N.G. Abdujabborov; I.A. Karimjanov and M.A. Kodirova

Rota-type operators on 3-dimensional nilpotent associative algebras

Communications in Mathematics, Vol. 29 (2021), No. 2, 227–241

Persistent URL: http://dml.cz/dmlcz/149191

Communications in Mathematics 29 (2021) 227–241
DOI: 10.2478/cm-2021-0020

Rota-type operators on 3-dimensional nilpotent associative algebras

N.G. Abdujabborov, I.A. Karimjanov and M.A. Kodirova

Abstract. We give the description of Rota–Baxter operators, Reynolds operators, Nijenhuis operators and average operators on 3-dimensional nilpotent associative algebras over $\mathbb{C}$.

1 Introduction

Rota-Baxter algebra appeared from the probability theory and has found applications in many fields of mathematics and physics, such as number theory, quasi-symmetric functions, Lie algebras, and Yang-Baxter equations. Rota-Baxter operators were defined by Baxter to solve an analytic formula in probability [2],[7], [9], [10]. It has been related to other areas in mathematics and mathematical physics [1], [3], [5], [15], [21]

A Rota-Baxter operator on an associative algebra A over a field F is defined to be a linear map $P : A \to A$ satisfying

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \qquad \forall x, y \in A, \quad \lambda \in F. \tag{1}$$

Note that, if P is a Rota-Baxter operator of weight $\lambda \neq 0$, then $\lambda^{-1}P$ is a Rota-Baxter operator of weight 1. Therefore, it is sufficient to consider Rota-Baxter operators of weight 0 and 1.

At the moment there are descriptions of all Rota–Baxter operators on 3-dimensional simple Lie algebra[19], [20], on 4-dimensional simple associative algebra [21] and

2020 Msc: 16W20; 16S50
Key words: Rota–Baxter operator; Reynolds operator; Nijenhuis operator; average operator; nilpotent; associative algebras
Affiliation:
N.G. Abdujabborov – Andijan Machine-Building Instiute, Uzbekistan
E-mail: nurulhaqabdujabborov@mail.ru
I.A. Karimjanov – Andijan State University, Andijan and Institute of Mathematics, Namangan, Uzbekistan
E-mail: iqboli@gmail.com
M.A. Kodirova – Andijan State University, Uzbekistan
E-mail: mahfuza.kodirova@inbox.ru

some other algebras [4], [11], [16]. The study of some particular cases of Rota–Baxter operators was initiated in [22], [23] and [8]. Recently, in [13] the authors presented all homogeneous Rota-type operators on null-filiform associative algebras.

Here we give also the definition of operators which we consider in the present paper.

- Reynolds operator: $P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))$
- Nijenhuis operator: $P(x)P(y) = P(xP(y) + P(x)y - P(xy))$
- average operator: $P(x)P(y) = P(xP(y))$

Low dimensional associative algebras have been classified in several works. Hazlett classified nilpotent algebras of dimension less than or equal to 4 over the complex numbers [12]. Afterwards, Kruse and Price classified nilpotent associative algebras of dimension less than or equal to 4 over any field [14]. Mazzola published his results on associative unitary algebras of dimension 5 over algebraically closed fields of characteristic different from 2, and on nilpotent commutative associative algebras of dimension less than or equal to 5 over algebraically closed fields of characteristic different from 2,3 [17], [18].

Throughout this paper algebras are considered over the field of complex numbers.

Theorem 1. *[6] Any three-dimensional complex nilpotent associative algebra A is isomorphic to one of the following pairwise non-isomorphic algebras with a basis $\{e_1, e_2, e_3\}$:*

$$\begin{aligned}
&A_1 : e_1e_2 = e_2e_1 = e_3,\\
&A_2 : e_1^2 = e_2, e_1e_2 = e_2e_1 = e_3,\\
&A_3 : e_1e_2 = -e_2e_1 = e_3,\\
&A_4^{\alpha} : e_1^2 = e_3, e_2^2 = \alpha e_3, e_1e_2 = e_3, \quad \text{with } \alpha \in \mathbb{C},\\
&A_5 : e_1e_1 = e_2.
\end{aligned}$$

and the omitted products vanish.

