 \end{aligned}$$

and

$$\begin{aligned} QC_n^2 - QC_{n+r}QC_{n-r} &= (C_{2r} - C_0) - (C_{2r-1} - C_1)e_1 - (C_{2r+2} - C_2)e_2 \\ &\quad + (2C_3 + C_{2r+1} - C_{2r-3} - C_{2r-1} - C_{2r+3}/2)e_3. \end{aligned}$$



*Proof.* Using the Binet formula for balancing quaternions and the fact  $\lambda_1\lambda_2 = 1$ , we have

$$\begin{aligned}
 QB_n^2 - QB_{n+r}QB_{n-r} &= \left(\frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2}\right)^2 - \left(\frac{A\lambda_1^{n+r} - B\lambda_2^{n+r}}{\lambda_1 - \lambda_2}\right)\left(\frac{A\lambda_1^{n-r} - B\lambda_2^{n-r}}{\lambda_1 - \lambda_2}\right) \\
 &= \frac{AB\left(\left(\frac{\lambda_1}{\lambda_2}\right)^r - 1\right) + BA\left(\left(\frac{\lambda_2}{\lambda_1}\right)^r - 1\right)}{(\lambda_1 - \lambda_2)^2} \\
 &= \frac{AB(\lambda_1^{2r} - 1) + BA(\lambda_2^{2r} - 1)}{(\lambda_1 - \lambda_2)^2} \\
 &= \frac{\{-2 + 2\lambda_2e_1 + 2\lambda_1^2e_2 + (\lambda_1^3 + \lambda_2^3 + \lambda_2 - \lambda_1)e_3\}(\lambda_1^{2r} - 1)}{(\lambda_1 - \lambda_2)^2} \\
 &\quad + \frac{\{-2 + 2\lambda_1e_1 + 2\lambda_2^2e_2 + (\lambda_1^3 + \lambda_2^3 + \lambda_1 - \lambda_2)e_3\}(\lambda_2^{2r} - 1)}{(\lambda_1 - \lambda_2)^2} \\
 &= \left(\frac{1}{8} - \frac{\lambda_1^{2r} + \lambda_2^{2r}}{16}\right) + \left(\frac{\lambda_1^{2r-1} + \lambda_2^{2r-1}}{16} - \frac{\lambda_1 + \lambda_2}{16}\right)e_1 \\
 &\quad + \left(\frac{\lambda_1^{2r+2} + \lambda_2^{2r+2}}{16} - \frac{\lambda_1^2 + \lambda_2^2}{16}\right)e_2 \\
 &\quad + \left(\frac{\lambda_1^{2r+3} + \lambda_2^{2r+3}}{32} + \frac{\lambda_1^{2r-3} + \lambda_2^{2r-3}}{32}\right. \\
 &\quad \left. + \frac{\lambda_1^{2r-1} + \lambda_2^{2r-1}}{32} - \frac{\lambda_1^{2r+1} + \lambda_2^{2r+1}}{32} - \frac{\lambda_1^3 + \lambda_2^3}{16}\right)e_3
 \end{aligned}$$

which completes the proof of first part. The second part follows analogously.  $\square$

Since Cassini’s identity is a special case of Catalan’s identity where  $r = 1$ , the following result immediately follows from Theorem ??.

**Corollary 1.** *For any positive integer  $n$ , the Cassini identity for balancing quaternions is*

$$QB_{n+1}QB_{n-1} - QB_n^2 = \frac{AB(\lambda_1^2 - 1) + BA(\lambda_2^2 - 1)}{(\lambda_1 - \lambda_2)^2} = -2 + 70e_2 + 192e_3,$$

whereas that for Lucas-balancing quaternions is

$$QC_{n+1}QC_{n-1} - QC_n^2 = \frac{AB(1 - \lambda_1^2) + BA(1 - \lambda_2^2)}{4} = 16 - 560e_2 - 1536e_3.$$

