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A Closer Look at the Clock

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A Closer Look at the Clock

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Has it ever happened to you, to wake up early in the morning, still half asleep, unable to recognize which of the hands on your alarm clock was the short hand and which one was the long hand? If you were supposed to get up at half past six, you might panic that it was actually half past seven. Yes, it really can happen, as described in Figures 1(b) and (c). We intend to explain this "possible confusion" in today's article and describe all such clock hand positions. This will happen in Section 3.

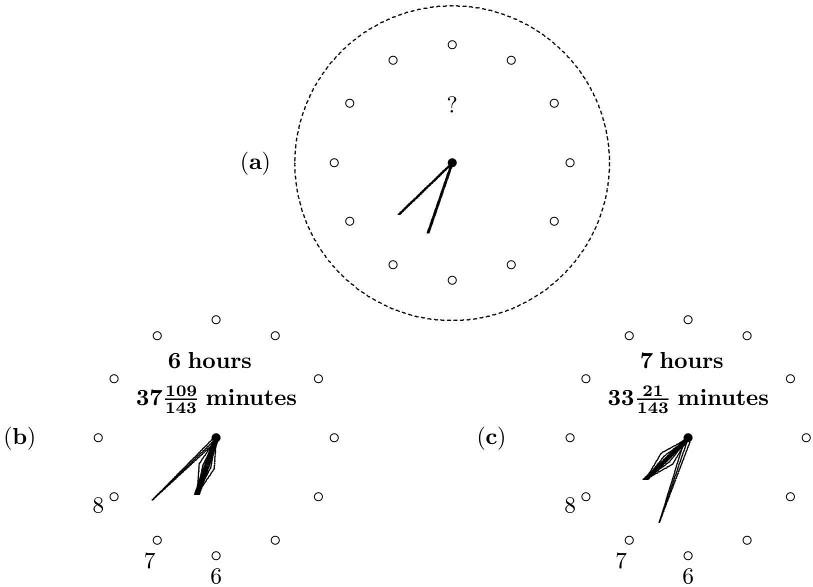
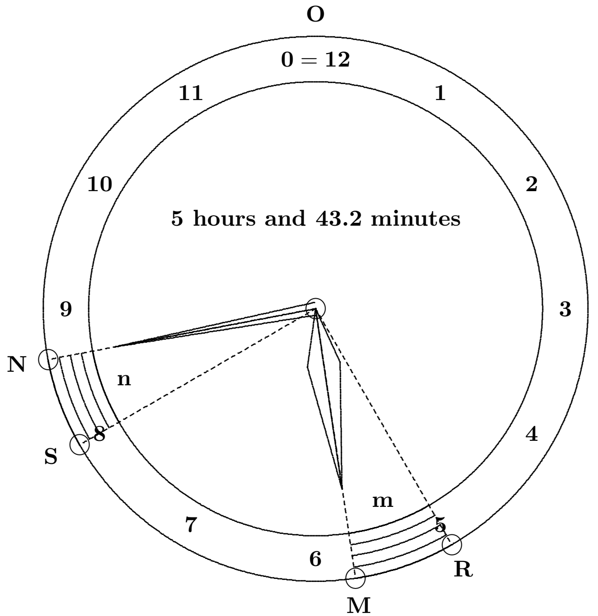


Figure 1. A morning dial

1. Let us emphasize that we are describing a classic dial with two moving hands: an hour hand and a minute hand (i.e. not a digital clock). In order to express ourselves accurately and concisely, we will call the clock hands the H-hand and the M-hand.

First, let us get acquainted with the description of the time on a circular dial, which is divided into 12 equal (hourly) parts, and each of these parts into 5 equal (minute) parts. The dial is therefore divided into 60 minute parts.

In common parlance, we describe time with two data points: hours and minutes. Typically, we report r hours ($0 \leq r < 12$ as measured in the hour scale) and v minutes ($0 \leq v < 60$ as measured in the minute scale). In Figure 2, $r = 5$ corresponds to the circular arc OR and $v = 43.2$ to the circular arc ON.



Hours	Minutes
$0 \sim 12$	$0 \sim 60$
1	5
2	10
3	15
4	20
5	25
6	30
7	35
8	40
9	45
10	50
11	55

Figure 2. Hour and minute scales

The position of the H-hand, corresponding in Figure 2 to the circular arc OM is thus in a minute system determined by coordinates

$$u = 5r + m, \quad \text{where } r \text{ is an integer and } 0 \leq m < 5.$$

Hence, we shall denote it (r, m), $r \in \{0, 1, 2, \ldots, 10, 11\}$, $0 \leq m < 5$. In Figure 2, $u = 28.6$ and $(r, m) = (5, 3.6)$.

