

Applications of Mathematics

Kassimu Mpungu; Tijani A. Apalara; Mukhiddin Muminov
On the stabilization of laminated beams with delay

Applications of Mathematics, Vol. 66 (2021), No. 5, 789–812

Persistent URL: <http://dml.cz/dmlcz/149083>

Terms of use:

© Institute of Mathematics AS CR, 2021

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON THE STABILIZATION OF LAMINATED BEAMS WITH DELAY

KASSIMU MPUNGU, Johor Bahru, TIJANI A. APALARA, Hafr Al-Batin,
MUKHIDDIN MUMINOV, Johor Bahru

Received February 26, 2020. Published online March 12, 2021.

Abstract. Of concern in this paper is the laminated beam system with frictional damping and an internal constant delay term in the transverse displacement. Under suitable assumptions on the weight of the delay, we establish that the system's energy decays exponentially in the case of equal wave speeds of propagation, and polynomially in the case of non-equal wave speeds.

Keywords: laminated beam; interfacial slip; delay; exponential and polynomial decay

MSC 2020: 35B40, 35L56, 93D15, 93D20, 93D23

1. INTRODUCTION

The laminated beam model describes a vibrating structure of an interfacial slip. It consists of two-layered beams of uniform thickness which are attached by an adhesive layer of small thickness in such a way that small amount of slip is possible while they are continuously in contact with each other. The model which consists of three coupled hyperbolic equations was derived by Hansen et al. [15] using the assumption of Timoshenko beam theory and is given as follows:

$$(1.1) \quad \begin{aligned} \rho w_{tt} + G(\psi - w_x)_x &= 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) &= 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t &= 0, \end{aligned}$$

with $x \in (0, 1)$ and $t \geq 0$. The subscripted t and x denote differentiation with respect to time and to the longitudinal spatial variable, respectively. The function $w = w(x, t)$ is the traverse displacement, $\psi = \psi(x, t)$ is the rotation angle, $s = s(x, t)$ is proportional to the amount of slip along the interface and $3s - \psi$ denotes the effective rotation angle. The positive parameters ρ , I_ρ , G , D , γ , and β are the density, mass

moment of inertia, shear stiffness, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. Laminated beams are considered to be very important especially in the field of engineering because of their applicability in building and construction of different structures. In recent years, researchers have focused on the study of the well-posedness and asymptotic stability properties of these structures by adding some damping mechanisms to the system. Let us mention some of the results.

Wang et al. [33] considered (1.1) with some boundary feedback controls and established an exponential decay result provided $\sqrt{\varrho/G} \neq \sqrt{I_\varrho/D}$. Tatar [32] and Mustafa [23] improved the result in [33] when they established the exponential stability for the system under weak assumptions on ϱ , G , I_ϱ , and D . Similar results were also established by Cao et al. [10] with different boundary controls. Apart from boundary control feedback, researchers also considered some other damping mechanisms like introducing additional damping terms in order to achieve the desired stability results. For instance, Raposo [30] proved exponential stability by introducing additional frictional damping in the form of αw_t and $\beta(3s - \psi)_t$ on the transverse displacement and rotation angle, respectively. Similarly, classical heat effect and second sound were also considered in the literature by Apalara [7] and [6], respectively. For other damping mechanisms, we refer the reader to [24], [11], [20], [21].

Often, delay effects appear in various physical problems. Thus, to exhaustively analyse such problems, delay differential equations are developed. Over the years, the control of PDEs with time delay effects has become a center of attraction to researchers. Generally, time delay is observed to have a significant effect on the stability of most of the systems. This effect may take different directions. For example, it was established that delayed positive feedback can stabilize purely oscillatory systems, see [1]. On the other hand, Zhang et al. [35] numerically illustrated a direct proportionality between time delay and diffusion in their study of a semi-linear fractional partial differential equation with time delay. This implied a negative effect of time delay on stability. Furthermore, it was established that the presence of an arbitrarily small delay may destabilize a system which is uniformly or asymptotically stable in the absence of delay. For instance, consider the system

$$(1.2) \quad \begin{cases} u_{tt}(x, t) - \Delta u(x, t) = 0, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau), & x \in \Gamma_1, t > 0. \end{cases}$$

It is known that in the absence of delay ($\mu_2 = 0$, $\mu_1 > 0$), the system is exponentially stable, see [17], [18], [36]. Whereas, in the presence of delay ($\mu_2 > 0$), Nicaise and Pignotti [25] proved, under the assumption $\mu_2 < \mu_1$, that the energy is exponentially stable. However, for the opposite case ($\mu_2 \geq \mu_1$), they were able to construct

a sequence of delays for which the corresponding solution is unstable. Similar conclusions were reached by [12], [34]. In some instances, the system's stability may not be affected by the delay, because the delay effect is insignificantly small to have any repercussions or the damping is strong enough to neutralize the delay effect. For instance, Mustafa [22] studied a thermoelastic system with boundary time-varying delay in one-dimensional space and showed that the damping effect through heat conduction is still strong enough to uniformly stabilize the system even in the presence of boundary time-varying delay. For more works regarding time delay, we refer the reader to [2], [16], [27], [29], [3], [5], [8], [14], [28], [26] and references therein.

Regarding the laminated beam with delay, very little has been done. We found the work of Feng [13] where he considered a laminated system with three internal constant time delays. Using boundary feedbacks coupled with some assumptions on internal parameters, he proved the well-posedness of the system and the exponential stability.

It is important to note that when $s(x, t)$ in (1.1) is identically zero, the standard Timoshenko system is obtained. Consequently, in this work, we extend the result in [4], which is on the Timoshenko system, to a laminated beam system and establish an exponential stability result. Precisely, we consider the following system of laminated beams with frictional damping and an internal constant delay term in the transverse displacement:

$$(1.3) \quad \begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu_1 w_t + \mu_2 w_t(x, t - \tau) = 0 & \text{in } (0, 1) \times (0, \infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0 & \text{in } (0, 1) \times (0, \infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0 & \text{in } (0, 1) \times (0, \infty), \\ w_t(x, t - \tau) = f_0(x, t - \tau) & \text{in } (0, 1) \times (0, \tau), \\ w(x, 0) = w_0, w_t(x, 0) = w_1, \psi(x, 0) = \psi_0, \psi_t(x, 0) = \psi_1 & \text{in } (0, 1), \\ s(x, 0) = s_0, s_t(x, 0) = s_1 & \text{in } (0, 1), \\ w(0, t) = w_x(1, t) = s_x(0, t) = s(1, t) = \psi_x(0, t) = \psi(1, t) = 0 & \text{in } (0, \infty). \end{cases}$$

Here, $w_0, w_1, \psi_0, \psi_1, s_0,$ and s_1 are initial data with f_0 being the history function in an appropriate space, and τ is time delay, μ_1 is a positive constant and μ_2 is a real number. Under suitable assumptions on the delay term and coefficients of wave propagation speed, we establish the exponential decay as well as the polynomial decay results of the energy of system (1.3).

The rest of the paper is organized as follows. We give some preliminaries which include the well-posedness result in Section 2. In Section 3, we state and prove some technical lemmas. In Sections 4 and 5, we establish exponential and polynomial decay results, respectively. Throughout this article $u_x = \partial u / \partial x$.

2. PRELIMINARIES

In this section, we introduce some necessary transformations. Moreover, we give the proof of the well-posedness result.