**Theorem 3 (d’Ocagne’s identity).** *If  $m, n \in \mathbb{N}$  with  $n \geq m$ , then*

$$\begin{aligned}
 QB_{m+1}QB_n - QB_mQB_{n+1} &= 2(-B_{n-m}e_0 + B_{n-m+1}e_1 + B_{n-m-2}e_2) \\
 &\quad + (B_{n-m+3} + B_{n-m-3} + B_{n-m+1} - B_{n-m-1})e_3
 \end{aligned}$$

and

$$\begin{aligned}
 QC_{m+1}QC_n - QC_mQC_{n+1} &= 16 [B_{n-m} - B_{n-m+1}e_1 - B_{n-m-2}e_2] \\
 &\quad - 8(B_{n-m+3} + B_{n-m+1} - B_{n-m-1} + B_{n-m+1})e_3.
 \end{aligned}$$

*Proof.* Using the Binet formula for balancing quaternions and since  $\lambda_1\lambda_2 = 1$ , we have

$$\begin{aligned}
 QB_{m+1}QB_n - QB_mQB_{n+1} &= \left( \frac{A\lambda_1^{m+1} - B\lambda_2^{m+1}}{\lambda_1 - \lambda_2} \right) \left( \frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2} \right) \\
 &\quad - \left( \frac{A\lambda_1^m - B\lambda_2^m}{\lambda_1 - \lambda_2} \right) \left( \frac{A\lambda_1^{n+1} - B\lambda_2^{n+1}}{\lambda_1 - \lambda_2} \right) \\
 &= \frac{BA\lambda_1^{n-m} - AB\lambda_2^{n-m}}{\lambda_1 - \lambda_2} \\
 &= -2B_{n-m}e_0 + 2B_{n-m+1}e_1 + 2B_{n-m-2}e_2 \\
 &\quad + (B_{n-m+3} + B_{n-m-3} + B_{n-m+1} - B_{n-m-1})e_3,
 \end{aligned}$$

which completes the proof of the first part. Similarly using the Binet formula for  $QC_n$ , the second result can be shown. □

An interesting observation from the above results is that the Catalan identities for balancing and Lucas-balancing quaternions are expressed in terms of Lucas-balancing numbers whereas the d’Ocagne identities for both these quaternions are in terms of balancing numbers.

**Theorem 4.** *The identity  $QC_n^2 - 8QB_n^2 = \frac{AB+BA}{2}$  holds for  $n \geq 1$ .*

*Proof.* Applying the Binet formulas for balancing and Lucas-balancing quaternions, we get

$$\begin{aligned}
 QC_n^2 - 8QB_n^2 &= \left( \frac{A\lambda_1^n + B\lambda_2^n}{2} \right)^2 - 8 \left( \frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2} \right)^2 \\
 &= \frac{(A^2\lambda_1^{2n} + AB + BA + B^2\lambda_2^{2n}) - (A^2\lambda_1^{2n} - AB - BA + B^2\lambda_2^{2n})}{4} \\
 &= \frac{AB + BA}{2},
 \end{aligned}$$

which completes the proof. □

### 3 Sum formulas of balancing and Lucas-balancing quaternions

In this section, we derive some sum formulas involving  $QB_n$  and  $QC_n$ .

The following identity is available in [??].

**Lemma 2.** *For all positive integers  $k$  and  $i$ ,*

$$\sum_{i=0}^n B_{k+i} = \frac{1}{4} [B_{(n+1)+k} - B_{n+k} - B_k + B_{k-1}]. \tag{4}$$

**Theorem 5.** If  $P_r$  is the  $r^{\text{th}}$  Pell number, then

$$\sum_{r=1}^n QB_r = \frac{1}{4} \left( QB_{n+2} - QB_{n+1} - \sum_{r=0}^3 P_{2r+1} e_r \right).$$

