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HOMOGENIZATION OF LINEAR PARABOLIC EQUATIONS WITH THREE SPATIAL AND THREE TEMPORAL SCALES FOR CERTAIN MATCHINGS BETWEEN THE MICROSCOPIC SCALES

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Abstract. In this paper we establish compactness results of multiscale and very weak multiscale type for sequences bounded in $L^2(0,T;H_0^1(\Omega))$, fulfilling a certain condition. We apply the results in the homogenization of the parabolic partial differential equation $\varepsilon^p\partial_t u_\varepsilon(x,t) - \nabla\cdot(a(x\varepsilon^{-1},x\varepsilon^{-2},t\varepsilon^{-q},t\varepsilon^{-r})\nabla u_\varepsilon(x,t)) = f(x,t)$, where $0<p<q<r$. The homogenization result reveals two special phenomena, namely that the homogenized problem is elliptic and that the matching for which the local problem is parabolic is shifted by p, compared to the standard matching that gives rise to local parabolic problems.

Keywords: homogenization; parabolic problem; multiscale convergence; very weak multiscale convergence; two-scale convergence

MSC 2020: 35B27, 35K20

1. Introduction

Let $T>0$ and $\Omega_T=\Omega\times(0,T)$, where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary and $(0,T)$ is an open bounded interval in $\mathbb{R}$. We consider the homogenization of the linear parabolic equation

$$\begin{aligned} (1.1)\qquad \varepsilon^p\partial_t u_\varepsilon(x,t) - \nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\Big) &= f(x,t) \quad \text{in } \Omega_T,\\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega,\\ u_\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T), \end{aligned}$$

where $0<p<q<r$ are real numbers, $f\in L^2(\Omega_T)$ and $u_0\in L^2(\Omega)$. The coefficient a is periodic with respect to the unit cube $Y=(0,1)^N$ in the first two variables and

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with respect to the unit interval $S = (0, 1)$ in the third and fourth variable. More detailed information on the equation will be provided in Section 3.

Homogenization means that we study the limit behavior as $\varepsilon \to 0$ and search for a weak $L^2(0, T; H_0^1(\Omega))$-limit u to $\{u_\varepsilon\}$ which is the solution to a so-called homogenized problem. This limit problem is governed by a coefficient b that unlike $a(x\varepsilon^{-1}, x\varepsilon^{-2}, t\varepsilon^{-q}, t\varepsilon^{-r})$ does not include rapid oscillations. In the homogenization procedure local problems are also extracted which include information about the microstructure and whose solutions are utilized to determine b.

The present paper is a further generalization of the work presented in [12]. In earlier works, like e.g. [10], boundedness in $W^{1,2}(0, T; H_0^1(\Omega), L^2(\Omega))$, meaning that $\{u_\varepsilon\}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{\partial_t u_\varepsilon\}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, has been required when compactness results have been established. In [12], compactness results of $(2, 2)$-scale and very weak $(2, 2)$-scale convergence type were proven by requiring boundedness of the sequence $\{u_\varepsilon\}$ in $L^2(0, T; H_0^1(\Omega))$ but replacing the assumption of boundedness of the time derivative in $L^2(0, T; H^{-1}(\Omega))$ by a certain condition. This new approach originates, up to the authors' knowledge, from [13] and will be used in the present work. Here we focus on establishing appropriate compactness results and a homogenization result for the parabolic partial differential equation (1.1). In particular, we generalize the results from [12] to the $(2, 3)$-scale and $(3, 3)$-scale convergence types, adapting to problem (1.1), and present compactness results for both multiscale and very weak multiscale convergence.

For the homogenization part of this paper we apply the convergence results to establish a homogenization result for (1.1) with 13 different outcomes, depending on the choices of parameters p, q and r. The homogenization result will reveal two phenomena, which also occurred in both [12] and the proceeding work [5], where the homogenization of parabolic problems of a similar kind, but with only one rapid scale in space and time each, was presented. The first phenomenon is that the homogenized problem is of elliptic type even though the original problem is a parabolic one and the second is that resonance occurs for different matchings between the microscopic scales than the standard ones. By resonance we mean that the local problem is parabolic, which only occurs for certain matchings between the microscopic scales. What we call the standard matching is when a temporal scale equals the square of a spatial one, as was the case in several other studies, see e.g. [3], [11], [17], [2], [7], [9], [20], [10] or [6] for more on this matter. However, in our case the matching for which we have resonance is shifted by p. Note that in our equation, (1.1), we would get resonance for the standard matching if $p = 0$, cf. Section 5.3.1 in [19].

The paper is organized as follows. In Section 2 we recall some of the key definitions, namely evolution multiscale convergence and very weak evolution multiscale convergence. We prove the main convergence results (see Theorems 2.5 and 2.8), which lay the foundation to establish the homogenization result. Theorem 2.5 is where we find characterizations of the $(2,3)$-scale and $(3,3)$-scale limits for $\{\nabla u_\varepsilon\}$ under certain assumptions. In Theorem 2.8 we consider very weak $(2,3)$-scale and $(3,3)$-scale convergence for the sequences $\{\varepsilon^{-1}u_\varepsilon\}$ and $\{\varepsilon^{-2}u_\varepsilon\}$, respectively. In Section 3, we state a homogenization result presented in Theorem 3.1.

We end the introduction with some essential notations used throughout this paper.

Notation 1.1. *We denote* $\mathcal{Y}_{n,m} = Y^n \times S^m$ *with* $Y^n = Y_1 \times Y_2 \times \ldots \times Y_n$ *and* $S^m = S_1 \times S_2 \times \ldots \times S_m$*, where* $Y_1 = Y_2 = \ldots = Y_n = Y = (0,1)^N$ *and* $S_1 = S_2 = \ldots = S_m = S = (0,1)$*. We let* $y^n = y_1, y_2, \ldots, y_n$*,* $\mathrm{d}y^n = \mathrm{d}y_1\,\mathrm{d}y_2\ldots\mathrm{d}y_n$*,* $s^m = s_1, s_2, \ldots, s_m$ *and* $\mathrm{d}s^m = \mathrm{d}s_1\,\mathrm{d}s_2\ldots\mathrm{d}s_m$*. We define the function space* $\mathcal{W}_{i,j} = \{u \in L^2_\sharp(S_j; H^1_\sharp(Y_i)/\mathbb{R})\colon\ \partial_{s_j}u \in L^2_\sharp(S_j;(H^1_\sharp(Y_i)/\mathbb{R})')\}$*. The subscript* $\sharp$ *is used to denote periodicity of the functions involved over the domain in question. Lastly, for* $k = 1,\ldots,n$ *and* $j = 1,\ldots,m$*, the scale functions* $\varepsilon_k(\varepsilon)$ *and* $\varepsilon'_j(\varepsilon)$ *are strictly positive functions that tend to zero as* ε *does and* $\{\varepsilon_1,\ldots,\varepsilon_n\}$ *and* $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ *denote lists of spatial and temporal scales, respectively.*

2. Multiscale and very weak multiscale convergence

The concept of multiscale convergence is a generalization of the classical two-scale convergence, originating from [15] and [16]. Two-scale convergence is suitable for sequences having one microscopic spatial scale and it has been generalized, first to include multiple spatial scales by Allaire and Briane in [1], and later to also include multiple temporal scales.

Definition 2.1. A sequence $\{u_\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1,m+1)$-*scale converge to* $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v\Big(x,t,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n},\frac{t}{\varepsilon'_1},\ldots,\frac{t}{\varepsilon'_m}\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} u_0(x,t,y^n,s^m)v(x,t,y^n,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t$$

for all $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. This is denoted by

$$u_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m).$$

We make some standard assumptions on the scales. We say that the *scales in a list* $\{\varepsilon_1,\ldots,\varepsilon_n\}$ *are separated* if

$$\lim_{\varepsilon\to 0}\frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

and *well-separated* if there exists a positive integer l such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^l = 0,$$

where $k = 1, \ldots, n-1$. Following the definition by Persson, see e.g. [18], the generalization of separatedness and well-separatedness to include two lists of scales reads as follows.