As in [25], we introduce a new variable

$$(2.1) \quad z(x, \sigma, t) = w_t(x, t - \tau\sigma) \quad \text{in } (0, 1) \times (0, 1) \times (0, \infty).$$

It easy to show that z satisfies

$$(2.2) \quad \tau z_t(x, \sigma, t) + z_\sigma(x, \sigma, t) = 0 \quad \text{in } (0, 1) \times (0, 1) \times (0, \infty).$$

As a result, system (1.3) is equivalent to

$$(2.3) \quad \left\{ \begin{array}{ll} \varrho w_{tt} + G(\psi - w_x)_x + \mu_1 w_t + \mu_2 z(x, 1, t) = 0 & \text{in } (0, 1) \times (0, \infty), \\ I_\varrho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0 & \text{in } (0, 1) \times (0, \infty), \\ 3I_\varrho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0 & \text{in } (0, 1) \times (0, \infty), \\ \tau z_t(x, \sigma, t) + z_\sigma(x, \sigma, t) = 0 & \text{in } (0, 1) \times (0, 1) \times (0, \infty), \\ z(x, 1, t) = f_0(x, t - \tau) & \text{in } (0, 1) \times (0, \tau), \\ z(x, 0, t) = w_t(x, t) & \text{in } (0, 1) \times (0, \infty), \\ z(x, \sigma, 0) = f_0(x, -\sigma\tau) & \text{in } (0, 1) \times (0, 1), \\ w(x, 0) = w_0, \quad s(x, 0) = s_0, \quad \psi(x, 0) = \psi_0 & \text{in } (0, 1), \\ w_t(x, 0) = w_1, \quad s_t(x, 0) = s_1, \quad \psi_t(x, 0) = \psi_1 & \text{in } (0, 1), \\ w(0, t) = s_x(0, t) = \psi_x(0, t) & \\ \quad = w_x(1, t) = s(1, t) = \psi(1, t) = 0 & \text{in } (0, \infty). \end{array} \right.$$

Thus, we consider (2.3) instead of (1.3). With respect to the weight of delay, we assume that

$$(2.4) \quad |\mu_2| < \mu_1$$

and prove that this condition is sufficient to establish the well-posedness and the stability of the system (2.3) with the energy E , defined by

$$(2.5) \quad E(t) = \frac{1}{2} \int_0^1 [\varrho w_t^2 + I_\varrho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\varrho s_t^2 + 3Ds_x^2 + 4\gamma s^2] dx \\ + \frac{1}{2} \int_0^1 \left[G(\psi - w_x)^2 + \tau|\mu_2| \int_0^1 z^2(x, \sigma, t) d\sigma \right] dx.$$

Regarding the existence, uniqueness, and smoothness of the solution of problem (2.3), let $\xi = 3s - \psi$ and then write (2.3) in the form of a system of first-order equations:

$$(2.6) \quad \begin{cases} \frac{\partial w}{\partial t} = u, \\ \frac{\partial u}{\partial t} = -\frac{1}{\varrho}(G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x, 1, t)), \\ \frac{\partial \xi}{\partial t} = v, \\ \frac{\partial v}{\partial t} = \frac{1}{I_\varrho}(D\xi_{xx} + G(3s - \xi - w_x)), \\ \frac{\partial s}{\partial t} = y, \\ \frac{\partial y}{\partial t} = \frac{1}{I_\varrho}\left(Ds_{xx} - G(3s - \xi - w_x) - \frac{4\gamma}{3}s - \frac{4\beta}{3}y\right), \\ \frac{\partial z}{\partial t} = -\frac{1}{\tau}z_\sigma(x, \sigma, t). \end{cases}$$

We set $\Phi = (w, u, \xi, v, s, y, z)^\top$, so that (2.6) becomes

$$(2.7) \quad \begin{cases} \frac{d\Phi}{dt} = \mathcal{A}\Phi, & t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^\top, \end{cases}$$

where

$$\mathcal{A}\Phi = \begin{pmatrix} u \\ -\frac{1}{\varrho}(G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x, 1, t)) \\ v \\ \frac{1}{I_\varrho}(D\xi_{xx} + G(3s - \xi - w_x)) \\ y \\ \frac{1}{I_\varrho}\left(Ds_{xx} - G(3s - \xi - w_x) - \frac{4\gamma}{3}s - \frac{4\beta}{3}y\right) \\ -\frac{1}{\tau}z_\sigma(x, \sigma, t) \end{pmatrix}.$$

Remark 2.1. System (1.3) is the main problem. The transformation to system (2.3) is well known in the literature; it is necessary (not compulsory) because of the delay term. System (2.6) is the semigroup setting necessary for the proof of the well-posedness result. See [6], [4], [31] for a similar approach.

We proceed by introducing a one-dimensional Sobolev space [9], pp. 202–203,

$$W^{1,p}(0,1) = \left\{ u \in L^p(0,1); \exists g \in L^p(0,1) \text{ such that} \right. \\ \left. \int_0^1 u \varphi_x \, dx = - \int_0^1 g \varphi \, dx \forall \varphi \in C_0^1(0,1) \right\}$$

with φ being a test function. The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}(0,1)} = (\|u\|_{L^p(0,1)}^p + \|u_x\|_{L^p(0,1)}^p)^{1/p}.$$

If $p = 2$, we have

$$W^{1,2}(0,1) = H^1(0,1).$$

The space $H^1(0,1)$ is equipped with the inner product

$$(u, v)_{H^1(0,1)} = (u, v)_{L^2(0,1)} + (u_x, v_x)_{L^2(0,1)} = \int_0^1 (uv + u_x v_x) \, dx$$

and with the associated norm

$$\|u\|_{H^1(0,1)} = (\|u\|_{L^2(0,1)}^2 + \|u_x\|_{L^2(0,1)}^2)^{1/2}.$$

Furthermore, the space H^2 (see [9], p. 216) is defined as

$$H^2(0,1) = \{u \in W^{1,2}(0,1); u_x \in W^{1,2}(0,1)\}$$

and is equipped with an inner product

$$(u, v)_{H^2(0,1)} = (u, v)_{L^2(0,1)} + (u_x, v_x)_{L^2(0,1)} + (u_{xx}, v_{xx})_{L^2(0,1)}.$$

Concerning our problem, we consider the spaces

$$H_a^1 = \{v: v \in H^1(0,1), v(0) = 0\}, \quad H_b^1 = \{v: v \in H^1(0,1), v(1) = 0\}$$

and let

$$\mathcal{H} := H_a^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1))$$

be the Hilbert space equipped with the inner product

$$(2.8) \quad (\Phi, \tilde{\Phi})_{\mathcal{H}} = \varrho \int_0^1 u \tilde{u} \, dx + G \int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x) \, dx + I_\varrho \int_0^1 v \tilde{v} \, dx \\ + 3I_\varrho \int_0^1 y \tilde{y} \, dx + D \int_0^1 \xi_x \tilde{\xi}_x \, dx + 4\gamma \int_0^1 s \tilde{s} \, dx + 3D \int_0^1 s_x \tilde{s}_x \, dx \\ + \tau |\mu_2| \int_0^1 \int_0^1 z(x, \sigma, t) \tilde{z}(x, \sigma, t) \, d\sigma \, dx.$$

Instead of dealing with (2.3), we will consider (2.7) with the domain of the operator \mathcal{A} given by

$$\begin{aligned} D(\mathcal{A}) = \{ \Phi = (w, u, \xi, v, s, y, z)^\top \in \mathcal{H}; & w \in H^2(0, 1) \cap H_a^1(0, 1), \\ & \xi, s \in H^2(0, 1) \cap H_b^1(0, 1), u \in H_a^1(0, 1), v, y \in H_b^1(0, 1), \\ & z, z_\sigma \in L^2((0, 1) \times (0, 1)), w_x(1) = \xi_x(0) = s_x(0) = 0 \}. \end{aligned}$$

Note that $D(\mathcal{A})$ is independent of time $t > 0$. Furthermore, it is obvious that $D(\mathcal{A})$ is dense in \mathcal{H} .

Proposition 2.1. \mathcal{A} is a dissipative operator.

Proof. Using the inner product defined by (2.8) and integration by parts, we have

$$\begin{aligned} (2.9) \quad (\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = & - \left(\mu_1 - \frac{|\mu_2|}{2} \right) \int_0^1 u^2 dx - \frac{|\mu_2|}{2} \int_0^1 z^2(x, 1, t) dx \\ & - 4\beta \int_0^1 y^2 dx - \mu_2 \int_0^1 uz(x, 1, t) dx. \end{aligned}$$

Using Young's inequality, the last term in (2.9) gives

$$(2.10) \quad -\mu_2 \int_0^1 uz(x, 1, t) dx \leq \frac{|\mu_2|}{2} \int_0^1 u^2 dx + \frac{|\mu_2|}{2} \int_0^1 z^2(x, 1, t) dx.$$

Combining (2.9) and (2.10), we end up with

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \leq -(\mu_1 - |\mu_2|) \int_0^1 u^2 dx - 4\beta \int_0^1 y^2 dx.$$

Moreover, by (2.4), it follows that $(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \leq 0$. Thus, \mathcal{A} is dissipative. □

Proposition 2.2. $I - \mathcal{A}$ is surjective.