*Proof.* Using (??), we have

$$\begin{aligned} \sum_{r=0}^n QB_r &= \left( \sum_{r=0}^n B_r \right) e_0 + \left( \sum_{r=0}^n B_{r+1} \right) e_1 \\ &\quad + \left( \sum_{r=0}^n B_{r+2} \right) e_2 + \left( \sum_{r=0}^n B_{r+3} \right) e_3 \\ &= \left[ \frac{1}{4} (B_{n+2} - B_{n+1} - 1) \right] e_0 + \left[ \frac{1}{4} (B_{n+3} - B_{n+2} - 5) \right] e_1 \\ &\quad + \left[ \frac{1}{4} (B_{n+4} - B_{n+3} - 29) \right] e_2 + \left[ \frac{1}{4} (B_{n+5} - B_{n+4} - 169) \right] e_3 \\ &= \frac{1}{4} \left( \sum_{r=0}^3 B_{n+2+r} e_r - \sum_{r=0}^3 B_{n+1+r} - \sum_{r=0}^3 P_{2r+1} e_r \right) \\ &= \frac{1}{4} \left( QB_{n+2} - QB_{n+1} - \sum_{r=0}^3 P_{2r+1} e_r \right), \end{aligned}$$

which completes the proof. □

The following result gives a general relation concerning balancing and Lucas-balancing quaternions.

**Theorem 6.** For  $m, n \geq 0$ ,

$$QB_{mn} = \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} QB_r,$$

and

$$QC_{mn} = \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} C_m^r C_{m-1}^{n-r} QC_r.$$

*Proof.* Using the identity

$$B_{km+n} = \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} B_k^r B_{k-1}^{m-r} B_{r+n} \quad ([??, \text{Eq. (11)}]),$$

we have

$$\begin{aligned}
 QB_{mn} &= \sum_{l=0}^3 B_{mn+l}e_l \\
 &= \sum_{l=0}^3 \left( \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} B_{r+l} \right) e_l \\
 &= \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} B_m^r B_{m-1}^{n-r} \left( \sum_{l=0}^3 B_{r+l} e_l \right),
 \end{aligned}$$

which completes the proof of the first part. The second part can be obtained similarly using the identity

$$C_{km+n} = \sum_{r=0}^m \binom{m}{r} (-1)^{m-r} B_k^r B_{k-1}^{m-r} C_{r+n} \quad ([??, \text{Theorem 2.1}]). \quad \square$$

The next result follows directly from Theorem ?? by setting  $m = 2$ .

**Corollary 2.** For  $n \geq 0$ ,

$$QB_{2n} = \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} 6^r QB_r,$$

and

$$QC_{2n} = \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} 6^r QC_r.$$

The balancing and Lucas-balancing sums involving binomial coefficients were studied in [??]. The following are analogous to the identities studied in Theorem 4.1, [??].

**Theorem 7.** For any positive integers  $m$  and  $k$  with  $m > k \geq 0$ , we have

$$\sum_{r=0}^n QB_{mr+k} = \frac{(QB_k - QB_{mn+m+k}) + (QB_{mn+k} - QB_{k-m})}{2(1 - QC_m)}$$

and

$$\sum_{r=0}^n QC_{mr+k} = \frac{(QC_k - QC_{mn+m+k}) + (QC_{mn+k} - QC_{k-m})}{2(1 - QC_m)}.$$