Now let P be a linear operator on A such that

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

2 Main result

2.1 Rota–Baxter operator

Theorem 2. *There are five types of Rota-Baxter operators of weight 0 for the 3-dimensional associative algebra A_1, which are as follows:*

$$P_1 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix} \qquad \text{where } a_{22} \neq -a_{11}$$

$$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{where } a_{21} \neq 0$$

$$P_4 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{where } a_{12} \neq 0$$

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix} \qquad \text{where } a_{11}a_{12} \neq 0$$

Proof. Since P is a linear operator, we only need to consider the base elements. We have the equations:

$$\begin{aligned} a_{31} = a_{32} &= 0, \\ a_{12}(a_{33} - a_{11}) &= 0, \\ a_{21}(a_{33} - a_{22}) &= 0, \\ a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) &= 0 \end{aligned} \tag{2}$$

from

$$\begin{aligned} &P(e_2)P(e_3) = P(e_2P(e_3) + P(e_2)e_3) \Rightarrow a_{31} = 0 \\ &P(e_1)P(e_3) = P(e_1P(e_3) + P(e_1)e_3) \Rightarrow a_{32} = 0 \\ &P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1) \Rightarrow a_{12}(a_{33} - a_{11}) = 0 \\ &P(e_2)P(e_2) = P(e_2P(e_2) + P(e_2)e_2) \Rightarrow a_{21}(a_{33} - a_{22}) = 0 \\ &P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2) \Rightarrow a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) = 0. \end{aligned}$$

The missing cases $P(e_3)P(e_1)$, $P(e_3)P(e_3)$ and etc., lead to equations equivalent to the ones in the system (2).

Now we consider all possible cases:

Case 1. If $a_{12} = 0$, then the system of equations (2) becomes

$$\begin{aligned} a_{31} = a_{32} = a_{12} &= 0, \\ a_{21}(a_{33} - a_{22}) &= 0, \\ a_{11}a_{22} - a_{33}(a_{11} + a_{22}) &= 0. \end{aligned} \tag{3}$$

Case 1.1. • If $a_{21} = 0$ and $a_{22} \neq -a_{11}$, then we obtain

$$P_1 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix}, \quad a_{22} \neq -a_{11}$$

- If $a_{21} = 0$ and $a_{22} = -a_{11}$, then from system of equation (3) we get $a_{22} = a_{11} = 0$ and we have

$$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Case 1.2. If $a_{21} \neq 0$, then from system (3) we derive $a_{33} = a_{22} = 0$ and

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{21} \neq 0.$$

Case 2. If $a_{12} \neq 0$, then system (2) becomes

$$\begin{aligned} a_{31} = a_{32} = 0, &\quad a_{12} \neq 0, \\ a_{33} = a_{11}&, \\ a_{21}(a_{11} - a_{22}) = 0&, \\ a_{12}a_{21} - a_{11}^2 = 0&. \end{aligned} \tag{4}$$

Case 2.1. If $a_{21} = 0$, then we get $a_{33} = a_{11} = 0$ and deduce

$$P_4 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{12} \neq 0.$$

Case 2.2. If $a_{21} \neq 0$, then system of equations (4) becomes

$$\begin{aligned} a_{31} = a_{32} = 0, &\quad a_{12}a_{21} \neq 0, \\ a_{33} = a_{22} = a_{11}&, \\ a_{12}a_{21} - a_{11}^2 = 0& \end{aligned}$$

which will yield us to the Rota-Baxter operator

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, \quad a_{11}a_{12} \neq 0.$$