Proof. Since \mathcal{A} is dissipative and $D(\mathcal{A})$ is dense in \mathcal{H} , it is sufficient to show that \mathcal{A} is maximal. Given $H = (f_1, \dots, f_7) \in \mathcal{H}$, we must show that there exists $\Phi = (w, u, \xi, v, s, y, z) \in D(\mathcal{A})$ satisfying

$$(2.11) \quad (I - \mathcal{A})\Phi = H,$$

which implies

$$(2.12) \quad w - u = f_1 \in H_a^1(0, 1),$$

$$(2.13) \quad \varrho u + G(3s - \xi - w_x)_x + \mu_1 u + \mu_2 z(x, 1) = \varrho f_2 \in L^2(0, 1),$$

$$(2.14) \quad \xi - v = f_3 \in H_b^1(0, 1),$$

$$(2.15) \quad I_\varrho v - D\xi_{xx} - G(3s - \xi - w_x) = I_\varrho f_4 \in L^2(0, 1),$$

$$(2.16) \quad s - y = f_5 \in H_b^1(0, 1),$$

$$(2.17) \quad 3I_\varrho y - 3Ds_{xx} + 3G(3s - \xi - w_x) + 4\gamma s + 4\beta y = 3I_\varrho f_6 \in L^2(0, 1),$$

$$(2.18) \quad \tau z(x, \sigma, t) + z_\sigma(x, \sigma, t) = \tau f_7 \in L^2((0, 1) \times (0, 1)).$$

We observe that (2.18) with $z(x, 0) = u$ has a unique solution given by

$$(2.19) \quad z(x, \sigma, t) = e^{-\tau\sigma} u + \tau e^{-\tau\sigma} \int_0^\sigma e^{\tau q} f_7(x, q) dq \in L^2((0, 1) \times (0, 1)).$$

Using (2.12), (2.14), and (2.16), we end up with

$$(2.20) \quad \begin{aligned} G(3s - \xi - w_x)_x + \bar{\varrho} w &= h_1 \in L^2(0, 1), \\ -D\xi_{xx} - G(3s - \xi - w_x) + I_\varrho \xi &= h_2 \in L^2(0, 1), \\ -3Ds_{xx} + 3G(3s - \xi - w_x) + \lambda s &= h_3 \in L^2(0, 1), \end{aligned}$$

where

$$(2.21) \quad \begin{aligned} h_1 &= \bar{\varrho} f_1 + \varrho f_2 - \tau e^{-\tau} \int_0^1 e^{\tau q} f_7(x, q) dq, \quad h_2 = I_\varrho(f_3 + f_4), \\ h_3 &= (4\beta + 3I_\varrho) f_5 + 3I_\varrho f_6, \quad \bar{\varrho} = \varrho + \mu_1 + \mu_2 e^{-\tau}, \quad \lambda = 4\gamma + 4\beta + 3I_\varrho. \end{aligned}$$

To solve (2.20) we consider

$$(2.22) \quad B((w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s})) = F(\tilde{w}, \tilde{\xi}, \tilde{s}),$$

where $B: [H_a^1(0, 1) \times H_b^1(0, 1) \times H_b^1(0, 1)]^2 \rightarrow \mathbb{R}$ is the bilinear form given by

$$\begin{aligned} B((w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s})) &= G \int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x) dx + D \int_0^1 \xi_x \tilde{\xi}_x dx \\ &\quad + 3D \int_0^1 s_x \tilde{s}_x dx + \bar{\varrho} \int_0^1 w \tilde{w} dx + \lambda \int_0^1 s \tilde{s} dx + I_\varrho \int_0^1 \xi \tilde{\xi} dx \end{aligned}$$

and $F: [H_a^1(0, 1) \times H_b^1(0, 1) \times H_b^1(0, 1)] \rightarrow \mathbb{R}$ is the linear form defined by

$$F(\tilde{w}, \tilde{\xi}, \tilde{s}) = \int_0^1 h_1 \tilde{w} dx + \int_0^1 h_2 \tilde{\xi} dx + \int_0^1 h_3 \tilde{s} dx.$$

Now, for $V = H_a^1(0, 1) \times H_b^1(0, 1) \times H_b^1(0, 1)$ equipped with the norm

$$\|(w, \xi, s, q)\|_V^2 = \|3s - \xi - w_x\|_2^2 + \|w\|_2^2 + \|\xi_x\|_2^2 + \|s_x\|_2^2,$$

one can easily see that B and F are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned} B((w, \xi, s), (w, \xi, s)) &= G \int_0^1 (3s - \xi - w_x)^2 dx + D \int_0^1 \xi_x^2 dx + 3D \int_0^1 s_x^2 dx \\ &\quad + \bar{\varrho} \int_0^1 w^2 dx + \lambda \int_0^1 s^2 dx + I_\varrho \int_0^1 \xi^2 dx \\ &\geq c \|(w, \xi, s)\|_V^2. \end{aligned}$$

Thus B is coercive. Consequently, by Lax-Milgram lemma, system (2.20) has a unique solution

$$w \in H_a^1(0, 1), \quad \xi, s \in H_b^1(0, 1).$$

Substituting w , ξ , and s into (2.12), (2.14), and (2.16), respectively, we obtain

$$u \in H_a^1(0, 1), \quad v, y \in H_b^1(0, 1).$$

Similarly, inserting u into (2.19), bearing in mind (2.18), we end up with

$$z, z_\sigma \in L^2((0, 1) \times (0, 1)).$$

Now, if $(\tilde{\xi}, \tilde{s}) \equiv (0, 0) \in H_a^1(0, 1) \times H_a^1(0, 1)$, then (2.22) reduces to

$$(2.23) \quad -G \int_0^1 (3s - \xi - w_x) \tilde{w}_x dx + \bar{\varrho} \int_0^1 w \tilde{w} dx = \int_0^1 h_1 \tilde{w} dx \quad \forall \tilde{w} \in H_a^1(0, 1),$$

which implies

$$(2.24) \quad -Gw_{xx} = -3Gs_x + G\xi_x - \varrho w + h_1 \in L^2(0, 1).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$w \in H^2(0, 1) \cap H_a^1(0, 1).$$

Moreover, (2.23) is also true for any $\phi \in C^1([0, 1])$, $\phi(0) = 0$, which is in $H_a^1(0, 1)$. Hence, we have

$$G \int_0^1 w_x \phi_x dx + \int_0^1 (3Gs_x - G\xi_x + \varrho w - h_1) \phi dx = 0 \quad \forall \phi \in C^1([0, 1]), \phi(0) = 0.$$

Thus, using integration by parts and bearing in mind (2.24), we obtain

$$w_x(1)\phi(1) = 0 \quad \forall \phi \in C^1([0, 1]), \quad \phi(0) = 0.$$

Therefore, $w_x(1) = 0$. Similarly, we obtain

$$\begin{aligned} -D\xi_{xx} &= G(3s - \xi - w_x) - I_\rho \xi + h_2 \in L^2(0, 1), \\ -3Ds_{xx} &= -3G(3s - \xi - w_x) - \lambda s + h_3 \in L^2(0, 1). \end{aligned}$$

Consequently, we have

$$\xi, s \in H^2(0, 1) \cap H_b^1(0, 1), \quad \xi_x(0) = s_x(0) = 0.$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of a unique $\Phi \in D(\mathcal{A})$ such that (2.11) is satisfied. Thus the operator $I - \mathcal{A}$ is surjective. \square

As a consequence of the Hille-Yosida theorem [19], Theorem 1.2.2, p. 3, we have that \mathcal{A} generates a C_0 -semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on \mathcal{H} . From the semigroup theory, $\Phi(t) = e^{t\mathcal{A}}\Phi_0$ is the unique solution of (2.7) satisfying the conditions of the following theorem.