It is very important to emphasize here that the position of the H-hand is decisive. Time is completely determined by its very position (position of one hand, the H-hand).

The position of the M-hand, expressed in the minutes scale by v and corresponding in Figure 2 to the circular arc ON, is already derived from this information provided by the H-hand. The change of the H-hand position by one minute segment corresponds to the time of 12 minutes, and therefore

$$v = 12m.$$

The position of the M-hand can be described by the pair (s, n), where s is an integer, $0 \leq s < 12$ and $0 \leq n < 5$. However, these data ($v = 43.2$, $s = 8$, $n = 3.2$ in Figure 2) as far as the M-hand is concerned do not provide full information (the hour count is missing).

The following simple example provides an illustration of our notation: Time of 3 hours and 43 minutes is described by the pair $(r, m) = (3, 3\frac{7}{12})$, because $\frac{43}{12} = 3\frac{7}{12}$. The minute coordinate of the H-hand is

$$u = 5r + m = 5 \times 3 + 3\frac{7}{12} = 18\frac{7}{12}.$$

This may be an appropriate place to point out that our time span covers a 12-hour period: either 12:00 p.m. to 12:00 a.m. or 12:00 a.m. to 12:00 p.m. You may apply the statements in this article to either time span.

Referring to the clocks 1(b) and 1(c) in Figure 1, apply our notation to the time information provided there.

In the case 1(b), $r_b = 6$ and $v_b = 37\frac{109}{143} = \frac{5400}{143}$ minutes. From here, $m_b = \frac{v_b}{12} = \frac{450}{143} = 3\frac{21}{143}$. Thus, the time is described by the pair $(6, 3\frac{21}{143})$.

In the case 1(c), $r_c = 7$ and $v_c = 33\frac{21}{143} = \frac{4740}{143}$ minutes. Thus $m_c = \frac{395}{143} = 2\frac{109}{143}$. The time is described by the pair $(7, 2\frac{109}{143})$. Notice that $12m_c = 5r_b + m_b$ and $12m_b = 5r_c + m_c$.

This means that the interchange of the clock hands indicated in Figure 1(a) is possible. Applying the minute scale, the H-hand shows $33\frac{21}{143}$ minutes and the M-hand shows $37\frac{109}{143}$ minutes in the first case 1(b). In the other case 1(c), the H-hand shows $37\frac{109}{143}$ minutes and the M-hand $33\frac{21}{143}$ minutes.

Let us point out that besides the mentioned hour or minute scales, one can also describe the time in seconds or degrees. Figure 3 compares the translation of the coordinates of one scale into the other. Here, we compare x hours, y minutes, z seconds and w degrees.

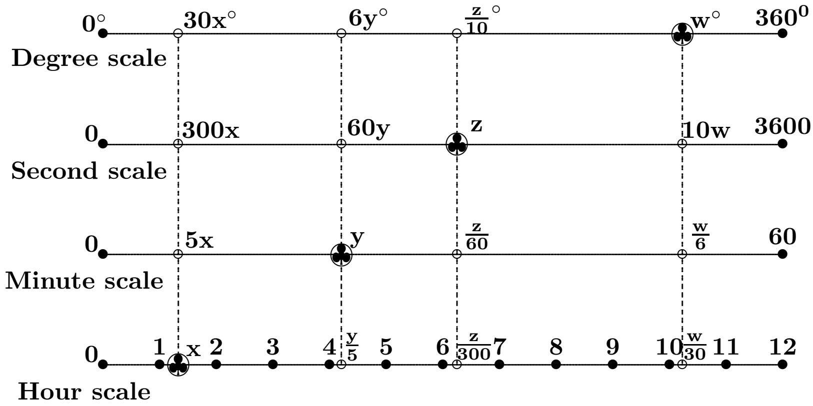


Figure 3. Clock (circular) scales

Remember: The time is determined by the position of the hour hand (H-hand) on the circular dial divided into 60 minutes, i.e. by the number u, $0 \leq u < 60$. For better visibility, we further express the time u in the form $u = 5r+m$, where r is a natural number and m satisfies $0 \leq m < 5$. The time is then registered by the pair (r, m) and expressed as r hours and $12m$ minutes.