Definition 2.2. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ be lists of (well-)separated scales. Collect all elements from both lists in one common list. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is (well-)separated, the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are said to be *jointly (well-)separated*.

We present a compactness result for evolution multiscale convergence.

Theorem 2.3. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and suppose that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are jointly separated. Then, up to a subsequence,*

$$u_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m),$$

where $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$.

Proof. See Theorem A.1 in [10]. □

As the next theorem states, the evolution multiscale limit is unique.

Theorem 2.4. *The $(n+1, m+1)$-scale limit is unique.*

Proof. The proof is analogous to the proof of the uniqueness of the two-scale limit given in the discussion below Definition 1 in [14]. □

We are now ready to give a compactness result for the gradient of a sequence $\{u_\varepsilon\}$. The following theorem will play a vital role in the homogenization of (1.1).

Theorem 2.5. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H^1_0(\Omega))$ and, for any $v \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C^\infty_\sharp(S_1)$, $c_3 \in C^\infty_\sharp(S_2)$ and $r > q > 0$,*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v(x) \partial_t \Big(\varepsilon^r c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.1}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v(x) \partial_t \Big(\varepsilon^q c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.2}$$

Then, with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_1' = \varepsilon^q$ and $\varepsilon_2' = \varepsilon^r$, up to a subsequence,

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{2.3}$$

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u(x,t), \tag{2.4}$$

$$\nabla u_\varepsilon(x,t) \overset{2,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) \tag{2.5}$$

and

$$\nabla u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2), \tag{2.6}$$

where $u \in L^2(0,T;H_0^1(\Omega))$, $u_1 \in L^2(\Omega_T \times S^2; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$.

Proof. From the boundedness of $\{u_\varepsilon\}$ in $L^2(0,T;H_0^1(\Omega))$, the weak convergence (2.3) follows immediately. It also implies that $\{\nabla u_\varepsilon\}$ is bounded in $L^2(\Omega_T)^N$ and hence, according to Theorems 2.3 and 2.4, we have

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u_0(x,t,y^2,s^2) \tag{2.7}$$

and

$$\nabla u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} \tau_0(x,t,y^2,s^2), \tag{2.8}$$

up to a subsequence, for some unique $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,2})$ and $\tau_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,2})^N$.

We proceed by characterizing u_0, where we first show that u_0 is independent of the local space and time variables y_1, y_2, s_1 and s_2. Letting $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y_1)$, $v_3 \in C_\sharp^\infty(Y_2)^N$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$ and $c_3 \in C_\sharp^\infty(S_2)$, it holds that

$$\begin{aligned}
&\int_{\Omega_T} \nabla u_\varepsilon(x,t)\varepsilon^2 v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = -\int_{\Omega_T} u_\varepsilon(x,t)\Big(\varepsilon^2 \nabla v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big) + \varepsilon v_1(x) \nabla_{y_1} v_2\Big(\frac{x}{\varepsilon}\Big) \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big) \\
&\qquad\qquad + v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) \nabla_{y_2} \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we have applied integration by parts and carried out the differentiation process. As $\varepsilon \to 0$, $\{\varepsilon^2 \nabla u_\varepsilon\}$ approaches 0 due to boundedness of $\{\nabla u_\varepsilon\}$ and we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\Omega_T} &- u_\varepsilon(x,t)\Big(\varepsilon^2 \nabla v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big) + \varepsilon v_1(x) \nabla_{y_1} v_2\Big(\frac{x}{\varepsilon}\Big) \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big) \\
&+ v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) \nabla_{y_2} \cdot v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0
\end{aligned}$$

and since all but the third term vanish, (2.7) gives

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} -u_0(x,t,y^2,s^2)v_1(x)v_2(y_1)\nabla_{y_2}\cdot v_3(y_2)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Applying the Variational Lemma we have

$$-\int_{Y_2} u_0(x,t,y^2,s^2)\nabla_{y_2}\cdot v_3(y_2)\,\mathrm{d}y_2 = 0$$

a.e. in $\Omega_T\times\mathcal{Y}_{1,2}$, showing that u_0 is independent of y_2. Next we let $v_1\in D(\Omega)$, $v_2\in C^\infty_\sharp(Y_1)^N$, $c_1\in D(0,T)$, $c_2\in C^\infty_\sharp(S_1)$ and $c_3\in C^\infty_\sharp(S_2)$. By integration by parts and after differentiation we have that

$$\begin{aligned}\int_{\Omega_T} &\nabla u_\varepsilon(x,t)\varepsilon v_1(x)\cdot v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{\Omega_T} u_\varepsilon(x,t)\Big(\varepsilon\nabla v_1(x)\cdot v_2\Big(\frac{x}{\varepsilon}\Big)+v_1(x)\nabla_{y_1}\cdot v_2\Big(\frac{x}{\varepsilon}\Big)\Big)\\ &\qquad\times c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and as $\varepsilon\to 0$, we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -u_0(x,t,y_1,s^2)v_1(x)\nabla_{y_1}\cdot v_2(y_1)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By the Variational Lemma

$$-\int_{Y_1} u_0(x,t,y_1,s^2)\nabla_{y_1}\cdot v_2(y_1)\,\mathrm{d}y_1 = 0$$

a.e. in $\Omega_T\times S^2$, which shows that u_0 is independent of y_1. To show independence of s_2 we carry out the differentiations in (2.1) and obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} &u_\varepsilon(x,t)v(x)\Big(\varepsilon^r\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\\ &+\varepsilon^{r-q}c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)+\varepsilon^{r-r}c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)\partial_{s_2}c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

Passing to the limit we arrive at

$$\int_{\Omega_T}\int_{S^2} u_0(x,t,s^2)v(x)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0$$

and the Variational Lemma gives

$$\int_{S_2} u_0(x,t,s^2)\partial_{s_2} c_3(s_2)\,\mathrm{d}s_2 = 0$$

a.e. in $\Omega_T \times S_1$. We conclude that u_0 does not depend on the local time variable s_2. For showing independence of s_1 we carry out the differentiations in (2.2) and obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v(x)\Big(\varepsilon^q\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big) + \varepsilon^{q-q}c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon^q}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

As ε tends to zero we have

$$\int_{\Omega_T}\int_{S_1} u_0(x,t,s_1)v(x)c_1(t)\partial_{s_1}c_2(s_1)\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t = 0$$

and by the Variational Lemma

$$\int_{S_1} u_0(x,t,s_1)\partial_{s_1}c_2(s_1)\,\mathrm{d}s_1 = 0$$

a.e. in Ω_T, hence u_0 is independent of s_1. In conclusion, we have shown that

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u_0(x,t), \tag{2.9}$$

where $u_0 \in L^2(\Omega_T)$, and the last step in the characterization of u_0 is to show that $u_0 \in L^2(0,T;H_0^1(\Omega))$. Observe that (2.9) means

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} & u_\varepsilon(x,t)v\Big(x,t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} u_0(x,t)v(x,t,y^2,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $v \in L^2(\Omega_T;C_\sharp(\mathcal{Y}_{2,2}))$ and since $L^2(\Omega_T) \subset L^2(\Omega_T;C_\sharp(\mathcal{Y}_{2,2}))$, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} u_0(x,t)v(x,t)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} u_0(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $v \in L^2(\Omega_T)$. Observing that the weak convergence (2.3) implies

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(\Omega_T)$$

for the same $u \in L^2(0,T;H_0^1(\Omega))$, we see that u_0 coincides with the weak limit u, hence $u_0 \in L^2(0,T;H_0^1(\Omega))$ and the proof of (2.4) is complete.