*Proof.* Using the Binet formula for balancing quaternions, we obtain

$$\begin{aligned}
 \sum_{r=0}^n QB_{mr+k} &= \sum_{r=0}^n \frac{A\lambda_1^{mr+k} - B\lambda_2^{mr+k}}{\lambda_1 - \lambda_2} \\
 &= \frac{1}{\lambda_1 - \lambda_2} \left( A\lambda_1^k \sum_{r=0}^n \lambda_1^{mr} - B\lambda_2^k \sum_{r=0}^n \lambda_2^{mr} \right) \\
 &= \frac{1}{\lambda_1 - \lambda_2} \left[ A\lambda_1^k \left( \frac{\lambda_1^{mn+m} - 1}{\lambda_1^m - 1} \right) - B\lambda_2^k \left( \frac{\lambda_2^{mn+m} - 1}{\lambda_2^m - 1} \right) \right] \\
 &= \frac{1}{\lambda_1 - \lambda_2} \left[ A \left( \frac{\lambda_1^{mn+m+k} - \lambda_1^k}{\lambda_1^m - 1} \right) - B \left( \frac{\lambda_2^{mn+m+k} - \lambda_2^k}{\lambda_2^m - 1} \right) \right] \\
 &= \frac{1}{\lambda_1 - \lambda_2} \left[ \frac{(A\lambda_1^k - B\lambda_2^k)}{2 - (\lambda_1^m + \lambda_2^m)} - \frac{(A\lambda_1^{mn+m+k} - B\lambda_2^{mn+m+k})}{2 - (\lambda_1^m + \lambda_2^m)} \right] \\
 &\quad + \left[ \frac{(A\lambda_1^{mn+k} - B\lambda_2^{mn+k})}{2 - (\lambda_1^m + \lambda_2^m)} - \frac{(A\lambda_1^{k-m} - B\lambda_2^{k-m})}{2 - (\lambda_1^m + \lambda_2^m)} \right] \\
 &= \frac{(QB_k - QB_{mn+m+k}) + (QB_{mn+k} - QB_{k-m})}{2(1 - QC_m)}.
 \end{aligned}$$

The proof for the Lucas-balancing quaternions is similar. □

The following result is an immediate consequence of the above result.

**Corollary 3.** For  $m \geq 0$ , we have

$$\sum_{r=0}^n QB_{mr} = \frac{(QB_0 - QB_{mn+m}) + (QB_{mn} + QB_m)}{2(1 - QC_m)},$$

and

$$\sum_{r=0}^n QB_r = \frac{(QB_0 - QB_{n+1}) + (QB_n + QB_1)}{2(1 - QC_1)}.$$

### 4 Generating functions for balancing and Lucas-balancing quaternions

Generating functions are used to solve linear recurrence relations with constant coefficients. Recall that a generating function for a sequence  $\{a_n\}$  of real numbers is defined by

$$L(s) = \sum_{n=0}^{\infty} a_n s^n.$$

The generating function for the balancing sequence is given by

$$G(s) = \frac{s}{1 - 6s + s^2} \quad ([??, \text{Theorem 6.1}])$$

whereas that for the Lucas-balancing sequence is

$$g(s) = \frac{1 - 3s}{1 - 6s + s^2} \quad ([??, Proposition 4.4]).$$

In order to find the generating functions for both  $QB_n$  and  $QC_n$ , we need the following result.

**Theorem 8.** For any natural numbers  $k$  and  $m$  with  $k > m \geq 0$  and  $n \in \mathbb{N}$ , the generating functions of  $QB_{kn+m}$  and  $QC_{kn+m}$  are respectively

$$\sum_{n=0}^{\infty} QB_{kn+m} s^n = \frac{QB_m - QB_{m-k}s}{1 - 2C_k s + s^2}$$

and

$$\sum_{n=0}^{\infty} QC_{kn+m} s^n = \frac{QC_m - QC_{m-k}s}{1 - 2C_k s + s^2}.$$

*Proof.* Using the Binet formula for  $QB_n$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} QB_{kn+m} s^n &= \sum_{n=0}^{\infty} \left( \frac{A\lambda_1^{kn+m} - B\lambda_2^{kn+m}}{\lambda_1 - \lambda_2} \right) s^n \\ &= \frac{1}{\lambda_1 - \lambda_2} \left( \frac{A\lambda_1^m}{1 - \lambda_1^k s} - \frac{B\lambda_2^m}{1 - \lambda_2^k s} \right) \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[ \frac{(A\lambda_1^m - B\lambda_2^m) - (A\lambda_1^{m-k} - B\lambda_2^{m-k})s}{1 - (\lambda_1^k + \lambda_2^k)s + s^2} \right] \\ &= \frac{QB_m - QB_{m-k}s}{1 - 2C_k s + s^2}, \end{aligned}$$

which is the desired result. For  $QC_{kn+m}$  the proof is similar as  $QB_{kn+m}$ . □

The following results are direct consequences of Theorem ??.