2. Here we recall a question that circulates in different versions on the internet (see e.g. [1] or [2]), and solve it by describing the time in degrees.

Exercise. *Determine the time after two o'clock in the afternoon at which the clock hands will be at right angles.*

In mathematics we aim to generalize formulations and solutions and not just stop at specific cases. This is one of the fundamental features of mathematics, for a general solution is often more illuminating and explains special cases in depth.

Here we expand the exercise by asking to determine within a certain hour r the time at which one clock hand will make an angle w, $0 \leq w < 360°$ with the other hand. This means in the minute scale that $0 \leq \frac{w}{6} < 60°$.

We are going to determine the time (r, m), when (in the minute system)

$$5r + m + \frac{w}{6} = 12m, \quad \text{i.e.} \quad 30r + w = 66m.$$

Hence

$$m = \frac{30r + w}{66} \quad \text{assuming that } m < 5.$$

This means that $0 \leq r \leq 10$ and $\frac{w}{30} < 11 - r$. The time required is

$$(r, m = \frac{30r + w}{66}).$$

Consequently, for an angle $w < 30°$ there is no restriction.

In the case when $\frac{w}{30} \geq 11 - r$, that is when $30r + w \geq 330$, we write

$$\frac{30r + w}{66} = 5 + m'.$$

Here, the time in question is

$$(r + 1, m' = \frac{30r + w}{66} - 5).$$

In our exercise (when $w = 90°$) this will happen for the values $r = 8, 9, 10$. The situation when the clock hands are at a right angle is achieved also for the angle $w = 270°$. In that case, this hourly shift will happen for all values with the exception $r = 0$ and $r = 1$.

Here is a complete list of all 22 times when the clock hands are at a right angle.

$(0, \frac{15}{11})$	i.e.	0	hours	$16\frac{4}{11}$	minutes	$(0, \frac{45}{11})$	i.e.	0	hours	$49\frac{1}{11}$	minutes
$(1, \frac{20}{11})$	i.e.	1	hour	$21\frac{9}{11}$	minutes	$(1, \frac{50}{11})$	i.e.	1	hour	$54\frac{6}{11}$	minutes
$(2, \frac{25}{11})$	i.e.	2	hours	$27\frac{3}{11}$	minutes	$(3, 0)$	i.e.	3	hours	0	minutes
$(3, \frac{30}{11})$	i.e.	3	hours	$32\frac{8}{11}$	minutes	$(4, \frac{5}{11})$	i.e.	4	hours	$5\frac{5}{11}$	minutes
$(4, \frac{35}{11})$	i.e.	4	hours	$38\frac{2}{11}$	minutes	$(5, \frac{10}{11})$	i.e.	5	hours	$10\frac{10}{11}$	minutes
$(5, \frac{40}{11})$	i.e.	5	hours	$43\frac{7}{11}$	minutes	$(6, \frac{15}{11})$	i.e.	6	hours	$16\frac{4}{11}$	minutes
$(6, \frac{45}{11})$	i.e.	6	hours	$49\frac{1}{11}$	minutes	$(6, \frac{20}{11})$	i.e.	7	hours	$21\frac{9}{11}$	minutes
$(7, \frac{50}{11})$	i.e.	7	hours	$54\frac{6}{11}$	minutes	$(8, \frac{25}{11})$	i.e.	8	hours	$27\frac{3}{11}$	minutes
$(9, 0)$	i.e.	9	hours	0	minutes	$(9, \frac{30}{11})$	i.e.	9	hours	$32\frac{8}{11}$	minutes
$(10, \frac{5}{11})$	i.e.	10	hours	$5\frac{5}{11}$	minutes	$(10, \frac{35}{11})$	i.e.	10	hours	$38\frac{2}{11}$	minutes
$(11, \frac{10}{11})$	i.e.	11	hours	$10\frac{10}{11}$	minutes	$(11, \frac{40}{11})$	i.e.	11	hours	$43\frac{7}{11}$	minutes

For instance, choosing $r = 2, w = 90°$, the clock hands are at a right angle at 2 hours and $27\frac{3}{11}$ minutes. The hour (short) hand is at the position $12\frac{3}{11}$ minutes.