Now we will identify τ_0. Let H denote the space of generalized divergence-free functions in $L^2(\Omega;L^2_\sharp(Y^2)^N)$ defined as

$$H = \Big\{ v \in L^2(\Omega;L^2_\sharp(Y^2)^N)\colon\ \nabla_{y_2}\cdot v(x,y^2) = 0 \text{ and } \int_{Y_2} \nabla_{y_1}\cdot v(x,y^2)\,\mathrm{d}y_2 = 0 \Big\}.$$

Using vc, where $v \in D(\Omega;C^\infty_\sharp(Y^2))^N \cap H$ and $c \in D(0,T;C^\infty_\sharp(S^2))$, as a test function in (2.8) we get, up to a subsequence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} &\nabla u_\varepsilon(x,t)\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)c\Big(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} \tau_0(x,t,y^2,s^2)\cdot v(x,y^2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for some $\tau_0 \in L^2(\Omega_T\times\mathcal{Y}_{2,2})^N$. By integration by parts on the left-hand side we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} &- u_\varepsilon(x,t)\nabla\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)c\Big(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\int_{\Omega_T} -u_\varepsilon(x,t)\Big(\nabla_x\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big) + \frac{1}{\varepsilon}\nabla_{y_1}\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big) \\ &\quad + \frac{1}{\varepsilon^2}\nabla_{y_2}\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)c\Big(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\int_{\Omega_T} -u_\varepsilon(x,t)\Big(\nabla_x\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big) \\ &\quad + \frac{1}{\varepsilon}\nabla_{y_1}\cdot v\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)c\Big(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where the last term has vanished due to the fact that $\nabla_{y_2}\cdot v = 0$. Since

$$\int_{Y_2}\nabla_{y_1}\cdot v(x,y^2)\,\mathrm{d}y_2 = 0,$$

Theorem 3.3 in [1] gives that $\{\varepsilon^{-2}\nabla_{y_1}\cdot v(x,x\varepsilon^{-1},x\varepsilon^{-2})\}$ is bounded in $H^{-1}(\Omega)$. Passing to the limit while using this boundedness yields

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} &- u(x,t)\nabla_x\cdot v(x,y^2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} \nabla u(x,t)\cdot v(x,y^2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $v \in D(\Omega; C_\sharp^\infty(Y^2))^N \cap H$ and $c \in D(0,T; C_\sharp^\infty(S^2))$. We conclude that

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} \tau_0(x,t,y^2,s^2) \cdot v(x,y^2) c(t,s^2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} \nabla u(x,t) \cdot v(x,y^2) c(t,s^2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t$$

or equivalently

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} (\tau_0(x,t,y^2,s^2) - \nabla u(x,t)) \cdot v(x,y^2) c(t,s^2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t = 0.$$

By the Variational Lemma we obtain

$$\int_{\Omega} \int_{Y^2} (\tau_0(x,t,y^2,s^2) - \nabla u(x,t)) \cdot v(x,y^2)\, \mathrm{d}y^2\, \mathrm{d}x = 0$$

a.e. in $(0,T) \times S^2$. This means that $\tau_0 - \nabla u$ belongs to the orthogonal of $D(\Omega; C_\sharp^\infty(Y^2))^N \cap H$ and by density (see property (i) of Lemma 3.7 in [1]) to the orthogonal of the whole space H. According to property (ii) of Lemma 3.7 in [1], we deduce that

$$\tau_0(x,t,y^2,s^2) - \nabla u(x,t) = \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2),$$

where $u_1 \in L^2(\Omega_T \times S^2; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$, which proves (2.6).

Now, choosing a test function $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{1,2}))$ on the left-hand side of (2.5), (2.6) gives

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \nabla u_\varepsilon(x,t) v\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2))$$
$$\times\, v(x,t,y_1,s^2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t.$$

Integrating over Y_2 while using the fact that

$$\int_{Y_2} \nabla_{y_2} u_2(x,t,y^2,s^2)\, \mathrm{d}y_2 = 0$$

we arrive at

$$\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2)) v(x,t,y_1,s^2)\, \mathrm{d}y_1\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t,$$

which proves (2.5). □

In the case of an appearance of sequences that are not bounded in any Lebesgue space, it might not be possible to obtain a multiscale limit. In [11], Holmbom introduced a concept of convergence that was improved by Nguetseng and Woukeng in [17] and further developed and named very weak multiscale convergence in [8]. The full generalization of the concept was given in e.g. [10], for which we provide the definition. This kind of convergence is crucial in the homogenization of (1.1), where unbounded sequences appear.

Definition 2.6. A sequence $\{w_\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1, m+1)$-*scale converge very weakly to* $w_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} w_\varepsilon(x,t) v_1\Big(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_{n-1}}\Big) v_2\Big(\frac{x}{\varepsilon_n}\Big) c\Big(t, \frac{t}{\varepsilon'_1}, \ldots, \frac{t}{\varepsilon'_m}\Big) \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T}\int_{\mathcal{Y}_{n,m}} w_0(x,t,y^n,s^m) v_1(x,y^{n-1}) v_2(y_n) c(t,s^m)\,\mathrm{d}y^n\,\mathrm{d}s^m\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for any $v_1 \in D(\Omega; C^\infty_\sharp(Y^{n-1}))$, $v_2 \in C^\infty_\sharp(Y_n)/\mathbb{R}$ and $c \in D(0,T; C^\infty_\sharp(S^m))$, where

$$\int_{Y_n} w_0(x,t,y^n,s^m)\,\mathrm{d}y_n = 0. \tag{2.10}$$

We write

$$w_\varepsilon(x,t) \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} w_0(x,t,y^n,s^m).$$

Remark 2.7. Due to (2.10) the limit is unique.

In earlier works, see e.g. [19] or [10], compactness results for very weak evolution multiscale convergence for $\{u_\varepsilon\}$ bounded in $W^{1,2}(0,T; H^1_0(\Omega), L^2(\Omega))$ have been established. Here, we will prove analogous results without requiring boundedness of the time derivative in $L^2(0,T; H^{-1}(\Omega))$. Note that conditions (2.11) and (2.12) are the same as (2.1) and (2.2) in Theorem 2.5.

Theorem 2.8. *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $L^2(0,T; H^1_0(\Omega))$ *and, for any* $v \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C^\infty_\sharp(S_1)$, $c_3 \in C^\infty_\sharp(S_2)$ *and* $r > q > 0$,

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t) v(x) \partial_t\Big(\varepsilon^r c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.11}$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t) v(x) \partial_t\Big(\varepsilon^q c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.12}$$

Then, with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_1' = \varepsilon^q$ and $\varepsilon_2' = \varepsilon^r$, up to a subsequence

$$\frac{1}{\varepsilon} u_\varepsilon(x,t) \overset{2,3}{\underset{vw}{\rightharpoonup}} u_1(x,t,y_1,s^2) \tag{2.13}$$

and

$$\frac{1}{\varepsilon^2} u_\varepsilon(x,t) \overset{3,3}{\underset{vw}{\rightharpoonup}} u_2(x,t,y^2,s^2), \tag{2.14}$$

where $u_1 \in L^2(\Omega_T \times S^2; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are the same as in (2.5) *and* (2.6) *in Theorem 2.5.*

P r o o f. We point out that to prove (2.13) and (2.14) means to show

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_T} \frac{1}{\varepsilon} u_\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ = \int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_1(x,t,y_1,s^2) v_1(x) v_2(y_1) c(t,s^2) \,\mathrm{d}y_1 \,\mathrm{d}s^2 \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{2.15}$$

for any $v_1 \in D(\Omega)$, $v_2 \in C^\infty_\sharp(Y_1)/\mathbb{R}$ and $c \in D(0,T; C^\infty_\sharp(S^2))$, and

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_T} \frac{1}{\varepsilon^2} u_\varepsilon(x,t) v_1\Big(x, \frac{x}{\varepsilon}\Big) v_2\Big(\frac{x}{\varepsilon^2}\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ = \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} u_2(x,t,y^2,s^2) v_1(x,y_1) v_2(y_2) c(t,s^2) \,\mathrm{d}y^2 \,\mathrm{d}s^2 \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{2.16}$$

for any $v_1 \in D(\Omega; C^\infty_\sharp(Y_1))$, $v_2 \in C^\infty_\sharp(Y_2)/\mathbb{R}$ and $c \in D(0,T; C^\infty_\sharp(S^2))$, respectively.