**Corollary 4.** The generating function  $G_Q(s)$  for balancing quaternions and  $g_Q(s)$  for Lucas-balancing quaternions are respectively

$$G_Q(s) = \frac{se_0 + e_1 + (6 - s)e_2 + (35 - 6s)e_3}{1 - 6s + s^2}$$

and

$$g_Q(s) = \frac{(1 - 3s)e_0 + (3 - s)e_1 + (17 - 3s)e_2 + (99 - 17s)e_3}{1 - 6s + s^2}.$$

*Proof.* Let the generating function for  $QB_n$  be

$$G_Q(s) = \sum_{n=0}^{\infty} QB_n s^n.$$

By putting  $k = 1$  and  $m = 0$ , Theorem ?? becomes

$$G_Q(s) = \frac{1}{1 - 6s + s^2} \sum_{r=0}^3 (B_r - B_{r-1}s)e_r$$

$$= \frac{se_0 + e_1 + (6 - s)e_2 + (35 - 6s)e_3}{1 - 6s + s^2},$$

which completes the proof. The proof is similar for Lucas-balancing quaternions. □

The next results demonstrate the exponential generating functions and Poisson generating functions for both balancing and Lucas-balancing quaternions.

**Theorem 9.** For  $m, n \in \mathbb{N}$ , the exponential generating functions of the quaternions  $QB_{m+n}$  and  $QC_{m+n}$  are respectively

$$\sum_{n=0}^{\infty} \frac{QB_{m+n}}{n!} s^n = \frac{A\lambda_1^m e^{\lambda_1 s} - B\lambda_2^m e^{\lambda_2 s}}{\lambda_1 - \lambda_2}$$

and

$$\sum_{n=0}^{\infty} \frac{QC_{m+n}}{n!} s^n = \frac{A\lambda_1^m e^{\lambda_1 s} + B\lambda_2^m e^{\lambda_2 s}}{2}.$$

*Proof.* Using Binet’s formula for  $QB_{m+n}$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} QB_{m+n} \frac{s^n}{n!} &= \sum_{n=0}^{\infty} \left( \frac{A\lambda_1^{m+n} - B\lambda_2^{m+n}}{\lambda_1 - \lambda_2} \right) \frac{s^n}{n!} \\ &= \frac{A\lambda_1^m}{\lambda_1 - \lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1 s)^n}{n!} - \frac{B\lambda_2^m}{\lambda_1 - \lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_2 s)^n}{n!} \\ &= \left( \frac{A\lambda_1^m}{\lambda_1 - \lambda_2} \right) e^{\lambda_1 s} - \left( \frac{B\lambda_2^m}{\lambda_1 - \lambda_2} \right) e^{\lambda_2 s}, \end{aligned}$$

and the result follows. Further simplification gives

$$\sum_{n=0}^{\infty} \frac{QB_{m+n}}{n!} s^n = QC_m \left( \sum_{n=0}^{\infty} B_n \frac{s^n}{n!} \right) + QB_m \left( \sum_{n=0}^{\infty} C_n \frac{s^n}{n!} \right).$$

The proof for the Lucas-balancing quaternions is similar. □

**Corollary 5.** The exponential generating functions for balancing and Lucas-balancing quaternions are respectively

$$\sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n = \frac{Ae^{\lambda_1 s} - Be^{\lambda_2 s}}{\lambda_1 - \lambda_2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{QC_n}{n!} s^n = \frac{Ae^{\lambda_1 s} + Be^{\lambda_2 s}}{2}.$$

The following result relating to Poisson generating functions is an immediate consequence of Theorem ?? because Poisson generating functions for balancing and Lucas-balancing quaternions can be obtained by multiplying  $e^{-s}$  to the exponential generating functions for both these quaternions.