A similar list of the times for the (unoriented) angle $w = 30°$ is as follows:

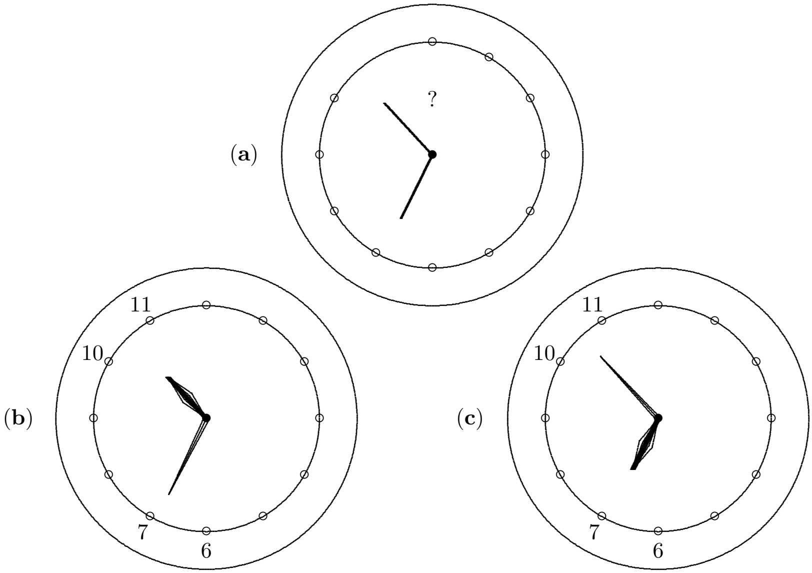
$(0, \frac{5}{11})$	i.e.	0	hours	$5\frac{5}{11}$	minutes		$(1, 0)$	i.e.	1	hour	0	minutes
$(1, \frac{10}{11})$	i.e.	1	hour	$10\frac{10}{11}$	minutes		$(2, \frac{5}{11})$	i.e.	2	hours	$5\frac{5}{11}$	minutes
$(2, \frac{15}{11})$	i.e.	2	hours	$16\frac{4}{11}$	minutes		$(3, \frac{10}{11})$	i.e.	3	hours	$10\frac{10}{11}$	minutes
$(3, \frac{20}{11})$	i.e.	3	hours	$21\frac{9}{11}$	minutes		$(4, \frac{15}{11})$	i.e.	4	hours	$16\frac{4}{11}$	minutes
$(4, \frac{25}{11})$	i.e.	4	hours	$27\frac{3}{11}$	minutes		$(5, \frac{20}{11})$	i.e.	5	hours	$21\frac{9}{11}$	minutes
$(5, \frac{30}{11})$	i.e.	5	hours	$32\frac{8}{11}$	minutes		$(6, \frac{25}{11})$	i.e.	6	hours	$27\frac{3}{11}$	minutes
$(6, \frac{35}{11})$	i.e.	6	hours	$38\frac{2}{11}$	minutes		$(7, \frac{30}{11})$	i.e.	7	hours	$32\frac{8}{11}$	minutes
$(7, \frac{40}{11})$	i.e.	7	hours	$43\frac{7}{11}$	minutes		$(8, \frac{35}{11})$	i.e.	8	hours	$38\frac{2}{11}$	minutes
$(8, \frac{45}{11})$	i.e.	8	hours	$49\frac{1}{11}$	minutes		$(9, \frac{40}{11})$	i.e.	9	hours	$43\frac{7}{11}$	minutes
$(9, \frac{50}{11})$	i.e.	9	hours	$54\frac{6}{11}$	minutes		$(10, \frac{45}{11})$	i.e.	10	hours	$49\frac{1}{11}$	minutes
$(11, 0)$	i.e.	11	hours	0	minutes		$(11, \frac{50}{11})$	i.e.	11	hours	$54\frac{6}{11}$	minutes

As an example, choosing $r = 6$, $w = 30°$, the clock hands make the angle $30°$ at 6 hours and $27\frac{3}{11}$ minutes, as well as at 6 hours and $38\frac{2}{11}$ minutes. The hour (short) hand is at the positions $22\frac{3}{11}$ and $33\frac{2}{11}$ minutes.