We start by proving (2.13). Note that any $v_2 \in C^\infty_\sharp(Y_1)/\mathbb{R}$ can be represented by

$$v_2(y_1) = \Delta_{y_1} \varrho(y_1) = \nabla_{y_1} \cdot (\nabla_{y_1} \varrho(y_1))$$

for some $\varrho \in C^\infty_\sharp(Y_1)/\mathbb{R}$. The left-hand side of (2.15) can now be expressed as

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_T} \frac{1}{\varepsilon} u_\varepsilon(x,t) v_1(x) \nabla_{y_1} \cdot \Big(\nabla_{y_1} \varrho\Big(\frac{x}{\varepsilon}\Big)\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ = \lim_{\varepsilon\to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \nabla \cdot \Big(\nabla_{y_1} \varrho\Big(\frac{x}{\varepsilon}\Big)\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ = \lim_{\varepsilon\to 0} \Big(\int_{\Omega_T} -\nabla u_\varepsilon(x,t) v_1(x) \cdot \nabla_{y_1} \varrho\Big(\frac{x}{\varepsilon}\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ - \int_{\Omega_T} u_\varepsilon(x,t) \nabla v_1(x) \cdot \nabla_{y_1} \varrho\Big(\frac{x}{\varepsilon}\Big) c\Big(t, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \Big), \end{aligned}$$

where we used antidifferentiation with respect to y_1 and integration by parts. By Theorem 2.5, as ε tends to zero we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2))v_1(x)\cdot\nabla_{y_1}\varrho(y_1)c(t,s^2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} u(x,t)\nabla v_1(x)\cdot\nabla_{y_1}\varrho(y_1)c(t,s^2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t.$$

Integration by parts in the last term with respect to x leaves us with

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -\nabla_{y_1}u_1(x,t,y_1,s^2)v_1(x)\cdot\nabla_{y_1}\varrho(y_1)c(t,s^2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t$$

and by integration by parts with respect to y_1 we arrive at

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} u_1(x,t,y_1,s^2)v_1(x)\nabla_{y_1}\cdot(\nabla_{y_1}\varrho(y_1))c(t,s^2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} u_1(x,t,y_1,s^2)v_1(x)v_2(y_1)c(t,s^2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t,$$

which proves (2.13).

We continue by proving (2.14). Observing that any $v_2\in C_\sharp^\infty(Y_2)/\mathbb{R}$ can be expressed as

$$v_2(y_2)=\Delta_{y_2}\varrho(y_2)=\nabla_{y_2}\cdot(\nabla_{y_2}\varrho(y_2))$$

for some $\varrho\in C_\sharp^\infty(Y_2)/\mathbb{R}$, following the same steps as above, the left-hand side of (2.16) can be written as

$$\lim_{\varepsilon\to 0}\biggl(\int_{\Omega_T} -\nabla u_\varepsilon(x,t)v_1\Bigl(x,\frac{x}{\varepsilon}\Bigr)\cdot\nabla_{y_2}\varrho\Bigl(\frac{x}{\varepsilon^2}\Bigr)c\Bigl(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Bigr)\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_T} u_\varepsilon(x,t)$$
$$\times\Bigl(\nabla_x v_1\Bigl(x,\frac{x}{\varepsilon}\Bigr)+\frac{1}{\varepsilon}\nabla_{y_1}v_1\Bigl(x,\frac{x}{\varepsilon}\Bigr)\Bigr)\cdot\nabla_{y_2}\varrho\Bigl(\frac{x}{\varepsilon^2}\Bigr)c\Bigl(t,\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Bigr)\,\mathrm{d}x\,\mathrm{d}t\biggr).$$

Since $\{\varepsilon^{-2}\nabla_{y_1}v_1(x,x\varepsilon^{-1})\cdot\nabla_{y_2}\varrho(x\varepsilon^{-2})\}$ is bounded in $H^{-1}(\Omega)$, the last term in the second integral vanishes as we pass to the limit, and applying Theorem 2.5, we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} -(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2)+\nabla_{y_2}u_2(x,t,y^2,s^2))$$
$$\times v_1(x,y_1)\cdot\nabla_{y_2}\varrho(y_2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} u(x,t)\nabla_x v_1(x,y_1)\cdot\nabla_{y_2}\varrho(y_2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t.$$

By observing that

$$\int_{Y_2}\nabla_{y_2}\varrho(y_2)\,\mathrm{d}y_2=0,$$

all but the last term in the first integral vanish, leaving us with

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} -\nabla_{y_2}u_2(x,t,y^2,s^2)v_1(x,y_1)\cdot\nabla_{y_2}\varrho(y_2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t$$

and integration by parts with respect to y_2 gives

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} & u_2(x,t,y^2,s^2)v_1(x,y_1)\nabla_{y_2}\cdot(\nabla_{y_2}\varrho(y_2))c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} u_2(x,t,y^2,s^2)v_1(x,y_1)v_2(y_2)c(t,s^2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which proves (2.14). □

3. Homogenization

This section is devoted to the homogenization of problem (1.1). We start by recalling the equation

$$\begin{aligned}\varepsilon^p\partial_t u_\varepsilon(x,t)-\nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\Big) &= f(x,t) && \text{in } \Omega_T,\\ u_\varepsilon(x,0) &= u_0(x) && \text{in } \Omega,\\ u_\varepsilon(x,t) &= 0 && \text{on } \partial\Omega\times(0,T),\end{aligned}\tag{3.1}$$

where $0<p<q<r$, $f\in L^2(\Omega_T)$ and $u_0\in L^2(\Omega)$. Under the assumption that the coefficient $a\in C_\sharp(\mathcal{Y}_{2,2})^{N\times N}$ satisfies the coercivity condition

$$a(y^2,s^2)\xi\cdot\xi\geqslant C_0|\xi|^2$$

for all $(y^2,s^2)\in\mathbb{R}^{2N}\times\mathbb{R}^2$, all $\xi\in\mathbb{R}^N$ and some $C_0>0$, (3.1) possesses a unique solution $u_\varepsilon\in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$ for every fixed ε, see Section 23.7 in [21]. Further, the a priori estimate

$$\|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\leqslant C_1\tag{3.2}$$

holds for some $C_1>0$ independent of ε, according to the reasoning in Section 3 in [4].

Before we are ready to give the homogenization result we show that assumptions (2.1) and (2.2) in Theorems 2.5 and 2.8 are satisfied, i.e. that for $v\in D(\Omega)$, $c_1\in D(0,T)$, $c_2\in C_\sharp^\infty(S_1)$, $c_3\in C_\sharp^\infty(S_2)$ and $r>q>0$,

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v(x)\partial_t\Big(\varepsilon^r c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t=0\tag{3.3}$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v(x)\partial_t\Big(\varepsilon^q c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\tag{3.4}$$

The weak form of (3.1) is

$$\int_{\Omega_T} -\varepsilon^p u_\varepsilon(x,t)v(x)\partial_t c(t) + a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.5}$$

where $0 < p < q < r$, for all $v \in H_0^1(\Omega)$ and $c \in D(0,T)$. Taking the test function

$$v(x)c(t) = \varepsilon^{r-p} v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)$$

with $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$ and $c_3 \in C_\sharp^\infty(S_2)$, we get, after rearranging,

$$\begin{aligned}\int_{\Omega_T} & u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^r c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} \varepsilon^{r-p} a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{\Omega_T} \varepsilon^{r-p} f(x,t)v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Passing to the limit while recalling that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, which implies boundedness of $\{\nabla u_\varepsilon\}$ in $L^2(\Omega_T)^N$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_T} & u_\varepsilon(x,t)v_1(x)\partial_t\Big(\varepsilon^r c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_T} \varepsilon^{r-p} a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{\Omega_T} \varepsilon^{r-p} f(x,t)v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big) = 0\end{aligned}$$

and (3.3) is fulfilled. Following the same steps again but taking the test function

$$v(x)c(t) = \varepsilon^{q-p} v_1(x)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big),$$

where $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S_1)$, in the weak form (3.5), yields that (3.4) is fulfilled.