Theorem 2.1 (well-posedness result). *Let $\Phi_0 \in \mathcal{H}$. Then there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.7). Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Remark 2.2. Theorem 2.1 guarantees the existence of system (2.3) in a weak sense.

3. TECHNICAL LEMMAS

In this section, we state and prove the necessary lemmas required to construct a suitable Lyapunov functional, which is used to establish some stability results for the energy of the solution of system (2.3). Throughout this section, c is a generic positive constant, precisely,

$$c \geq \max \left\{ \frac{\mu_1^2}{G}, \frac{\mu_2^2}{G}, \frac{\rho^2}{4}, \frac{9\rho^2}{4}, \frac{G^2}{2D}, \frac{9G^2}{4\gamma}, \frac{3I_\rho\mu_1^2}{2D\rho^2}, \frac{3I_\rho\mu_2^2}{2D\rho^2}, \frac{3G^2}{2DI_\rho} \right\}.$$

Lemma 3.1. *Let (w, ψ, s, z) be a solution of (2.3). Then the energy functional E , defined by (2.5), satisfies*

$$(3.1) \quad E'(t) \leq -m_0 \int_0^1 w_t^2 dx - 4\beta \int_0^1 s_t^2 dx, \quad t \geq 0,$$

for some positive constant m_0 .

Proof. We begin with multiplying the first three equations in system (2.3) by w_t , $(3s_t - \psi_t)$ and s_t , respectively, then integrate by parts over $(0, 1)$ using the boundary conditions. This leads to

$$(3.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 [\varrho w_t^2 + I_\varrho (3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 \\ & \quad + 3I_\varrho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + G(\psi - w_x)^2] dx \\ & = -\mu_1 \int_0^1 w_t^2 dx - \mu_2 \int_0^1 z(x, 1, t) w_t dx - 4\beta \int_0^1 s_t^2 dx. \end{aligned}$$

Next, we multiply (2.3)₄ by $|\mu_2|z$, and then integrate the product over $(0, 1) \times (0, 1)$. Using the fact that $z(x, 0, t) = w_t$, we obtain

$$(3.3) \quad \frac{\tau|\mu_2|}{2} \frac{d}{dt} \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx = -\frac{|\mu_2|}{2} \int_0^1 z^2(x, 1, t) dx + \frac{|\mu_2|}{2} \int_0^1 w_t^2 dx.$$

Putting together (3.2) and (3.3) leads to

$$(3.4) \quad \begin{aligned} E'(t) = & -\left(\mu_1 - \frac{|\mu_2|}{2}\right) \int_0^1 w_t^2 dx - \mu_2 \int_0^1 z(x, 1, t) w_t dx \\ & - \frac{|\mu_2|}{2} \int_0^1 z^2(x, 1, t) dx - 4\beta \int_0^1 s_t^2 dx. \end{aligned}$$

Applying Young's inequality on the second term of (3.4) gives

$$(3.5) \quad -\mu_2 \int_0^1 w_t z(x, 1, t) dx \leq \frac{|\mu_2|}{2} \int_0^1 z^2(x, 1, t) dx + \frac{|\mu_2|}{2} \int_0^1 w_t^2 dx$$

and lastly, substituting (3.5) in (3.4), and using (2.5) completes the proof of (3.1). \square

Lemma 3.2. *If (w, ψ, s, z) is a solution of (2.3), then the functional F_1 , defined by*

$$F_1(t) := \varrho \int_0^1 w w_t dx - \varrho \int_0^1 w_t \int_0^x \psi(y) dy dx, \quad t \geq 0,$$

satisfies, for any $\varepsilon_1, \varepsilon_2 > 0$, the estimate

$$(3.6) \quad \begin{aligned} \frac{d}{dt} F_1(t) \leq & -\frac{G}{2} \int_0^1 (\psi - w_x)^2 dx + \varepsilon_1 \int_0^1 (3s_t - \psi_t)^2 dx \\ & + c \left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) \int_0^1 w_t^2 dx + \varepsilon_2 \int_0^1 s_t^2 dx + c \int_0^1 z^2(x, 1, t) dx. \end{aligned}$$

Proof. Differentiating F_1 and using (2.3)₁, i.e., $-\varrho w_{tt} = G(\psi - w_x)_x + \mu_1 w_t + \mu_2 z(x, 1, t)$,

$$\begin{aligned} \frac{d}{dt} F_1(t) &= -G \int_0^1 (\psi - w_x)_x w \, dx - \mu_1 \int_0^1 w_t w \, dx + \varrho \int_0^1 w_t^2 \, dx \\ &\quad + \mu_1 \int_0^1 w_t \int_0^x \psi(y) \, dy \, dx - \mu_2 \int_0^1 z(x, 1, t) w \, dx \\ &\quad + G \int_0^1 (\psi - w_x)_x \int_0^x \psi(y) \, dy \, dx - \varrho \int_0^1 w_t \int_0^x \psi_t(y) \, dy \, dx \\ &\quad + \mu_2 \int_0^1 z(x, 1, t) \int_0^x \psi(y) \, dy \, dx. \end{aligned}$$

Integrating by parts the terms involving G and using $w_x = -(\psi - w_x) + \psi$ leads to

$$\begin{aligned} \frac{d}{dt} F_1(t) &= -G \int_0^1 (\psi - w_x)^2 \, dx + \varrho \int_0^1 w_t^2 \, dx - \mu_1 \int_0^1 w_t w \, dx \\ &\quad + \mu_1 \int_0^1 w_t \int_0^x \psi(y) \, dy \, dx - \mu_2 \int_0^1 z(x, 1, t) w \, dx \\ &\quad + \mu_2 \int_0^1 z(x, 1, t) \int_0^x \psi(y) \, dy \, dx - \varrho \int_0^1 w_t \int_0^x \psi_t(y) \, dy \, dx. \end{aligned}$$

Using $\psi_t = -(3s_t - \psi_t) + 3s_t$, we have

$$\begin{aligned} (3.7) \quad \frac{d}{dt} F_1(t) &= -G \int_0^1 (\psi - w_x)^2 \, dx + \varrho \int_0^1 w_t^2 \, dx \\ &\quad + \mu_1 \int_0^1 w_t \left(\int_0^x \psi(y) \, dy - w \right) \, dx \\ &\quad + \mu_2 \int_0^1 \left(\int_0^x \psi(y) \, dy - w \right) z(x, 1, t) \, dx \\ &\quad + \varrho \int_0^1 w_t \int_0^x (3s_t - \psi_t)(y) \, dy \, dx \\ &\quad - 3\varrho \int_0^1 w_t \int_0^x s_t(y) \, dy \, dx. \end{aligned}$$

By Young's, Poincaré's and Cauchy-Schwarz inequalities, we have the right-hand side of (3.7) as follows:

$$\begin{aligned} (3.8) \quad \mu_1 \int_0^1 w_t \left(\int_0^x \psi(y) \, dy - w \right) \, dx \\ \leq \frac{G}{4} \int_0^1 \left(\int_0^x \psi(y) \, dy - w \right)^2 \, dx + \frac{\mu_1^2}{G} \int_0^1 w_t^2 \, dx \\ \leq \frac{G}{4} \int_0^1 (\psi - w_x)^2 \, dx + c \int_0^1 w_t^2 \, dx, \end{aligned}$$

$$\begin{aligned}
(3.9) \quad & \mu_2 \int_0^1 \left(\int_0^x \psi(y) \, dy - w \right) z(x, 1, t) \, dx \\
& \leq \frac{G}{4} \int_0^1 \left(\int_0^x \psi(y) \, dy - w \right)^2 \, dx + \frac{\mu_2^2}{G} \int_0^1 z^2(x, 1, t) \, dx \\
& \leq \frac{G}{4} \int_0^1 (\psi - w_x)^2 \, dx + c \int_0^1 z^2(x, 1, t) \, dx,
\end{aligned}$$

and for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,

$$\begin{aligned}
(3.10) \quad & \varrho \int_0^1 w_t \int_0^x (3s_t - \psi_t)(y) \, dy \, dx \\
& \leq \varepsilon_1 \int_0^1 \left(\int_0^x (3s_t - \psi_t)(y) \, dy \right)^2 \, dx + \frac{\varrho^2}{4\varepsilon_1} \int_0^1 w_t^2 \, dx \\
& \leq \varepsilon_1 \int_0^1 (3s_t - \psi_t)^2 \, dx + \frac{c}{\varepsilon_1} \int_0^1 w_t^2 \, dx,
\end{aligned}$$

and

$$\begin{aligned}
(3.11) \quad & -3\varrho \int_0^1 w_t \left(\int_0^x s_t(y) \, dy \right) \, dx \leq \varepsilon_2 \int_0^1 \left(\int_0^x s_t(y) \, dy \right)^2 \, dx + \frac{9\varrho^2}{4\varepsilon_2} \int_0^1 w_t^2 \, dx \\
& \leq \varepsilon_2 \int_0^1 s_t^2 \, dx + \frac{c}{\varepsilon_2} \int_0^1 w_t^2 \, dx,
\end{aligned}$$

respectively. Consequently, the estimate (3.6) follows by substituting (3.8)–(3.10) into (3.7). \square