**Corollary 6.** *The Poisson generating function for balancing and Lucas-balancing quaternions are*

$$\sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n e^{-s} = \frac{Ae^{\lambda_1 s} - Be^{\lambda_2 s}}{e^s(\lambda_1 - \lambda_2)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{QC_n}{n!} s^n e^{-s} = \frac{Ae^{\lambda_1 s} + Be^{\lambda_2 s}}{2e^s},$$

respectively.

The various generating functions discussed above are applied to derive the following identities.

**Lemma 3.** *For any natural number  $n$ ,  $QB_{n+r+1} - QB_{n+r-1} = 2QC_{n+r}$ .*

*Proof.* Using Proposition ??, we get

$$\begin{aligned} \sum_{n=0}^{\infty} QB_{n+r+1} s^n - \sum_{n=0}^{\infty} QB_{n+r-1} s^n &= \frac{QB_{r+1} - QB_r s}{1 - 6s + s^2} - \frac{QB_{r-1} - QB_{r-2} s}{1 - 6s + s^2} \\ &= \frac{(QB_{r+1} - QB_{r-1}) - (QB_r - QB_{r-2})s}{1 - 6s + s^2} \\ &= \frac{2QC_r - 2QC_{r-1} s}{1 - 6s + s^2} \\ &= 2 \sum_{n=0}^{\infty} QC_{n+r} s^n, \end{aligned}$$

which completes the proof. □

**Lemma 4.** *For any natural number  $n$ ,*

$$\sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n = \frac{e^{3s}}{\sqrt{8}} \left[ QC_0 \sinh(\sqrt{8}s) + \sqrt{8}QB_0 \cosh(\sqrt{8}s) \right]$$

and

$$\sum_{n=0}^{\infty} \frac{QC_n}{n!} s^n = e^{3s} \left[ \sqrt{8}QB_0 \sinh(\sqrt{8}s) + QC_0 \cosh(\sqrt{8}s) \right].$$



*Proof.* For any natural number  $n$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{QB_n}{n!} s^n &= \frac{Ae^{\lambda_1 s} - Be^{\lambda_2 s}}{\lambda_1 - \lambda_2} \\ &= \frac{e^{3s}}{2\sqrt{8}} (Ae^{\sqrt{8}s} - Be^{-\sqrt{8}s}) \\ &= \frac{e^{3s}}{2\sqrt{8}} \left[ \left( \frac{QC_r + \sqrt{8}QB_r}{\lambda_1^r} \right) e^{\sqrt{8}s} - \left( \frac{QC_r - \sqrt{8}QB_r}{\lambda_2^r} \right) e^{-\sqrt{8}s} \right] \\ &= \frac{e^{3s}}{\sqrt{8}} \left[ C_r QC_r \left( \frac{e^{\sqrt{8}s} - e^{-\sqrt{8}s}}{2} \right) - \sqrt{8}B_r QC_r \left( \frac{e^{\sqrt{8}s} + e^{-\sqrt{8}s}}{2} \right) \right. \\ &\quad \left. + \sqrt{8}C_r QB_r \left( \frac{e^{\sqrt{8}s} + e^{-\sqrt{8}s}}{2} \right) - 8B_r QB_r \left( \frac{e^{\sqrt{8}s} - e^{-\sqrt{8}s}}{2} \right) \right] \\ &= \frac{e^{3s}}{\sqrt{8}} \left[ \sinh(\sqrt{8}s) (C_r QC_r - 8B_r QB_r) \right. \\ &\quad \left. + \sqrt{8} \cosh(\sqrt{8}s) (C_r QB_r - B_r QC_r) \right]. \end{aligned}$$

Using the Proposition ?? in the above expression we get the desired result. □

Binet’s formulas for balancing and Lucas-balancing quaternions was already shown in Theorem ?. However, these formulas can also be derived by applying generating functions for both balancing and Lucas-balancing quaternions as follows.