We are now prepared to prove the homogenization result. Depending on the choices of p, q and r ($0 < p < q < r$) in (3.1), we get different outcomes. In Theorem 3.1 we present 13 possible cases arising from different combinations of p, q and r. Here we will see that the local problems are parabolic when the matching between the microscopic scales that give resonance is shifted by p compared to the standard case (cf. Section 5.3.1 in [19]). This means that resonance appears when the temporal scale multiplied by ε^{-p} is the square of a spatial scale.

Theorem 3.1. *Let $\{u_\varepsilon\}$ be a sequence of solutions to* (3.1) *in $W^{1,2}(0,T;H_0^1(\Omega), L^2(\Omega))$. Then it holds that*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{3.6}$$

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u(x,t), \tag{3.7}$$

$$\nabla u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2), \tag{3.8}$$

where $u \in L^2(0,T;H_0^1(\Omega))$ is the unique solution to the homogenized problem

$$\begin{aligned} -\nabla \cdot (b\nabla u(x,t)) &= f(x,t) \quad \text{in } \Omega_T, \\ u(x,t) &= 0 \qquad\quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{3.9}$$

where the coefficient b is characterized by the formulas below. For all 13 *cases we assume that $0 < p < q < r$.*

(1) *Letting $r < 2 + p$, the homogenized coefficient is given by*

$$\begin{aligned} b\nabla u(x,t) = \int_{\mathcal{Y}_{2,2}} & a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) \\ & + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y^2\,\mathrm{d}s^2, \end{aligned} \tag{3.10}$$

and $u_1 \in L^2(\Omega_T \times S^2; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$ are given by the local problems

$$-\nabla_{y_2} \cdot (a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2))) = 0, \tag{3.11}$$

$$-\nabla_{y_1} \cdot \int_{Y_2} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y_2 = 0. \tag{3.12}$$

(2) *Choosing $r = 2 + p$, the coefficient b is determined by* (3.10) *while $u_1 \in L^2(\Omega_T \times S_1; \mathcal{W}_{1,2})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$ are the solutions to the local problems*

$$-\nabla_{y_2} \cdot (a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2))) = 0, \tag{3.13}$$

$$\begin{aligned} \partial_{s_2} u_1(x,t,y_1,s^2) - \nabla_{y_1} \cdot \int_{Y_2} & a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) \\ & + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y_2 = 0. \end{aligned} \tag{3.14}$$

(3) *If* $2+p<r<4+p$ *while* $q<2+p$, *we have*

$$\text{(3.15)}\qquad b\nabla u(x,t)=\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1) + \nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y^2\,\mathrm{d}s^2,$$

where $u_1\in L^2(\Omega_T\times S_1;H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2\in L^2(\Omega_T\times\mathcal{Y}_{1,2};H^1_\sharp(Y_2)/\mathbb{R})$ *are given by the system*

$$\text{(3.16)}\qquad -\nabla_{y_2}\cdot(a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)))=0,$$

$$\text{(3.17)}\qquad -\nabla_{y_1}\cdot\int_{Y_2\times S_2} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1) + \nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s_2=0.$$

(4) *Taking* $2+p<r<4+p$ *and* $q=2+p$, *the homogenized coefficient is given by* (3.15) *and* $u_1\in L^2(\Omega_T;\mathcal{W}_{1,1})$ *and* $u_2\in L^2(\Omega_T\times\mathcal{Y}_{1,2};H^1_\sharp(Y_2)/\mathbb{R})$ *are determined by*

$$\text{(3.18)}\qquad -\nabla_{y_2}\cdot(a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)))=0,$$

$$\text{(3.19)}\qquad \partial_{s_1}u_1(x,t,y_1,s_1)-\nabla_{y_1}\cdot\int_{Y_2\times S_2} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1) +\nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s_2=0.$$

(5) *When* $q<r<4+p$ *and* $q>2+p$, *the coefficient* b *is determined by*

$$\text{(3.20)}\qquad b\nabla u(x,t)=\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1) + \nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y^2\,\mathrm{d}s^2$$

and the local problems are

$$\text{(3.21)}\qquad -\nabla_{y_2}\cdot(a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)))=0,$$

$$\text{(3.22)}\qquad -\nabla_{y_1}\cdot\int_{Y_2\times S^2} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1) + \nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s^2=0,$$

where $u_1\in L^2(\Omega_T;H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2\in L^2(\Omega_T\times\mathcal{Y}_{1,2};H^1_\sharp(Y_2)/\mathbb{R})$.

(6) *In the case when* $r = 4 + p$ *while* $q < 2 + p$, *the homogenized coefficient is characterized by* (3.15) *while* $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$ *are given by the system of local problems*

$$\partial_{s_2} u_2(x,t,y^2,s^2) - \nabla_{y_2} \cdot (a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))) = 0, \tag{3.23}$$

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S_2} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s_2 = 0. \tag{3.24}$$

(7) *When* $r = 4 + p$ *and* $q = 2 + p$, *the coefficient* b *is given by* (3.15), *where* $u_1 \in L^2(\Omega_T; \mathcal{W}_{1,1})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$ *are the solutions to*

$$\partial_{s_2} u_2(x,t,y^2,s^2) - \nabla_{y_2} \cdot (a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))) = 0, \tag{3.25}$$

$$\partial_{s_1} u_1(x,t,y_1,s_1) - \nabla_{y_1} \cdot \int_{Y_2 \times S_2} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s_2 = 0. \tag{3.26}$$

(8) *Letting* $r = 4 + p$ *while* $q > 2 + p$ *gives us the homogenized coefficient* (3.20) *defined by the system of local problems*

$$\partial_{s_2} u_2(x,t,y^2,s^2) - \nabla_{y_2} \cdot (a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))) = 0, \tag{3.27}$$

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S^2} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s^2))\,\mathrm{d}y_2\,\mathrm{d}s^2 = 0, \tag{3.28}$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$.

(9) *Choosing* $r > 4 + p$ *and* $q < 2 + p$, *we have the homogenized coefficient*

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{2,1}} \left(\int_{S_2} a(y^2,s^2)\,\mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\,\mathrm{d}y^2\,\mathrm{d}s_1, \tag{3.29}$$

where $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$ *are the solutions to the local problems*

$$-\nabla_{y_2} \cdot \left(\left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s_1)) \right) = 0, \tag{3.30}$$

$$-\nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\, \mathrm{d}y_2 = 0. \tag{3.31}$$

(10) *When* $r > 4 + p$ *while* $q = 2 + p$, *the homogenized coefficient is given by* (3.29) *and the local problems are*

$$-\nabla_{y_2} \cdot \left(\left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s_1)) \right) = 0, \tag{3.32}$$

$$\partial_{s_1} u_1(x,t,y_1,s_1) - \nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\, \mathrm{d}y_2 = 0 \tag{3.33}$$

with $u_1 \in L^2(\Omega_T; \mathcal{W}_{1,1})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$.