Lemma 3.3. *If (w, ψ, s, z) is a solution of (2.3), then the functional F_2 , defined by*

$$F_2(t) := -I_\varrho \int_0^1 (3s_t - \psi_t)(3s - \psi) \, dx, \quad t \geq 0,$$

satisfies

$$\begin{aligned}
(3.12) \quad & \frac{d}{dt} F_2(t) \leq -I_\varrho \int_0^1 (3s_t - \psi_t)^2 \, dx \\
& \quad + \frac{3D}{2} \int_0^1 (3s_x - \psi_x)^2 \, dx + c \int_0^1 (\psi - w_x)^2 \, dx.
\end{aligned}$$

Proof. First we differentiate F_2 and use (2.3)₂ to obtain

$$\begin{aligned}
(3.13) \quad & \frac{d}{dt} F_2(t) = -I_\varrho \int_0^1 (3s_t - \psi_t)^2 \, dx + D \int_0^1 (3s_x - \psi_x)^2 \, dx \\
& \quad - G \int_0^1 (\psi - w_x)(3s - \psi) \, dx.
\end{aligned}$$

Exploiting Young's and Poincaré's inequality, we have

$$(3.14) \quad -G \int_0^1 (\psi - w_x)(3s - \psi) \, dx \leq \frac{G^2}{2D} \int_0^1 (\psi - w_x)^2 \, dx + \frac{D}{2} \int_0^1 (3s - \psi)^2 \, dx \\ \leq c \int_0^1 (\psi - w_x)^2 \, dx + \frac{D}{2} \int_0^1 (3s_x - \psi_x)^2 \, dx.$$

Finally, substituting (3.14) into (3.13) completes the proof. \square

Lemma 3.4. *Let (w, ψ, s, z) be a solution of (2.3). Then the functional F_3 , defined by*

$$F_3(t) := 3I_\varrho \int_0^1 s_t s \, dx + 2\beta \int_0^1 s^2 \, dx, \quad t \geq 0,$$

satisfies

$$(3.15) \quad \frac{d}{dt} F_3(t) \leq -3D \int_0^1 s_x^2 \, dx + \frac{9G^2}{4\gamma} \int_0^1 (\psi - w_x)^2 \, dx \\ - 3\gamma \int_0^1 s^2 \, dx + 3I_\varrho \int_0^1 s_t^2 \, dx.$$

Proof. Direct computations involving simple differentiation of F_3 , followed by substitution for the integral of s_{tt} using (2.3)₃, then integrating by parts the term containing $s_{xx}s$, lead to

$$(3.16) \quad \frac{d}{dt} F_3(t) = -3D \int_0^1 s_x^2 \, dx - 4\gamma \int_0^1 s^2 \, dx \\ + 3I_\varrho \int_0^1 s_t^2 \, dx - 3G \int_0^1 (\psi - w_x)s \, dx.$$

Using Young's inequality, we estimate the last terms of (3.16) as follows:

$$-3G \int_0^1 (\psi - w_x)s \, dx \leq \frac{9G^2}{4\gamma} \int_0^1 (\psi - w_x)^2 \, dx + \gamma \int_0^1 s^2 \, dx \\ \leq c \int_0^1 (\psi - w_x)^2 \, dx + \gamma \int_0^1 s^2 \, dx.$$

Consequently, we obtain (3.15), which completes the proof. \square

Lemma 3.5. *If (w, ψ, s, z) is a solution of (2.3), then the functional F_4 , defined by*

$$F_4(t) := - \int_0^1 (3s_t - \psi_t)w_x \, dx - \int_0^1 (3s_x - \psi_x)w_t \, dx + 3 \int_0^1 (3s_t - \psi_t)s \, dx, \quad t \geq 0,$$

satisfies for any $\varepsilon_3 > 0$ the estimate

$$(3.17) \quad \begin{aligned} \frac{d}{dt} F_4(t) \leq & -\frac{D}{2I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3} \int_0^1 s_t^2 dx \\ & + c \int_0^1 w_t^2 dx + c \int_0^1 (\psi - w_x)^2 dx + c \int_0^1 z^2(x, 1, t) dx \\ & - \left(\frac{D}{I_\varrho} - \frac{G}{\varrho} \right) \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx. \end{aligned}$$

P r o o f. Differentiating F_4 , making use of the first two equations in system (2.3) coupled with the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, and then integrating by parts the terms involving $(3s_{xx} - \psi_{xx})$, we arrive at

$$(3.18) \quad \begin{aligned} \frac{d}{dt} F_4(t) = & \frac{\mu_1}{\varrho} \int_0^1 (3s_x - \psi_x) w_t dx + \frac{\mu_2}{\varrho} \int_0^1 (3s_x - \psi_x) z(x, 1, t) dx \\ & + \frac{G}{I_\varrho} \int_0^1 (\psi - w_x)^2 dx + \frac{G}{I_\varrho} \int_0^1 (\psi - w_x)(3s - \psi) dx + 3 \int_0^1 (3s_t - \psi_t) s_t dx \\ & - \frac{D}{I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx - \left(\frac{D}{I_\varrho} - \frac{G}{\varrho} \right) \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx. \end{aligned}$$

Next, we exploit Young's and Poincaré's inequalities to estimate the non-square terms of (3.18),

$$\begin{aligned} \frac{\mu_1}{\varrho} \int_0^1 (3s_x - \psi_x) w_t dx & \leq \frac{D}{6I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + \frac{3I_\varrho \mu_1^2}{2D\varrho^2} \int_0^1 w_t^2 dx \\ & \leq \frac{D}{6I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + c \int_0^1 w_t^2 dx, \\ \frac{\mu_2}{\varrho} \int_0^1 (3s_x - \psi_x) z(x, 1, t) dx & \leq \frac{D}{6I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + \frac{3I_\varrho \mu_2^2}{2D\varrho^2} \int_0^1 z^2(x, 1, t) dx \\ & \leq \frac{D}{6I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + c \int_0^1 z^2(x, 1, t) dx, \\ \frac{G}{I_\varrho} \int_0^1 (\psi - w_x)(3s - \psi) dx & \leq \frac{3G^2}{2DI_\varrho} \int_0^1 (\psi - w_x)^2 dx + \frac{D}{6I_\varrho} \int_0^1 (3s - \psi)^2 dx \\ & \leq c \int_0^1 (\psi - w_x)^2 dx + \frac{D}{6I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx, \end{aligned}$$

and for $\varepsilon_3 > 0$

$$3 \int_0^1 (3s_t - \psi_t) s_t dx \leq \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3} \int_0^1 s_t^2 dx.$$

Combining the above four estimates with (3.18) concludes our proof. \square

Lemma 3.6. Let (w, ψ, s, z) be a solution of (2.3). Then the functional F_5 , defined by

$$F_5(t) := \tau \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x, \sigma, t) \, d\sigma \, dx, \quad t \geq 0,$$

satisfies, for some $m_1 > 0$, the estimate

$$(3.19) \quad \frac{d}{dt} F_5(t) \leq -m_1 \int_0^1 z^2(x, 1, t) \, dx - m_1 \tau \int_0^1 \int_0^1 z^2(x, \sigma, t) \sigma \, dx + \int_0^1 w_t^2 \, dx.$$