□

Theorem 3. *The Rota-Baxter operators of weight 1 on the 3-dimensional associative algebra A_1 are the following:*

$$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & -1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$P_2 = \begin{pmatrix} -1 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{1+a_{11}+a_{22}} \end{pmatrix}, \qquad a_{11}+a_{22} \neq -1$$

$$P_4 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad a_{21} \neq 0$$

$$P_5 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & -1 & a_{23} \\ 0 & 0 & -1 \end{pmatrix}, \qquad a_{21} \neq 0$$

$$P_6 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad a_{12} \neq 0$$

$$P_7 = \begin{pmatrix} -1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & -1 \end{pmatrix}, \qquad a_{12} \neq 0$$

$$P_8 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}(a_{11}+1)}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, \qquad a_{11}a_{12}(1+a_{11}) \neq 0.$$

Proof. Since P is a linear operator, we only need to consider the base elements which P satisfying equation (1) with $\lambda = 1$. We get the equations:

$$\begin{aligned} a_{31} = a_{32} &= 0, \\ a_{12}(a_{33} - a_{11}) &= 0, \\ a_{21}(a_{33} - a_{22}) &= 0, \\ a_{12}a_{21} + a_{11}a_{22} - a_{33}(1 + a_{11} + a_{22}) &= 0. \end{aligned} \tag{5}$$

So, the solutions of the system of equations (5) give us all Rota-Baxter operators of weight 1 for A_1. □

We present the list of Rota-Baxter operators of weights 0 and 1 on the A_2, A_3, A_4^α and A_5 algebras.

Algebra	Rota-Baxter Operators of weight 0	Restrictions
A_2	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{1}{2}a_{11} & \frac{2}{3}a_{12} \\ 0 & 0 & \frac{1}{3}a_{11} \end{pmatrix}$	$a_{11} \neq 0$
A_3	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}} \end{pmatrix}$	$a_{22} \neq -a_{11}$
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}^2}{a_{12}} & -a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12} \neq 0$
	$P_3 = \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
A_4^α	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2 + a_{11}^2 + a_{11}a_{12} = (2a_{11} + a_{12})a_{33};$ $\alpha a_{12}a_{22} + a_{11}(a_{21} + a_{22}) = (a_{11} + a_{21} + a_{22} + \alpha a_{12})a_{33};$ $a_{21}(a_{11} + a_{12}) + \alpha a_{12}a_{22} = (\alpha a_{12} + a_{21})a_{33};$ $\alpha a_{22}^2 + a_{21}^2 + a_{21}a_{22} = (2\alpha a_{22} + a_{21})a_{33}$
A_5	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{1}{2}a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11} \neq 0$

Algebra	Rota-Baxter Operators of weight 1	Restrictions
A_2	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}^2}{2a_{11}+1} & \frac{2a_{11}a_{12}(a_{11}+1)}{3a_{11}^2+3a_{11}+1} \\ 0 & 0 & \frac{a_{11}^3}{3a_{11}^2+3a_{11}+1} \end{pmatrix}$	$a_{11}\notin\{-\frac{1}{2},\frac{1}{6}(-3\pm i\sqrt{3})\}$
A_3	$P_1=\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & -1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
	$P_2=\begin{pmatrix} -1 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
	$P_3=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}(1+a_{11})}{a_{12}} & -1-a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12}\neq 0$
	$P_4=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{1+a_{11}+a_{22}} \end{pmatrix}$	$a_{11}+a_{22}\neq -1$
A_4^α	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2+a_{11}^2+a_{11}a_{12}$ $=(1+2a_{11}+a_{12})a_{33};$ $\alpha a_{12}a_{22}+a_{11}(a_{21}+a_{22})$ $=(1+a_{11}+a_{21}+a_{22}+\alpha a_{12})a_{33};$ $a_{21}(a_{11}+a_{12})+\alpha a_{12}a_{22}$ $=(\alpha a_{12}+a_{21})a_{33};$ $\alpha a_{22}^2+a_{21}^2+a_{21}a_{22}$ $=(\alpha(1+2a_{22})+a_{21})a_{33}$
A_5	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}^2}{1+2a_{11}} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$1+2a_{11}\neq 0$