(11) *When* $r > 4 + p$ *and* $2 + p < q < 4 + p$, *we have*

$$b \nabla u(x,t) = \int_{\mathcal{Y}_{2,1}} \left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\, \mathrm{d}y^2\, \mathrm{d}s_1 \tag{3.34}$$

together with the local problems

$$-\nabla_{y_2} \cdot \left(\left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1)) \right) = 0, \tag{3.35}$$

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S_1} \left(\int_{S_2} a(y^2, s^2)\, \mathrm{d}s_2 \right) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\, \mathrm{d}y_2\, \mathrm{d}s_1 = 0, \tag{3.36}$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$.

(12) *Taking $q = 4 + p$, the coefficient in the homogenized problem is given by* (3.34) *and $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times Y_1; \mathcal{W}_{2,1})$ are determined by*

$$\partial_{s_1} u_2(x,t,y^2,s_1) - \nabla_{y_2} \cdot \biggl(\biggl(\int_{S_2} a(y^2,s^2)\,\mathrm{d}s_2\biggr) \times(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\biggr) = 0, \tag{3.37}$$

$$-\nabla_{y_1} \cdot \int_{Y_2\times S_1} \biggl(\int_{S_2} a(y^2,s^2)\,\mathrm{d}s_2\biggr) \times(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1))\,\mathrm{d}y_2\,\mathrm{d}s_1 = 0. \tag{3.38}$$

(13) *In the case when $q > 4 + p$, the coefficient is characterized by*

$$b\nabla u(x,t) = \int_{Y^2} \biggl(\int_{S^2} a(y^2,s^2)\,\mathrm{d}s^2\biggr)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2))\,\mathrm{d}y^2$$

and the local problems are given by

$$-\nabla_{y_2} \cdot \biggl(\biggl(\int_{S^2} a(y^2,s^2)\,\mathrm{d}s^2\biggr) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2))\biggr) = 0, \tag{3.39}$$

$$-\nabla_{y_1} \cdot \int_{Y_2} \biggl(\int_{S^2} a(y^2,s^2)\,\mathrm{d}s^2\biggr) \times (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2))\,\mathrm{d}y_2 = 0, \tag{3.40}$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times Y_1; H^1_\sharp(Y_2)/\mathbb{R})$.

Proof. Since $\{u_\varepsilon\}$ satisfies the a priori estimate (3.2) and conditions (3.3) and (3.4), Theorem 2.5 gives us (3.6), (3.7) and (3.8). The continuation of this proof will be divided into three parts. We start by finding the homogenized problem (3.9) followed by proving independencies of local time variables and determining the local problems, which together will provide us with the characterizations of the homogenized coefficient for all 13 cases.

Taking the test function

$$v(x)c(t) = v_1(x)c_1(t),$$

where $v_1 \in H^1_0(\Omega)$ and $c_1 \in D(0,T)$, in the weak form (3.5) and letting ε tend to zero, Theorem 2.5 yields

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s^2) + \nabla_{y_2} u_2(x,t,y^2,s^2)) \cdot\nabla v_1(x)c_1(t)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)v_1(x)c_1(t)\,\mathrm{d}x\,\mathrm{d}t.$$

By the Variational Lemma we arrive at

$$\int_\Omega\Big(\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2)+\nabla_{y_2}u_2(x,t,y^2,s^2))\,\mathrm{d}y^2\,\mathrm{d}s^2\Big) \cdot\nabla v_1(x)\,\mathrm{d}x = \int_\Omega f(x,t)v_1(x)\,\mathrm{d}x \tag{3.41}$$

a.e. in $(0,T)$, which is the weak form of (3.9).

We start by deriving a common ground, divided into two paths, for the reasoning about independencies and the local problems. For the first path, in the weak form (3.5), we choose a test function which captures the oscillations from the second microscopic scale $\varepsilon_2=\varepsilon^2$, more precisely, we choose

$$v(x)c(t)=\varepsilon^k v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big), \tag{3.42}$$

where $k>0$, $v_1\in D(\Omega)$, $v_2\in C^\infty_\sharp(Y_1)$, $v_3\in C^\infty_\sharp(Y_2)/\mathbb{R}$, $c_1\in D(0,T)$, $c_2\in C^\infty_\sharp(S_1)$ and $c_3\in C^\infty_\sharp(S_2)$. After differentiations we arrive at

$$\begin{aligned}\int_{\Omega_T} &-u_\varepsilon(x,t)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big(\varepsilon^{k+p}\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\\
&+\varepsilon^{k+p-q}c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)+\varepsilon^{k+p-r}c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)\partial_{s_2}c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\\
&+a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\cdot\Big(\varepsilon^k\nabla v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)\\
&+\varepsilon^{k-1}v_1(x)\nabla_{y_1}v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)+\varepsilon^{k-2}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\nabla_{y_2}v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big)\\
&\times c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
=\int_{\Omega_T}&f(x,t)\varepsilon^k v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Passing to the limit, omitting terms that obviously tend to zero, we have

$$\begin{aligned}\lim_{\varepsilon\to0}\Big(\int_{\Omega_T} &-u_\varepsilon(x,t)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big(\varepsilon^{k+p-q}c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\\
&+\varepsilon^{k+p-r}c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)\partial_{s_2}c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big)\\
&+a\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\frac{t}{\varepsilon^q},\frac{t}{\varepsilon^r}\Big)\nabla u_\varepsilon(x,t)\cdot\Big(\varepsilon^{k-1}v_1(x)\nabla_{y_1}v_2\Big(\frac{x}{\varepsilon}\Big)v_3\Big(\frac{x}{\varepsilon^2}\Big)\\
&+\varepsilon^{k-2}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\nabla_{y_2}v_3\Big(\frac{x}{\varepsilon^2}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^q}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big)=0.\end{aligned} \tag{3.43}$$

For the second path, i.e. the one with respect to the first spatial microscopic scale $\varepsilon_1 = \varepsilon$, we let

$$v(x)c(t) = \varepsilon^k v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big), \tag{3.44}$$

where $k > 0$, $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y_1)/\mathbb{R}$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$ and $c_3 \in C_\sharp^\infty(S_2)$, act as a test function in the weak form (3.5). Differentiating leads to

$$\begin{aligned}
&\int_{\Omega_T} - u_\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^{k+p}\partial_t c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) \\
&\quad + \varepsilon^{k+p-q} c_1(t) \partial_{s_1} c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) + \varepsilon^{k+p-r} c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) \partial_{s_2} c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big) \\
&\quad + a\Big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \nabla u_\varepsilon(x,t) \cdot \Big(\varepsilon^k \nabla v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) + \varepsilon^{k-1} v_1\Big(x\Big) \nabla_{y_1} v_2\Big(\frac{x}{\varepsilon}\Big)\Big) \\
&\quad \times c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} f(x,t) \varepsilon^k v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

and as $\varepsilon \to 0$, after omitting terms that vanish, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\Big(&\int_{\Omega_T} - u_\varepsilon(x,t) v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^{k+p-q} c_1(t) \partial_{s_1} c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) \\
&\quad + \varepsilon^{k+p-r} c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) \partial_{s_2} c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big) \\
&\quad + a\Big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^q}, \frac{t}{\varepsilon^r}\Big) \nabla u_\varepsilon(x,t) \cdot \varepsilon^{k-1} v_1(x) \nabla_{y_1} v_2\Big(\frac{x}{\varepsilon}\Big) \\
&\quad \times c_1(t) c_2\Big(\frac{t}{\varepsilon^q}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big) = 0.
\end{aligned} \tag{3.45}$$

Now we are ready to prove the independencies of local time variables and we start by showing when u_2 is independent of s_2. Let $r > 4 + p$ and choose $k = r - p - 2$ in (3.42). As $\varepsilon \to 0$, applying Theorems 2.5 and 2.8, the limit of (3.43) becomes

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} - u_2(x,t,y^2,s^2) v_1(x) v_2(y_1) v_3(y_2) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0$$

and by the Variational Lemma

$$\int_{S_2} -u_2(x,t,y^2,s^2) \partial_{s_2} c_3(s_2)\,\mathrm{d}s_2 = 0$$

a.e. in $\Omega_T \times \mathcal{Y}_{2,1}$, which indicates that u_2 is independent of s_2.