Proof. Differentiating F_5 and using (2.3)₄ and $z(x, 0, t) = w_t$, we get

$$\begin{aligned} \frac{d}{dt} F_5(t) &= -2 \int_0^1 \int_0^1 e^{-\tau\sigma} z(x, \sigma, t) z_\sigma(x, \sigma, t) \, d\sigma \, dx \\ &= - \int_0^1 \int_0^1 \frac{d}{d\sigma} [e^{-\tau\sigma} z^2(x, \sigma, t)] \, d\sigma \, dx - \tau \int_0^1 \int_0^1 e^{-\tau\sigma} z^2(x, \sigma, t) \, d\sigma \, dx \\ &= - \int_0^1 [e^{-\tau} z^2(x, 1, t) - z^2(x, 0, t)] \, dx - \tau \int_0^1 \int_0^1 e^{-\tau\sigma} z^2(x, \sigma, t) \, d\sigma \, dx \\ &= - \int_0^1 e^{-\tau} z^2(x, 1, t) \, dx + \int_0^1 w_t^2 \, dx - \tau \int_0^1 \int_0^1 e^{-\tau\sigma} z^2(x, \sigma, t) \, d\sigma \, dx. \end{aligned}$$

Next, exploiting the inequality $e^{-\tau} \leq e^{-\sigma\tau} \leq 1$ for any $\sigma \in (0, 1)$, for some $m_1 = e^{-\tau}$, we arrive at the estimate (3.19). \square

4. EXPONENTIAL STABILITY

Our next task is to define a Lyapunov functional \mathcal{L} and show that it is equivalent to the energy functional E .

Lemma 4.1. For $N > 0$, $N_k > 0$ ($k = 1, \dots, 5$), the Lyapunov functional defined by

$$(4.1) \quad \mathcal{L}(t) := NE(t) + \sum_{k=1}^5 N_k F_k(t), \quad t \geq 0,$$

satisfies the equivalence relation ($\mathcal{L} \sim E$)

$$(4.2) \quad c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t) \quad \forall t \geq 0$$

for some positive constants c_1 and c_2 .

Proof.

$$\begin{aligned}
 |\mathcal{L}(t) - NE(t)| &\leq \varrho N_1 \int_0^1 |ww_t| dx + \varrho N_1 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
 &\quad + I_\varrho N_2 \int_0^1 |(3s_t - \psi_t)(3s - \psi)| dx + 3I_\varrho N_3 \int_0^1 |s_t s| dx \\
 &\quad + 2\beta N_3 \int_0^1 s^2 dx + N_4 \int_0^1 |(3s_t - \psi_t)w_x| dx \\
 &\quad + N_4 \int_0^1 |(3s_x - \psi_x)w_t| dx + 3N_4 \int_0^1 |(3s_t - \psi_t)s| dx \\
 &\quad + \tau N_5 \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x, \sigma, t) d\sigma dx.
 \end{aligned}$$

Furthermore, using Young's, Poincaré's, Cauchy-Schwarz inequalities, the facts that $\psi = -(3s - \psi) + 3s$, $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, and $e^{-\sigma\tau} \leq 1$ for all $\sigma \in (0, 1)$, we obtain

$$\begin{aligned}
 |\mathcal{L}(t) - NE(t)| &\leq b \int_0^1 [w_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_t^2 + s_x^2 + s^2 + (\psi - w_x)^2] dx \\
 &\quad + b \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx,
 \end{aligned}$$

for some constant $b > 0$. Using (2.5), we obtain

$$(4.3) \quad |\mathcal{L}(t) - NE(t)| \leq b_1 E(t),$$

where $b_1 \geq \frac{1}{2} \max\{2b, \varrho, 3I_\varrho, 3D, 4\gamma, G, \tau|\mu_2|\}$. Inequality (4.3) yields

$$(N - b_1)E(t) \leq \mathcal{L}(t) \leq (N + b_1)E(t).$$

Taking N is sufficiently large, the estimate (4.2) follows accordingly. \square

At this point, we are ready to state and prove the first part of our main results.

Theorem 4.1. *Let (w, ψ, s, z) be a solution of (2.3). Then the energy functional (2.5) satisfies, for all $t \geq 0$,*

$$(4.4) \quad E(t) \leq k_0 e^{-k_1 t} \quad \text{if} \quad \frac{G}{\varrho} = \frac{D}{I_\varrho},$$

where k_0 and k_1 are positive constants.

Proof. We begin by differentiating (4.1), and then substitute the estimates (3.6), (3.12), (3.15), (3.17), and (3.19) to obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left[m_0 N - c N_1 \left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) - c N_4 - N_5 \right] \int_0^1 w_t^2 dx \\
& - 3 D N_3 \int_0^1 s_x^2 dx - 3 \gamma N_3 \int_0^1 s^2 dx \\
& - \left[4 \beta N - \varepsilon_2 N_1 - 3 I_\varrho N_3 - \frac{9 N_4}{\varepsilon_3} \right] \int_0^1 s_t^2 dx \\
& - [I_\varrho N_2 - \varepsilon_1 N_1 - \varepsilon_3 N_4] \int_0^1 (3 s_t - \psi_t)^2 dx \\
& - \left[\frac{G N_1}{2} - c N_2 - c N_3 - c N_4 \right] \int_0^1 (\psi - w_x)^2 dx \\
& - \left[\frac{D N_4}{2 I_\varrho} - \frac{3 D N_2}{2} \right] \int_0^1 (3 s_x - \psi_x)^2 dx \\
& - [m_1 N_5 - c N_1 - c N_4] \int_0^1 z^2(x, 1, t) dx - m_1 \tau N_5 \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx \\
& - N_4 \left(\frac{D}{I_\varrho} - \frac{G}{\varrho} \right) \int_0^1 (3 s_x - \psi_x)(\psi - w_x)_x dx.
\end{aligned}$$

Next, we carefully choose our constants. We begin by setting

$$N_2 = N_3 = \varepsilon_2 = 1, \quad \varepsilon_1 = \frac{I_\varrho}{3 N_1}, \quad \varepsilon_3 = \frac{I_\varrho}{3 N_4} \quad \text{and} \quad N_4 = 4 I_\varrho,$$

to arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & - [m_0 N - c_3 N_1 (1 + N_1) - c_3 - N_5] \int_0^1 w_t^2 dx \\
& - [4 \beta N - N_1 - 3 I_\varrho] \int_0^1 s_t^2 dx - \left[\frac{G N_1}{2} - c_3 \right] \int_0^1 (\psi - w_x)^2 dx \\
& - 3 D \int_0^1 s_x^2 dx - \frac{I_\varrho}{3} \int_0^1 (3 s_t - \psi_t)^2 dx - 3 \gamma \int_0^1 s^2 dx - \frac{D}{2} \int_0^1 (3 s_x - \psi_x)^2 dx \\
& - [m_1 N_5 - c_3 N_1 - c_3] \int_0^1 z^2(x, 1, t) dx - m_1 \tau N_5 \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx \\
& - N_4 \left(\frac{D}{I_\varrho} - \frac{G}{\varrho} \right) \int_0^1 (3 s_x - \psi_x)(\psi - w_x)_x dx,
\end{aligned}$$

for some $c_3 > 0$. Now, we choose N_1 large enough such that

$$\frac{G N_1}{2} - c_3 > 0.$$

Once N_1 is fixed, we proceed to choose N_5 large enough such that

$$m_1 N_5 - c_3 N_1 - c_3 > 0.$$

Lastly, we choose N so large that (4.2) remains valid, and furthermore,

$$m_0 N - c_3 N_1 (1 + N_1) - N_5 - c_3 > 0 \quad \text{and} \quad 4\beta N - N_1 - 3I_\varrho > 0.$$