2.2 Reynolds operator

Theorem 4. *All Reynolds operators on the 3-dimensional associative algebra A_1 are listed below:*

$$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}-a_{11}a_{22}} \end{pmatrix} \qquad \text{where } a_{11}+a_{22}-a_{11}a_{22} \neq 0$$

$$P_3 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{where } a_{12}a_{22} \neq 0$$

$$P_4 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{where } a_{21} \neq 0$$

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & \frac{a_{11}}{1-a_{11}} \end{pmatrix} \qquad \text{where } (1-a_{11})a_{12} \neq 0$$

Proof. Consider

$$P(e_3)P(e_3) = (a_{31}e_1 + a_{32}e_2 + a_{33}e_3)(a_{31}e_1 + a_{32}e_2 + a_{33}e_3) = 2a_{31}a_{32}e_3.$$

On the other hand,

$$\begin{aligned} P(e_3)P(e_3) &= P(e_3P(e_3) + P(e_3)e_3 - P(e_3)P(e_3)) = -2a_{31}a_{32}P(e_3) \\ &= -2a_{31}^2a_{32}e_1 - 2a_{31}a_{32}^2e_2 - 2a_{31}a_{32}a_{33}e_3. \end{aligned}$$

Comparing the coefficients of the basic elements we derive

$$a_{31}a_{32} = 0\,.$$

Now we consider the next cases:

Case 1. If $a_{31} = 0$, then identity

$$P(e_3)P(e_2) = P(e_3P(e_2) + P(e_3)e_2 - P(e_3)P(e_2))$$

implies that $a_{21}a_{32} = 0$.

Case 1.1. If $a_{21} = 0$, from

$$P(e_3)P(e_1) = P(e_3P(e_1) + P(e_3)e_1 - P(e_3)P(e_1))$$

we obtain $a_{32}(a_{11} - 1) = 0$.

Case 1.1.1. If $a_{32} = 0$, then from

$$P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1 - P(e_1)P(e_1))$$

we have $a_{12}(a_{11} - a_{33} + a_{11}a_{33}) = 0$.

1. If $a_{12} = 0$, considering $P(e_2)P(e_1) = P(e_2P(e_1)+P(e_2)e_1-P(e_2)P(e_1))$ we deduce that $a_{11}a_{22} = (a_{11} + a_{22} - a_{11}a_{22})a_{33}$.
   - If $a_{11} + a_{22} - a_{11}a_{22} = 0$, then we deduce the operator P_1;
   - If $a_{11} + a_{22} - a_{11}a_{22} \neq 0$, then we derive the operator P_2.
2. If $a_{12} \neq 0$, then we get operator P_3.

Case 1.1.2. If $a_{32} \neq 0$, then we have $a_{11} = 1$. Moreover, from

$$P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2 - P(e_1)P(e_2))$$

we derive that $a_{32} = 0$, that is a contradiction.

Case 1.2. If $a_{21} \neq 0$, then we have $a_{32} = 0$ and

$$P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1 - P(e_1)P(e_1)) \Rightarrow a_{12}(a_{11} - a_{33} + a_{11}a_{33}) = 0;$$

$$P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2 - P(e_1)P(e_2)) \Rightarrow \begin{array}{c} a_{12}a_{21}(1 + a_{33}) + a_{11}a_{22} \\ = (a_{11} + a_{22} - a_{11}a_{22})a_{33}; \end{array}$$

$$P(e_2)P(e_2) = P(e_2P(e_2) + P(e_2)e_2 - P(e_2)P(e_2)) \Rightarrow a_{22} - a_{33} + a_{22}a_{33} = 0.$$

Therefore,

- If $a_{12} = 0$, we have operator P_4;
- If $a_{12} \neq 0$, we obtain operator P_5.