Let us remark that for every value of $w \neq 0°, 180°$ the number of cases when the clock hands form the angle w is 22. In the case when $w = 0°$ (when the hands overlap, i.e. when they lie directly on the top of each other) or when $w = 180°$ (when the hands lie in a straight line) this number is reduced to a half, i.e. to 11 cases. These times are recorded in the following two columns:

		$w = 0°$							$w = 180°$		
$(0, 0)$	i.e.	0	hours	0	minutes	$(0, \frac{30}{11})$	i.e.	0	hours	$32\frac{8}{11}$	minutes
$(1, \frac{5}{11})$	i.e.	1	hour	$5\frac{5}{11}$	minutes	$(1, \frac{35}{11})$	i.e.	1	hour	$38\frac{2}{11}$	minutes
$(2, \frac{10}{11})$	i.e.	2	hours	$10\frac{10}{11}$	minutes	$(2, \frac{40}{11})$	i.e.	2	hours	$43\frac{7}{11}$	minutes
$(3, \frac{15}{11})$	i.e.	3	hours	$16\frac{4}{11}$	minutes	$(3, \frac{45}{11})$	i.e.	3	hours	$49\frac{1}{11}$	minutes
$(4, \frac{20}{11})$	i.e.	4	hours	$21\frac{9}{11}$	minutes	$(4, \frac{50}{11})$	i.e.	4	hours	$54\frac{6}{11}$	minutes
$(5, \frac{25}{11})$	i.e.	5	hours	$27\frac{3}{11}$	minutes	$(6, 0)$	i.e.	6	hours	0	minutes
$(6, \frac{30}{11})$	i.e.	6	hours	$32\frac{8}{11}$	minutes	$(7, \frac{5}{11})$	i.e.	7	hours	$5\frac{5}{11}$	minutes
$(7, \frac{35}{11})$	i.e.	7	hours	$38\frac{2}{11}$	minutes	$(8, \frac{10}{11})$	i.e.	8	hours	$10\frac{10}{11}$	minutes
$(8, \frac{40}{11})$	i.e.	8	hours	$43\frac{7}{11}$	minutes	$(9, \frac{15}{11})$	i.e.	9	hours	$16\frac{4}{11}$	minutes
$(9, \frac{45}{11})$	i.e.	9	hours	$49\frac{1}{11}$	minutes	$(10, \frac{20}{11})$	i.e.	10	hours	$21\frac{9}{11}$	minutes
$(10, \frac{50}{11})$	i.e.	10	hours	$54\frac{6}{11}$	minutes	$(11, \frac{25}{11})$	i.e.	11	hours	$27\frac{3}{11}$	minutes

3. The case $w = 0°$ brings us back to the original question of our article: namely to determine the positions of the clock hands that allow us to interchange the functions of the hands.



10 hours $34\frac{58}{143}$ minutes **6 hours $52\frac{124}{143}$ minutes**

Figure 4. Evening and morning dials

Let us formulate yet another exercise that relates, in a specific situation, to our question.

Exercise. *When I went to bed yesterday evening after ten o'clock, the hands on my alarm clok were exactly in the same position as when I got up this morning before seven o'clock: Just as we can see it in* Figure 4(a)*. Is it possible?*

Figure 4 provides both answer and explanation. Yes, it is possible. It was 10 hours and $34\frac{58}{143}$ minutes in the evening; we have agreed to describe this time by a pair of numbers $(10, 2\frac{124}{143})$. This morning, the time was 6 hours and $52\frac{124}{143}$ minutes, thus the time described by the pair $(6, 4\frac{58}{143})$. The minute coordinates of the H-hand were $52\frac{124}{143}$ in the evening and $34\frac{58}{143}$ in the morning, while the coordinates of the M-hand were $34\frac{58}{143}$ in the evening and $52\frac{124}{143}$ this morning.

As we have already described, every position (r, m) of the H-hand determines the position (s, n) of the M-hand defined by the relation $12m = 5s + n$. This position (s, n) can be considered to be a new position of the H-hand that further determines the respective position (t, p) of the M-hand defined by $12n = 5t + p$, $t \in \{0, 1, 2 \ldots, 10, 11\}$ and $0 \leq p < 5$. Call the clock hands “associated” if $t = r$ and $p = m$. Thus, our original question can be formulated as follows.