By virtue of Corollary ??, we have

$$G_Q(s) = \frac{1}{1 - 6s + s^2} \sum_{r=0}^3 (B_r - B_{r-1}s)e_r .$$

Further simplification using partial fractions reduces the above identity to

$$\begin{aligned} G_Q(s) &= \frac{1}{\lambda_1 - \lambda_2} \left[ \frac{QB_1 - \lambda_2 QB_0}{1 - s\lambda_1} - \frac{QB_1 - \lambda_1 QB_0}{1 - s\lambda_2} \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[ \sum_{s=0}^3 (B_{s+1} - \lambda_2 B_s)e_s \sum_{n=0}^{\infty} \lambda_1^n s^n \right. \\ &\quad \left. - \sum_{s=0}^3 (B_{s+1} - \lambda_1 B_s)e_s \sum_{n=0}^{\infty} \lambda_2^n s^n \right] \\ &= \frac{1}{\lambda_1 - \lambda_2} \left[ \sum_{s=0}^3 \lambda_1^s e_s \sum_{n=0}^{\infty} \lambda_1^n s^n - \sum_{s=0}^3 \lambda_2^s e_s \sum_{n=0}^{\infty} \lambda_2^n s^n \right]. \end{aligned}$$

That is,

$$G_Q(s) = \sum_{n=0}^{\infty} QB_n s^n = \sum_{n=0}^{\infty} \left( \frac{A\lambda_1^n - B\lambda_2^n}{\lambda_1 - \lambda_2} \right) s^n ,$$

and hence the Binet formula for  $QB_n$  is obtained. Similarly, the Binet formula for  $QC_n$  can also be obtained by using the generating function for Lucas-balancing quaternions.

## References

- [1] M. Akyiğit, H.H. Kösal, M. Tosun: Fibonacci generalized quaternions. *Advances in Applied Clifford Algebras* 24 (3) (2014) 631–641.
- [2] A. Behera, G.K. Panda: On the square roots of triangular numbers. *Fibonacci Quarterly* 37 (1999) 98–105.
- [3] P. Catarino, H. Campos, P. Vasco: On some identities for balancing and cobalancing numbers. *Annales Mathematicae et Informaticae* 45 (2015) 11–14.
- [4] P. Catarino: The modified Pell and the modified  $k$ -Pell quaternions and octonions. *Advances in Applied Clifford Algebras* 26 (2) (2016) 577–590.
- [5] C.B. Çimen, A. İpek: On Pell quaternions and Pell-Lucas quaternions. *Advances in Applied Clifford Algebras* 26 (1) (2016) 39–51.
- [6] R.K. Davala, G.K. Panda: On sum and ratio formulas for balancing numbers. *Journal of Indian Mathematical Society* 82 (1–2) (2015) 23–32.
- [7] S. Halici: On fibonacci quaternions. *Advances in Applied Clifford Algebras* 22 (2) (2012) 321–327.
- [8] A.F. Horadam: Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly* 70 (3) (1963) 289–291.
- [9] A.F. Horadam: Quaternion recurrence relations. *Ulam Quarterly* 2 (2) (1993) 23–33.
- [10] M.R. Iyer: A note on Fibonacci quaternions. *Fibonacci Quarterly* 7 (3) (1969) 225–229.
- [11] G.K. Panda, P.K. Ray: Some links of balancing and cobalancing numbers with Pell and associated Pell numbers. *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)* 6 (1) (2011) 41–72.
- [12] P.K. Ray: Curious congruences for balancing numbers. *International Journal of Contemporary Mathematical Sciences* 7 (18) (2012) 881–889.
- [13] P.K. Ray: Balancing and Lucas-balancing sums by matrix methods. *Mathematical Reports* 17 (2) (2015) 225–233.
- [14] A. Szynal-Liana, I. Włoch: The Pell quaternions and the Pell octonions. *Advances in Applied Clifford Algebras* 26 (1) (2016) 435–440.

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