Case 2. If $a_{31} \neq 0$, then we have $a_{32} = 0$ and from identities

$$\begin{array}{lcl} P(e_3)P(e_1) = P(e_3P(e_1) + P(e_3)e_1 - P(e_3)P(e_1)) & \Rightarrow & a_{12} = 0 \\ P(e_3)P(e_2) = P(e_3P(e_2) + P(e_3)e_2 - P(e_3)P(e_2)) & \Rightarrow & a_{22} = 1,\ a_{31} = 0 \end{array}$$

we obtain contradiction with $a_{31} \neq 0$. □

The Reynolds operators on algebras A_2, A_3, A_4^{α} and A_5 listed below.

Algebra	Reynolds Operators	Restrictions
A_2	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}}{2-a_{11}} & \frac{2a_{12}}{3-2a_{11}} \\ 0 & 0 & \frac{a_{11}}{3-2a_{11}} \end{pmatrix}$	$a_{11}(a_{11} - 2)(2a_{11} - 3) \neq 0$

Algebra	Reynolds Operators	Restrictions
A_3	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12}a_{21}(a_{33}+1) = a_{11}a_{22} + (a_{11}a_{22} - a_{11} - a_{22})a_{33}$
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{11} & a_{23} \\ 0 & 0 & -1 \end{pmatrix}$	
A_4^α	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2 + a_{11}^2 + a_{11}a_{12} = (2a_{11} + a_{12} - a_{11}^2 - a_{11}a_{12} - \alpha a_{12}^2)a_{33}$; $\alpha a_{12}a_{22} + a_{11}(a_{21}+a_{22}) = (a_{11} + a_{21} + a_{22} + \alpha a_{12} - a_{11}(a_{21}+a_{22}) - \alpha a_{12}a_{22})a_{33}$; $a_{21}(a_{11}+a_{12}) + \alpha a_{12}a_{22} = (\alpha a_{12} + a_{21} - a_{11}a_{21} - a_{12}a_{21} - \alpha a_{12}a_{22})a_{33}$; $\alpha a_{22}^2 + a_{21}^2 + a_{21}a_{22} = (2\alpha a_{22} + a_{21} - a_{21}^2 - a_{21}a_{22} - \alpha a_{22}^2)a_{33}$
A_5	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}}{2-a_{11}} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11}(a_{11}-2) \neq 0$

2.3 Nijenhuis operator

Theorem 5. *There are three types of Nijenhuis operators on the 3-dimensional associative algebra A_1, which are as follows:*

$$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix} \quad \text{where } a_{21} \neq 0$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix} \quad \text{where } a_{22} \neq a_{11}$$

Proof. Since P is a linear operator, we only need to consider the base elements which are satisfying in the equation

$$P(x)P(y) = P(xP(y) + P(x)y - P(xy))$$

and we obtain the equations:

$$\begin{aligned} a_{31} = a_{32} = 0, \\ a_{12}(a_{33} - a_{11}) = 0, \\ a_{21}(a_{33} - a_{22}) = 0, \\ a_{12}a_{21} + (a_{11} - a_{33})(a_{22} - a_{33}) = 0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} &P(e_3)P(e_2) = P(e_3P(e_2) + P(e_3)e_2 - P(e_3e_2)) \Rightarrow a_{31} = 0 \\ &P(e_3)P(e_1) = P(e_3P(e_1) + P(e_3)e_1 - P(e_3e_1)) \Rightarrow a_{32} = 0 \\ &P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1 - P(e_1e_1)) \Rightarrow a_{12}(a_{33} - a_{11}) = 0 \\ &P(e_2)P(e_2) = P(e_2P(e_2) + P(e_2)e_2 - P(e_2e_2)) \Rightarrow a_{21}(a_{33} - a_{22}) = 0 \\ &P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2 - P(e_1e_2)) \Rightarrow a_{12}a_{21} + (a_{11} - a_{33})(a_{22} - a_{33}) = 0. \end{aligned}$$

We have the following cases:

Case 1. If $a_{33} = a_{11}$, then (6) becomes

$$\begin{aligned} a_{21}(a_{33} - a_{22}) = 0, \\ a_{12}a_{21} = 0 \end{aligned}$$

- If $a_{21} = 0$, then we have the operator P_1
- If $a_{21} \neq 0$, then we obtain the operator P_2.