Original exercise. *Determine all associated positions of the clock hands.*

The exercise requires us to solve the following system of two linear equations:

$$\begin{aligned} 12m - n &= 5s, \\ -m + 12n &= 5r, \end{aligned}$$

where r and s are two arbitrary numbers of the set $\{0, 1, 2, \ldots, 10, 11\}$. Such systems have the following unique solution for every choice of numbers r, s

$$m = \frac{5r + 60s}{143} \quad \text{and} \quad n = \frac{60r + 5s}{143},$$

representing the uniquely determined positions (r, m) and (s, n) of the associated clock hands. These positions of the H-hands have in the minute scale the coordinates

$$u = \frac{60r + 720s}{143} \quad \text{and} \quad v = \frac{720r + 60s}{143}.$$

The respective times are “r hours and v minutes” and “s hours and u minutes” within the time period from 0 to 12 o’clock. Of course, the described positions of the H-hands can be interpreted in the time period of the entire day from 0 to 24 o’clock. In such a case the pair (r, m) can mean “r hours and v minutes” or “$(r + 12)$ hours and v minutes”. Both positions coincide for the choice $r = s$.

Figure 5 provides a list of all positions of the associated H-hands in the minute scale. Thus there are 143 positions of the associated clock hands within the 12-hour time period. The associated pairs are listed symmetrically along the diagonal. The diagonal consists of the 11 positions of the hands when they overlap. The intersection of the r-th row and s-th column specify the position $(r, \frac{5r+60s}{143})$ of the H-hand.

Hours	0	1	2	3	4	5	6	7	8	9	10	11
0	$\mathbf{0}$	$\frac{60}{143}$	$\frac{120}{143}$	$1\frac{37}{143}$	$1\frac{97}{143}$	$2\frac{14}{143}$	$2\frac{74}{143}$	$2\frac{134}{143}$	$3\frac{51}{143}$	$3\frac{111}{143}$	$4\frac{28}{143}$	$4\frac{88}{143}$
1	$5\frac{5}{143}$	$\mathbf{5\frac{65}{143}}$	$5\frac{125}{143}$	$6\frac{42}{143}$	$6\frac{102}{143}$	$7\frac{19}{143}$	$7\frac{79}{143}$	$7\frac{139}{143}$	$8\frac{56}{143}$	$8\frac{116}{143}$	$9\frac{33}{143}$	$9\frac{93}{143}$
2	$10\frac{10}{143}$	$10\frac{70}{143}$	$\mathbf{10\frac{130}{143}}$	$11\frac{47}{143}$	$11\frac{107}{143}$	$12\frac{24}{143}$	$12\frac{84}{143}$	$13\frac{1}{143}$	$13\frac{61}{143}$	$13\frac{121}{143}$	$14\frac{38}{143}$	$14\frac{98}{143}$
3	$15\frac{15}{143}$	$15\frac{75}{143}$	$15\frac{135}{143}$	$\mathbf{16\frac{52}{143}}$	$16\frac{112}{143}$	$17\frac{29}{143}$	$17\frac{89}{143}$	$18\frac{6}{143}$	$18\frac{66}{143}$	$18\frac{126}{143}$	$19\frac{43}{143}$	$19\frac{103}{143}$
4	$20\frac{20}{143}$	$20\frac{80}{143}$	$20\frac{140}{143}$	$21\frac{57}{143}$	$\mathbf{21\frac{117}{143}}$	$22\frac{34}{143}$	$22\frac{94}{143}$	$23\frac{11}{143}$	$23\frac{71}{143}$	$23\frac{131}{143}$	$24\frac{48}{143}$	$24\frac{108}{143}$
5	$25\frac{25}{143}$	$25\frac{85}{143}$	$26\frac{2}{143}$	$26\frac{62}{143}$	$26\frac{122}{143}$	$\mathbf{27\frac{39}{143}}$	$27\frac{99}{143}$	$28\frac{16}{143}$	$28\frac{76}{143}$	$28\frac{136}{143}$	$29\frac{53}{143}$	$29\frac{113}{143}$
6	$30\frac{30}{143}$	$30\frac{90}{143}$	$31\frac{7}{143}$	$31\frac{67}{143}$	$31\frac{127}{143}$	$32\frac{44}{143}$	$\mathbf{32\frac{104}{143}}$	$33\frac{21}{143}$	$33\frac{81}{143}$	$33\frac{141}{143}$	$34\frac{58}{143}$	$34\frac{118}{143}$
7	$35\frac{35}{143}$	$35\frac{95}{143}$	$36\frac{12}{143}$	$36\frac{72}{143}$	$36\frac{132}{143}$	$37\frac{49}{143}$	$37\frac{109}{143}$	$\mathbf{38\frac{26}{143}}$	$38\frac{86}{143}$	$39\frac{3}{143}$	$39\frac{63}{143}$	$39\frac{123}{143}$
8	$40\frac{40}{143}$	$40\frac{100}{143}$	$41\frac{17}{143}$	$41\frac{77}{143}$	$41\frac{137}{143}$	$42\frac{54}{143}$	$42\frac{114}{143}$	$43\frac{31}{143}$	$\mathbf{43\frac{91}{143}}$	$44\frac{8}{143}$	$44\frac{68}{143}$	$44\frac{128}{143}$
9	$45\frac{45}{143}$	$45\frac{105}{143}$	$46\frac{22}{143}$	$46\frac{82}{143}$	$46\frac{142}{143}$	$47\frac{59}{143}$	$47\frac{119}{143}$	$48\frac{36}{143}$	$48\frac{96}{143}$	$\mathbf{49\frac{13}{143}}$	$49\frac{73}{143}$	$49\frac{133}{143}$
10	$50\frac{50}{143}$	$50\frac{110}{143}$	$51\frac{27}{143}$	$51\frac{87}{143}$	$52\frac{4}{143}$	$52\frac{64}{143}$	$52\frac{124}{143}$	$53\frac{41}{143}$	$53\frac{101}{143}$	$54\frac{18}{143}$	$\mathbf{54\frac{78}{143}}$	$54\frac{138}{143}$
11	$55\frac{55}{143}$	$55\frac{115}{143}$	$56\frac{32}{143}$	$56\frac{92}{143}$	$57\frac{9}{143}$	$57\frac{69}{143}$	$57\frac{129}{143}$	$58\frac{46}{143}$	$58\frac{106}{143}$	$59\frac{23}{143}$	$59\frac{83}{143}$	$\mathbf{0}$