Now we show independence of s_1 in u_2. Let $q > 4 + p$ and since $r > q$, this implies that u_2 is independent of s_2. Therefore we let $c_3 \equiv 1$ in (3.42) and we choose $k = q - p - 2$. Passing to the limit in (3.43), Theorems 2.5 and 2.8 yield

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} -u_2(x,t,y^2,s_1)v_1(x)v_2(y_1)v_3(y_2)c_1(t)\partial_{s_1}c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0$$

and integrating over S_2 and applying the Variational Lemma on $\Omega_T \times Y^2$, we obtain that u_2 is independent of s_1.

Next we show independence of s_2 in u_1. Let $r > 2 + p$ and choose $k = r - p - 1$ in (3.44). Letting ε tend to zero in (3.45), applying Theorems 2.5 and 2.8, we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -u_1(x,t,y_1,s^2)v_1(x)v_2(y_1)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0$$

and the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,1}$ shows that u_1 is independent of s_2.

The last independence to show is when u_1 is independent of s_1. Here we let $q > 2 + p$ and recall that since $r > q$, u_1 is independent of s_2. In (3.44) we choose $k = q - p - 1$ and set $c_3 \equiv 1$. As $\varepsilon \to 0$ in (3.45), Theorems 2.5 and 2.8 give

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -u_1(x,t,y_1,s_1)v_1(x)v_2(y_1)c_1(t)\partial_{s_1}c_2(s_1)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Integrating over S_2 and using the Variational Lemma on $\Omega_T \times Y_1$ we have that u_1 is independent of s_1.

To sum up, we know that u_1 is independent of s_2 whenever $r > 2 + p$ and that u_1 is independent of both s_1 and s_2 when $q > 2+p$. In the case when $r > 4+p$, u_2 (and of course also u_1) is independent of s_2 and if $q > 4 + p$, we have that u_2 (and u_1) is independent of both s_1 and s_2. These independencies together with (3.41) give the formulas for the homogenized coefficient in the cases (1)–(13).

Now we are going to derive the system of local problems for each of the 13 cases. Each case has a system consisting of two local problems. The first local problem is with respect to the faster microscopic scale $\varepsilon_2 = \varepsilon^2$ and our point of departure is always (3.43), where we have chosen $k = 2$ in (3.42). The second local problem is with respect to the slower microscopic scale $\varepsilon_1 = \varepsilon$ and the point of departure here is (3.45), where we have taken $k = 1$ in (3.44).

Case (1): $r < 2 + p$. To obtain the first local problem we let $\varepsilon \to 0$ in (3.43) and applying Theorem 2.5 we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t) + \nabla_{y_1}u_1(x,t,y_1,s^2) + \nabla_{y_2}u_2(x,t,y^2,s^2))$$
$$\times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,2}$, we obtain the weak form of (3.11).

For the second local problem, passing to the limit in (3.45), using Theorems 2.5 and 2.8, we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t=0$$

and the Variational Lemma on $\Omega_T \times S^2$ gives us the weak form of (3.12).

Case (2): $r = 2 + p$. Passing to the limit in (3.43) yields the same result as for the first local problem in Case (1), which is the weak form of (3.13).

For the second local problem, we apply Theorems 2.5 and 2.8 as we pass to the limit in (3.45) to get

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -u_1(x,t,y_1,s^2)v_1(x)v_2(y_1)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ +\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

Using the Variational Lemma on $\Omega_T \times S_1$, we get the weak form of (3.14).

Case (3): $2 + p < r < 4 + p$ and $q < 2 + p$. Passing to the limit in (3.43) and applying Theorems 2.5 and 2.8, recalling that u_1 is independent of s_2, we arrive at

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ \times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

Applying the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,2}$ we have the weak form of (3.16).

Because of the independence of s_2 in u_1, we can let $c_3 \equiv 1$ in (3.44). As $\varepsilon \to 0$ in (3.45), by Theorems 2.5 and 2.8 we obtain

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t=0$$

and the Variational Lemma on $\Omega_T \times S_1$ gives the weak form of (3.17).

Case (4): $2 + p < r < 4 + p$ and $q = 2 + p$. Passing to the limit in (3.43), remembering that u_1 is independent of s_2, by Theorems 2.5 and 2.8 we arrive at the same local problem as the first one in Case (3), which is the weak form of (3.18).

Letting $c_3 \equiv 1$ in (3.44) and passing to the limit in (3.45), applying Theorems 2.5 and 2.8, we get

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} & -u_1(x,t,y_1,s_1)v_1(x)v_2(y_1)c_1(t)\partial_{s_1}c_2(s_1)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ & \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

Integrating over S_2 in the first integral and applying the Variational Lemma on Ω_T we get the weak form of (3.19).

Case (5): $q < r < 4+p$ and $q > 2+p$. Remembering that u_1 is independent of both s_1 and s_2, when $\varepsilon \to 0$ in (3.43), we apply Theorems 2.5 and 2.8 and have

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} & a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ & \times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

By using the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,2}$ we arrive at the weak form of (3.21).

Because of the independencies, we can let $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.44). Applying Theorem 2.5 as ε tends to zero in (3.45) yields

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} & a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ & \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0\end{aligned}$$

and by the Variational Lemma on Ω_T we get the weak form of (3.22).

Case (6): $r = 4+p$ and $q < 2+p$. Noting that u_1 is independent of s_2, passing to the limit in (3.43), Theorems 2.5 and 2.8 give us

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} & -u_2(x,t,y^2,s^2)v_1(x)v_2(y_1)v_3(y_2)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s^2)) \\ & \times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

Applying the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,1}$ we have the weak form of (3.23).

Because of the independence in u_1, we can let $c_3 \equiv 1$ in (3.44) and as $\varepsilon \to 0$, (3.45) becomes, due to Theorems 2.5 and 2.8,

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2)) \\ \times v_1(x) \cdot \nabla_{y_1} v_2(y_1) c_1(t) c_2(s_1)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Using the Variational Lemma on $\Omega_T \times S_1$ we obtain the weak form of (3.24).

Case (7): $r = 4 + p$ and $q = 2 + p$. As $\varepsilon \to 0$ in (3.43), we end up with the same local problem as the first one in Case (6), which is the weak form of (3.25).

Letting ε tend to zero in (3.45), recalling that u_1 is independent of s_2 so that $c_3 \equiv 1$, Theorems 2.5 and 2.8 yield

$$\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} - u_1(x,t,y_1,s_1) v_1(x) v_2(y_1) c_1(t) \partial_{s_1} c_2(s_1)\, \mathrm{d}y_1\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t \\ + \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s_1) + \nabla_{y_2} u_2(x,t,y^2,s^2)) \\ \times v_1(x) \cdot \nabla_{y_1} v_2(y_1) c_1(t) c_2(s_1)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Integrating over S_2 in the first integral and taking the Variational Lemma on Ω_T gives us the weak form of (3.26).

Case (8): $r = 4 + p$ and $q > 2 + p$. Letting ε tend to zero in (3.43), observing that u_1 is independent of both s_1 and s_2, Theorems 2.5 and 2.8 give

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} - u_2(x,t,y^2,s^2) v_1(x) v_2(y_1) v_3(y_2) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t \\ + \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s^2)) \\ \times v_1(x) v_2(y_1) \cdot \nabla_{y_2} v_3(y_2) c_1(t) c_2(s_1) c_3(s_2)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t = 0$$

and by applying the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,1}$ we get the weak form of (3.27).