Case 2. If $a_{33} \neq a_{11}$, we obtain P_3. □

Now we present the list of Nijenhuis operators on algebras A_2, A_3, A_4^α and A_5.

Algebra	Nijenhuis Operators	Restrictions
A_2	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	
A_3	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$	$a_{22} \neq a_{11}$
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{(a_{11}-a_{33})(a_{22}-a_{33})}{a_{12}} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12} \neq 0$

Algebra	Nijenhuis Operators	Restrictions
A_4^0	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{33} - a_{11} & a_{13} \\ a_{33} - a_{22} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	
$A_4^\alpha, \alpha \neq 0$	$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{12} = \frac{(a_{33}-a_{11})(1\pm\sqrt{1-4\alpha})}{2\alpha}$ $a_{21} = \frac{(a_{33}-a_{22})(1\mp\sqrt{1-4\alpha})}{2}$
A_5	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & 2a_{11} - a_{22} \end{pmatrix}$	$a_{23}a_{32} = -(a_{11} - a_{22})^2$
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{33} \neq a_{11}$

2.4 Average operator

Theorem 6. *The average operators on the 3-dimensional associative algebra A_1 are the following:*

$$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{22} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$$

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & 2a_{11} \end{pmatrix} \qquad \text{where } a_{12} \neq 0.$$

Proof. Since P is a linear operator, we only need to consider the base elements which are satisfying in the equation

$$P(x)P(y) = P(xP(y))$$

and also we have the equations:

$$\begin{aligned} a_{31} = a_{32} &= 0,\\ a_{12}(a_{33} - 2a_{11}) &= 0,\\ a_{21}(a_{33} - 2a_{22}) &= 0,\\ a_{12}a_{21} + a_{22}(a_{11} - a_{33}) &= 0,\\ a_{12}a_{21} + a_{11}(a_{22} - a_{33}) &= 0. \end{aligned} \tag{7}$$

The solutions of the system of equations (7) give us all average operators on A_1 algebras. □

Finally, we give the list of average operators on algebras A_2, A_3, A_4^{α} and A_5.

Algebra	average Operators	Restrictions
A_2	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{11} \end{pmatrix}$	$a_{11} \neq 0$
A_3	$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	$a_{11} \neq 0$
	$P_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	$a_{12}a_{21} = a_{11}a_{22}$
A_4^0	$P_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & -a_{11} & a_{13} \\ -a_{22} & a_{22} & a_{23} \\ -a_{32} & a_{32} & a_{33} \end{pmatrix}$	

Algebra	average Operators	Restrictions
$A_4^{\alpha}, \alpha \neq 0$	$P_1 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$a_{33} \neq 0$
	$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ -a_{22} & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$	$a_{22} \neq 0$
	$P_3 = \begin{pmatrix} -\frac{a_{12}(1 \pm \sqrt{1-4\alpha})}{2} & a_{12} & a_{13} \\ -\frac{a_{22}(1 \pm \sqrt{1-4\alpha})}{2} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	
	$P_4 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\alpha a_{11} a_{12}}{a_{11}+a_{12}} & \frac{\alpha a_{12}^2}{a_{11}+a_{22}} & a_{23} \\ 0 & 0 & \frac{a_{11}^2 + a_{11}a_{12} + \alpha a_{12}^2}{a_{11}+a_{12}} \end{pmatrix}$	$a_{12} \neq -a_{11}$ $a_{11}^2 + a_{11}a_{12} + \alpha a_{12}^2 \neq 0$
	$P_5 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}$	$a_{11} \neq 0$
A_5	$P_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$	
	$P_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11} \neq 0$

Acknowledgments

We thank the referee for the helpful comments and suggestions that contributed to improving this paper.