Figure 5. Record of the "associated" hand positions (in minute scale)

Conclude the article by illustrating one of the associated positions.

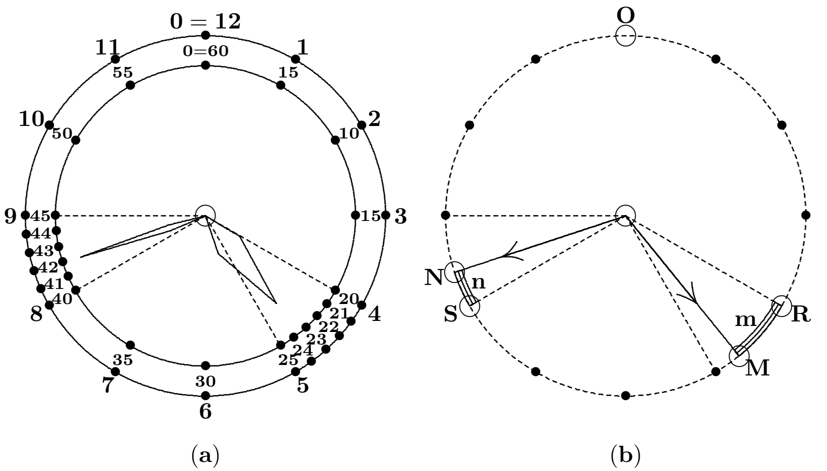


Figure 6. r hours and v $(= 12m = 5s + n)$ minutes

If we choose $r = 4$ and $s = 8$, the position of the clock hands is given in Figure 6:

$$m = \frac{500}{143}, \quad u = 23\frac{71}{143}, \quad v = 41\frac{137}{143}, \quad n = \frac{280}{143}$$

(check these entries and identify them in Figure 6).

As a final exercise, describe the positions of the associated clock hands for another choice of r and s, for example for $r = 2$ and $s = 10$.

Moreover, check that the angle between the clock hands in Figure 6 is $(110\frac{10}{13})^\circ$. Describe the angles between the associated clock hands in general.

References

[1] https://en.wikipedia.org/wiki/Clock angle problem

[2] Problem 49 J at https://math.naboj.org/problems.php?prob=2011

Slovníček: A Closer Look at the Clock

alarm clock = budík
associated = přidružený
concisely = stručně
to conclude = uzavřít
coordinates = souřadnice
decisive = rozhodující
dial = ciferník
fundamental feature = základní rys
to get acquainted with = seznámit se s
illuminating = osvětlující
in common parlance = v běžné řeči
to intend = zamýšlet, hodlat
interchange = (vzájemná) výměna
to overlap = překrývat se
scale = škála, měřítko, stupnice
time span = časové období