For the second local problem, due to independencies in u_1, we can let both $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.44). Letting $\varepsilon \to 0$ in (3.45), from Theorem 2.5 we obtain

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s^2)) \\ \times v_1(x) \cdot \nabla_{y_1} v_2(y_1) c_1(t)\, \mathrm{d}y^2\, \mathrm{d}s^2\, \mathrm{d}x\, \mathrm{d}t = 0$$

and the Variational Lemma on Ω_T gives us the weak form of (3.28).

Case (9): $r > 4+p$ and $q < 2+p$. Recalling that u_2 (and u_1) is independent of s_2, we can let $c_3 \equiv 1$ in (3.42). Passing to the limit in (3.43), Theorem 2.5 gives us

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s_1)) \\ \times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0$$

and using the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,1}$ we obtain the weak form of (3.30).

Due to the independence in u_1 we can let $c_3 \equiv 1$ in (3.44) and as $\varepsilon \to 0$ in (3.45), Theorems 2.5 and 2.8 yield

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s_1)) \\ \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

By the Variational Lemma on $\Omega_T \times S_1$ we have the weak form of (3.31).

Case (10): $r > 4+p$ and $q = 2+p$. Because of the independence of s_2 in u_2 we let $c_3 \equiv 1$ in (3.42) and as ε tends to zero in (3.43), recalling that also u_1 is independent of s_2, Theorems 2.5 and 2.8 give the same first local problem as in Case (9), which is the weak form of (3.32).

Again we can let $c_3 \equiv 1$ in (3.44), due to independence in u_1. Letting $\varepsilon \to 0$ in (3.45), from Theorems 2.5 and 2.8 we have

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} -u_1(x,t,y_1,s_1)v_1(x)v_2(y_1)c_1(t)\partial_{s_1}c_2(s_1)\,\mathrm{d}y_1\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t \\ +\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s_1)+\nabla_{y_2}u_2(x,t,y^2,s_1)) \\ \times v_1(x)\cdot\nabla_{y_1}v_2(y_1)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Integrating over S_2 in the first integral and using the Variational Lemma on Ω_T we get the weak form of (3.33).

Case (11): $r > 4+p$ and $2+p < q < 4+p$. Since u_2 is independent of s_2, we let $c_3 \equiv 1$ in (3.42). We also have independence of s_1 and s_2 in u_1, so as $\varepsilon \to 0$ in (3.43), Theorems 2.5 and 2.8 give

$$\int_{\Omega_T}\int_{\mathcal{Y}_{2,2}} a(y^2,s^2)(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1)+\nabla_{y_2}u_2(x,t,y^2,s_1)) \\ \times v_1(x)v_2(y_1)\cdot\nabla_{y_2}v_3(y_2)c_1(t)c_2(s_1)\,\mathrm{d}y^2\,\mathrm{d}s^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Applying the Variational Lemma on $\Omega_T \times \mathcal{Y}_{1,1}$ we get the weak form of (3.35).

Because of the independencies in u_1, for the second local problem, we can let both $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.44). Passing to the limit in (3.45), applying Theorem 2.5, we end up with

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1)) \\ \times v_1(x) \cdot \nabla_{y_1} v_2(y_1) c_1(t) \, \mathrm{d}y^2 \, \mathrm{d}s^2 \, \mathrm{d}x \, \mathrm{d}t = 0$$

and from the Variational Lemma on Ω_T we obtain the weak form of (3.36).

Case (12): $q = 4 + p$. Since u_2 is independent of s_2, we can take $c_3 \equiv 1$ in (3.42). Recalling that u_1 is independent of s_1 and s_2, passing to the limit in (3.43), from Theorems 2.5 and 2.8 we have

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} - u_2(x,t,y^2,s_1) v_1(x) v_2(y_1) v_3(y_2) c_1(t) \partial_{s_1} c_2(s_1) \, \mathrm{d}y^2 \, \mathrm{d}s^2 \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2,s_1)) \\ \times v_1(x) v_2(y_1) \cdot \nabla_{y_2} v_3(y_2) c_1(t) c_2(s_1) \, \mathrm{d}y^2 \, \mathrm{d}s^2 \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Integrating over S_2 in the first integral and applying the Variational Lemma on $\Omega_T \times Y_1$ we have the weak form of (3.37).

Because of the independencies in u_1 we can let $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.44) and as ε tends to zero in (3.45), we get the same result as for the second local problem in Case (11), sharing the weak form of (3.38).

Case (13): $q > 4 + p$. Recalling that u_2 is independent of s_1 and s_2, we can set $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.42). Noting that also u_1 is independent of both s_1 and s_2, letting $\varepsilon \to 0$ in (3.43), Theorem 2.5 yields

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2)) \\ \times v_1(x) v_2(y_1) \cdot \nabla_{y_2} v_3(y_2) c_1(t) \, \mathrm{d}y^2 \, \mathrm{d}s^2 \, \mathrm{d}x \, \mathrm{d}t = 0$$

and applying the Variational Lemma on $\Omega_T \times Y_1$ gives the weak form of (3.39).

For the second local problem, we again let $c_2 \equiv 1$ and $c_3 \equiv 1$ in (3.44) and as $\varepsilon \to 0$ in (3.45), Theorem 2.5 gives

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,2}} a(y^2, s^2)(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1) + \nabla_{y_2} u_2(x,t,y^2)) \\ \times v_1(x) \cdot \nabla_{y_1} v_2(y_1) c_1(t) \, \mathrm{d}y^2 \, \mathrm{d}s^2 \, \mathrm{d}x \, \mathrm{d}t = 0.$$

From the Variational Lemma on Ω_T we get the weak form of (3.40). □

R e m a r k 3.2. Since we are treating linear problems, it is possible to write the local problems and the homogenized coefficient explicitly. We demonstrate this for Case (1).

Following the approach in [1] we let

$$u_1(x,t,y_1,s^2) = z(y_1,s^2)\cdot\nabla u(x,t)$$

and

$$\begin{aligned} u_2(x,t,y^2,s^2) &= w(y^2,s^2)\cdot(\nabla u(x,t)+\nabla_{y_1}u_1(x,t,y_1,s^2)) \\ &= w(y^2,s^2)\cdot((I+\nabla_{y_1}z(y_1,s^2))\nabla u(x,t)), \end{aligned}$$

where $z\in L^\infty(S^2;H^1_\sharp(Y_1)/\mathbb{R})^N$ and $w\in L^\infty(\mathcal{Y}_{1,2};H^1_\sharp(Y_2)/\mathbb{R})^N$. Here I denotes the $N\times N$ identity matrix and $\nabla_{y_2}w$ and $\nabla_{y_1}z$ are the transposed $N\times N$ Jacobians $(\partial w_j/\partial y_{2,i})_{i,j}$ and $(\partial z_j/\partial y_{1,i})_{i,j}$, respectively. The local problems (3.11) and (3.12) can then be expressed as

$$-\nabla_{y_2}\cdot(a(y^2,s^2)(I+\nabla_{y_2}w(y^2,s^2)))=0$$

and

$$-\nabla_{y_1}\cdot\Big(\Big(\int_{Y_2}a(y^2,s^2)(I+\nabla_{y_2}w(y^2,s^2))\,\mathrm{d}y_2\Big)(I+\nabla_{y_1}z(y_1,s^2))\Big)=0,$$

respectively. The homogenized coefficient (3.10) then takes the form

$$b=\int_{\mathcal{Y}_{1,2}}\Big(\int_{Y_2}a(y^2,s^2)(I+\nabla_{y_2}w(y^2,s^2))\,\mathrm{d}y_2\Big)(I+\nabla_{y_1}z(y_1,s^2))\,\mathrm{d}y_1\,\mathrm{d}s^2.$$

Choosing appropriate function spaces for z and w and taking the independencies of local time variables in u_1 and u_2 into account, the procedure for Case (2)–(13) is analogous.

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