References

[1] R. Bai, L. Guo, J. Li, Y. Wu: Rota-Baxter 3-Lie algebras. *J. Math. Phys.* 54 (6) (2013) 063504.

[2] G. Baxter: An analytic problem whose solution follows from a simple algebraic identity. *Pac. J. Math.* 10 (1960) 731–742.

[3] A.A. Belavin, V.G. Drinfel'd: Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funct. Anal. its Appl.* 16 (3) (1982) 159–180.

[4] P. Benito, V. Gubarev, A. Pozhidaev: Rota-Baxter operators on quadratic algebras. *Mediterr. J. Math* 15 (2018) 1–23.

[5] Y. Chengand, Y. Su: Quantum deformations of the Heisenberg-Virasoro algebra. *Algebra Colloq.* 20 (2) (2013) 299–308.

[6] W.A. De Graaf: Classification of nilpotent associative algebras of small dimension. *Int. J. Algebra Comput.* 28 (1) (2018) 133–161.

[7] K. Ebrahimi-Fard: Loday-type algebras and the Rota-Baxter relation. *Lett. Math. Phys.* 61 (2) (2002) 139–147.

[8] X. Gao, M. Liu, C Bai, N. Jing: Rota-Baxter operators on Witt and Virasoro algebras. *J. Geom. Phys.* 108 (2016) 1–20.

[9] L. Guo: *An Introdction to Rota-Baxter Algebra*. International Press, Beijing, China (2012).

[10] L. Guo, W. Keigher: Baxter algebras and shuffle products. *Adv. Math.* 150 (1) (2000) 117–149.

[11] L. Guo, Z. Liu: Rota-Baxter operators on generalized power series rings. *J. Algebra Its Appl.* 8 (4) (2009) 557–564.

[12] O.C. Hazlett: On the classification and invariantive characterization of nilpotent algebras. *Am. J. Math.* 38 (2) (1916) 109–138.

[13] I. Karimjanov, I. Kaygorodov, M. Ladra: Rota-type operators on null-filiform associative algebras. *Linear and Multilinear algebra* 68 (1) (2020) 205–219.

[14] R.L. Kruse, D.T. Price: *Nilpotent Rings*. Gordon and Breach Science Publishers, New York (1969).

[15] A. Makhlouf, D. Yau: Rota-Baxter Hom-Lie-admissible algebras. *Communications in Algebra* 42 (3) (2014) 1231–1257.

[16] R. Mazurek: Rota–Baxter operators on skew generalized power series rings. *J. Algebra Its Appl.* 13 (7) (2014) 1450048.

[17] G. Mazzolla: The algebraic and geometric classification of associative algebras of dimension five. *Manuscr. Math.* 27 (1) (1979) 81–101.

[18] G. Mazzolla: Generic finite schemes and Hochschild cocycles. *Comment. Math. Helv.* 55 (2) (1980) 267–293.

[19] Y. Pan, Q. Liu, C. Bai, L. Guo: Post Lie algebra structures on the Lie algebra $sl(2, \mathbb{C})$. *Electron. J. Linear Algebra* 23 (2012) 180–197.

[20] J. Pei, C. Bai, L. Guo: Rota–Baxter operators on $sl(2, \mathbb{C})$ and solutions of the classical Yang-Baxter equation. *J. Math. Phys.* 55 (2) (2014) 021701.

[21] X. Tang, Y. Zhang, and Q. Sun.: Rota-Baxter operators on 4-dimensional complex simple associative algebras. *Appl. Math. Comput.* 229 (2014) 173–186.

[22] H. Yu: Classification of monomial Rota–Baxter operators on $k[x]$. *J. Algebra Its Appl.* 15 (5) (2016) 1650087.

[23] S. Zheng, L. Guo, M. Rosenkranz: Rota–Baxter operators on the polynomial algebra, integration, and averaging operators. *Pac. J. Math.* 275 (2) (2015) 481–507.

Received: April 21, 2020
Accepted for publication: June 12, 2020
Communicated by: Ivan Kaygorodov