Hence, for some $\alpha_0 > 0$, we end up with

$$(4.5) \quad \begin{aligned} \mathcal{L}'(t) \leq & -\alpha_0 \int_0^1 [w_t^2 + s_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + (\psi - w_x)^2] dx \\ & - \alpha_0 \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx \\ & - 4I_\varrho \alpha_0 \left(\frac{D}{I_\varrho} - \frac{G}{\varrho} \right) \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx. \end{aligned}$$

Hence, from (2.5) and the fact that $G/\varrho = D/I_\varrho$, we arrive at

$$(4.6) \quad \mathcal{L}'(t) \leq -\alpha_1 E(t) \quad \forall t > 0$$

for some $\alpha_1 > 0$. It follows directly from (4.2) and (4.6) that

$$(4.7) \quad \mathcal{L}'(t) \leq -k_1 \mathcal{L}(t) \quad \forall t > 0,$$

where $k_1 = \alpha_1/c_2$. A simple integration of (4.7) over $(0, t)$ yields

$$(4.8) \quad \mathcal{L}(t) \leq \mathcal{L}(0)e^{-k_1 t} \quad \forall t > 0.$$

Consequently, the relation (4.4) follows from (4.8) and (4.2) with $k_0 = c_2 E(0)/c_1$. □

5. POLYNOMIAL STABILITY

In this section, we consider the case of non-equal wave speeds and establish a polynomial stability result.

Theorem 5.1. *Let (w, ψ, s, z) be a strong solution of (2.3). If $G/\varrho \neq D/I_\varrho$, then the energy functional (2.5) satisfies*

$$(5.1) \quad E(t) \leq \frac{k_2}{t} \quad \forall t > 0,$$

where k_2 is a positive constant.

Proof. To prove estimate (5.1), we require the following second-order energy functional.

$$(5.2) \quad \begin{aligned} \mathcal{E}(t) = & \frac{1}{2} \int_0^1 [\varrho w_{tt}^2 + I_\varrho (3s_{tt} - \psi_{tt})^2 + D(3s_{xt} - \psi_{xt})^2 + 3I_\varrho s_{tt}^2 + 3Ds_{xt}^2 + 4\gamma s_t^2] dx \\ & + \frac{1}{2} \int_0^1 [G(\psi_t - w_{xt})^2 + \tau|\mu_2| \int_0^1 z_t^2(x, \sigma, t) d\sigma] dx. \end{aligned}$$

As in Lemma 3.1, it follows that \mathcal{E} satisfies the relation

$$(5.3) \quad \mathcal{E}'(t) \leq -m_0 \int_0^1 w_{tt}^2 dx - 4\beta \int_0^1 s_{tt}^2 dx \quad \forall t \geq 0.$$

Similarly to Lemma 4.1, we define a Lyapunov functional $\tilde{\mathcal{L}}$ as follows:

$$(5.4) \quad \tilde{\mathcal{L}}(t) = N[E(t) + \mathcal{E}(t)] + \sum_{k=1}^5 N_k F_k(t),$$

where F_k , $k = 1, \dots, 5$, and their respective derivatives remain as defined in Lemmas 3.2–3.6. It is important to note that the relation (4.2) does not hold for $\tilde{\mathcal{L}}$. To this effect, therefore, the major task ahead of us now is to find an estimate for the term

$$\left(\frac{G}{\varrho} - \frac{D}{I_\varrho}\right) \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx$$

in (3.18) and consequently a new estimate for the derivative of the functional F_4 . Let us begin by setting $\chi = (G/\varrho - D/I_\varrho)$. Using (2.3)₁, we obtain

$$(5.5) \quad \begin{aligned} \chi \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx = & -\frac{\varrho\chi}{G} \int_0^1 (3s_x - \psi_x)w_{tt} dx \\ & -\frac{\mu_1\chi}{G} \int_0^1 (3s_x - \psi_x)w_t dx \\ & -\frac{\mu_2\chi}{G} \int_0^1 (3s_x - \psi_x)z(x, 1, t) dx. \end{aligned}$$

Next, exploiting the Cauchy-Schwarz and Young's inequalities, we estimate the three terms on the right-hand side of (5.5) as follows:

$$\begin{aligned} -\frac{\varrho\chi}{G} \int_0^1 (3s_x - \psi_x)w_{tt} dx & \leq \delta \int_0^1 (3s_x - \psi_x)^2 dx + \frac{\varrho^2\chi^2}{4\delta G^2} \int_0^1 w_{tt}^2 dx, \\ -\frac{\mu_1\chi}{G} \int_0^1 (3s_x - \psi_x)w_t dx & \leq \delta \int_0^1 (3s_x - \psi_x)^2 dx + \frac{\mu_1^2\chi^2}{4\delta G^2} \int_0^1 w_t^2 dx, \\ -\frac{\mu_2\chi}{G} \int_0^1 (3s_x - \psi_x)z(x, 1, t) dx & \leq \delta \int_0^1 (3s_x - \psi_x)^2 dx + \frac{\mu_2^2\chi^2}{4\delta G^2} \int_0^1 z^2(x, 1, t) dx. \end{aligned}$$

Therefore, for some

$$c_4 \geq \max \left\{ \frac{\varrho^2 \chi^2}{4\delta G^2}, \frac{\mu_1^2 \chi^2}{4\delta G^2}, \frac{\mu_2^2 \chi^2}{4\delta G^2} \right\},$$

we have

$$(5.6) \quad \chi \int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx \leq 3\delta \int_0^1 (3s_x - \psi_x)^2 dx + c_4 \int_0^1 w_t^2 dx \\ + c_4 \int_0^1 w_{tt}^2 dx + c_4 \int_0^1 z^2(x, 1, t) dx.$$

Thus, using (5.6), choosing $\delta = D/(12I_\varrho)$ and setting $c_5 = c + c_4$, we note that the derivative of F_4 satisfies the new estimate

$$(5.7) \quad \frac{d}{dt} F_4(t) \leq -\frac{D}{4I_\varrho} \int_0^1 (3s_x - \psi_x)^2 dx + c_5 \int_0^1 w_t^2 dx + c \int_0^1 (\psi - w_x)^2 dx \\ + \frac{9}{\varepsilon_3} \int_0^1 s_t^2 dx + \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx \\ + c_4 \int_0^1 w_{tt}^2 dx + c_5 \int_0^1 z^2(x, 1, t) dx.$$

Similarly, differentiating $\tilde{\mathcal{L}}(t)$ defined in (5.4) and then substituting the estimates (3.6), (3.12), (3.15) (3.19), and (5.7), we obtain

$$\begin{aligned} \tilde{\mathcal{L}}'(t) \leq & - \left[4\beta N - \varepsilon_2 N_1 - 3I_\varrho N_3 - \frac{9N_3}{\varepsilon_3} \right] \int_0^1 s_t^2 dx \\ & - 3DN_3 \int_0^1 s_x^2 dx - 3\gamma N_3 \int_0^1 s^2 dx - [m_0 N - c_4 N_4] \int_0^1 w_{tt}^2 dx \\ & - \left[m_0 N - c_3 N_1 \left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) - c_5 N_4 - N_5 \right] \int_0^1 w_t^2 dx \\ & - \left[\frac{GN_1}{2} - c_3 N_2 - c_3 N_3 - c_3 N_5 \right] \int_0^1 (\psi - w_x)^2 dx \\ & - \left[\frac{DN_4}{4I_\varrho} - \frac{3DN_2}{2} \right] \int_0^1 (3s_x - \psi_x)^2 dx \\ & - [I_\varrho N_2 - \varepsilon_1 N_1 - \varepsilon_3 N_4] \int_0^1 (3s_t - \psi_t)^2 dx \\ & - [m_1 N_5 - cN_1 - c_5 N_4] \int_0^1 z^2(x, 1, t) dx \\ & - m_1 \tau N_5 \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx. \end{aligned}$$

With similar choices of constants N_2 , N_3 , ε_1 , ε_2 , and ε_3 as in the proof of Theorem 4.1, and setting $N_4 = 8I_\rho$ coupled with suitable choices of N_1 , N_5 , and N , we deduce that

$$\begin{aligned} \tilde{\mathcal{L}}'(t) \leq & -\alpha_2 \int_0^1 [w_t^2 + (\psi - w_x)^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + s_t^2] dx \\ & - \alpha_2 \tau \int_0^1 \int_0^1 z^2(x, \sigma, t) d\sigma dx, \end{aligned}$$

where α_2 is a positive constant. Comparing with (2.5), we have, for some $\alpha_3 > 0$,

$$(5.8) \quad \tilde{\mathcal{L}}'(t) \leq -\alpha_3 E(t) \quad \forall t > 0.$$

Since E is non-increasing, integrating (5.8) over $(0, t)$ yields

$$tE(t) \leq \int_0^t E(s) ds \leq -\frac{1}{\alpha_3} \int_0^t \tilde{\mathcal{L}}'(s) ds = \frac{1}{\alpha_3} [\tilde{\mathcal{L}}(0) - \tilde{\mathcal{L}}(t)] \leq \frac{\tilde{\mathcal{L}}(0)}{\alpha_3}.$$

Finally, for $k_2 = \tilde{\mathcal{L}}(0)/\alpha_3 = (E(0) + \mathcal{E}(0))/\alpha_3$, we have

$$E(t) \leq \frac{k_2}{t} \quad \forall t > 0,$$

which concludes the proof. □

Acknowledgements. The first and third authors appreciate the Research Management Center (RMC) of Universiti Teknologi Malaysia (UTM) for its continuous support. Likewise, the second author expresses sincere thanks to the University of Hafr Al-Batin (UHB).

References

- [1] *C. Abdallah, P. Dorato, J. Benitez-Read, R. Byrne*: Delayed positive feedback can stabilize oscillatory systems. American Control Conference (ACC). IEEE, Piscataway, 1993, pp. 3106–3107. doi
- [2] *E. M. Ait Benhassi, K. Ammari, S. Boulite, L. Maniar*: Feedback stabilization of a class of evolution equations with delay. *J. Evol. Equ.* *9* (2009), 103–121. zbl MR doi
- [3] *T. A. Apalara*: Well-posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay. *Electron. J. Differ. Equ.* *2014* (2014), Article ID 254, 15 pages. zbl MR
- [4] *T. A. Apalara*: Asymptotic behavior of weakly dissipative Timoshenko system with internal constant delay feedbacks. *Appl. Anal.* *95* (2016), 187–202. zbl MR doi
- [5] *T. A. Apalara*: Uniform decay in weakly dissipative Timoshenko system with internal distributed delay feedbacks. *Acta Math. Sci., Ser. B, Engl. Ed.* *36* (2016), 815–830. zbl MR doi

- [6] *T. A. Apalara*: Uniform stability of a laminated beam with structural damping and second sound. *Z. Angew. Math. Phys.* *68* (2017), Article ID 41, 16 pages. [zbl](#) [MR](#) [doi](#)
- [7] *T. A. Apalara*: On the stability of a thermoelastic laminated beam. *Acta Math. Sci., Ser. B, Engl. Ed.* *39* (2019), 1517–1524. [MR](#) [doi](#)
- [8] *T. A. Apalara, S. A. Messaoudi*: An exponential stability result of a Timoshenko system with thermoelasticity with second sound and in the presence of delay. *Appl. Math. Optim.* *71* (2015), 449–472. [zbl](#) [MR](#) [doi](#)
- [9] *H. Brezis*: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, New York, 2011. [zbl](#) [MR](#) [doi](#)
- [10] *X.-G. Cao, D.-Y. Liu, G.-Q. Xu*: Easy test for stability of laminated beams with structural damping and boundary feedback controls. *J. Dyn. Control Syst.* *13* (2007), 313–336. [zbl](#) [MR](#) [doi](#)
- [11] *Z. Chen, W. Liu, D. Chen*: General decay rates for a laminated beam with memory. *Taiwanese J. Math.* *23* (2019), 1227–1252. [zbl](#) [MR](#) [doi](#)
- [12] *R. Datko, J. Lagnese, M. P. Polis*: An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* *24* (1986), 152–156. [zbl](#) [MR](#) [doi](#)
- [13] *B. Feng*: Well-posedness and exponential decay for laminated Timoshenko beams with time delays and boundary feedbacks. *Math. Methods Appl. Sci.* *41* (2018), 1162–1174. [zbl](#) [MR](#) [doi](#)
- [14] *E. Fridman, S. Nicaise, J. Valein*: Stabilization of second order evolution equations with unbounded feedback with time-dependent delay. *SIAM J. Control Optim.* *48* (2010), 5028–5052. [zbl](#) [MR](#) [doi](#)
- [15] *S. W. Hansen, R. D. Spies*: Structural damping in laminated beams due to interfacial slip. *J. Sound Vib.* *204* (1997), 183–202. [doi](#)
- [16] *M. Kirane, B. Said-Houari*: Existence and asymptotic stability of a viscoelastic wave equation with a delay. *Z. Angew. Math. Phys.* *62* (2011), 1065–1082. [zbl](#) [MR](#) [doi](#)
- [17] *V. Komornik, E. Zuazua*: A direct method for the boundary stabilization of the wave equation. *J. Math. Pures Appl., IX. Sér.* *69* (1990), 35–54. [zbl](#) [MR](#)
- [18] *I. Lasiecka*: Global uniform decay rates for the solutions to wave equation with nonlinear boundary conditions. *Appl. Anal.* *47* (1992), 191–212. [zbl](#) [MR](#) [doi](#)
- [19] *Z. Liu, S. Zheng*: *Semigroups Associated with Dissipative Systems*. Chapman & Hall/CRC Research Notes in Mathematics 398. Chapman & Hall/CRC, Boca Raton, 1999. [zbl](#) [MR](#)
- [20] *A. Lo, N.-E. Tatar*: Stabilization of laminated beams with interfacial slip. *Electron. J. Differ. Equ.* *2015* (2015), Article ID 129, 14 pages. [zbl](#) [MR](#)
- [21] *A. Lo, N.-E. Tatar*: Uniform stability of a laminated beam with structural memory. *Qual. Theory Dyn. Syst.* *15* (2016), 517–540. [zbl](#) [MR](#) [doi](#)
- [22] *M. I. Mustafa*: Uniform stability for thermoelastic systems with boundary time-varying delay. *J. Math. Anal. Appl.* *383* (2011), 490–498. [zbl](#) [MR](#) [doi](#)
- [23] *M. I. Mustafa*: Boundary control of laminated beams with interfacial slip. *J. Math. Phys.* *59* (2018), Article ID 051508, 9 pages. [zbl](#) [MR](#) [doi](#)
- [24] *M. I. Mustafa*: Laminated Timoshenko beams with viscoelastic damping. *J. Math. Anal. Appl.* *466* (2018), 619–641. [zbl](#) [MR](#) [doi](#)
- [25] *S. Nicaise, C. Pignotti*: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.* *45* (2006), 1561–1585. [zbl](#) [MR](#) [doi](#)
- [26] *S. Nicaise, C. Pignotti*: Interior feedback stabilization of wave equations with time dependent delay. *Electron. J. Differ. Equ.* *2011* (2011), Article ID 41, 20 pages. [zbl](#) [MR](#)
- [27] *S. Nicaise, C. Pignotti, J. Valein*: Exponential stability of the wave equation with boundary time-varying delay. *Discrete Contin. Dyn. Syst., Ser. S* *4* (2011), 693–722. [zbl](#) [MR](#) [doi](#)
- [28] *S. Nicaise, J. Valein, E. Fridman*: Stability of the heat and of the wave equations with boundary time-varying delays. *Discrete Contin. Dyn. Syst., Ser. S* *2* (2009), 559–581. [zbl](#) [MR](#) [doi](#)

- [29] *C. Pignotti*: A note on stabilization of locally damped wave equations with time delay. *Syst. Control Lett.* *61* (2012), 92–97. [zbl](#) [MR](#) [doi](#)
- [30] *C. A. Raposo*: Exponential stability for a structure with interfacial slip and frictional damping. *Appl. Math. Lett.* *53* (2016), 85–91. [zbl](#) [MR](#) [doi](#)
- [31] *B. Said-Houari, Y. Laskri*: A stability result of a Timoshenko system with a delay term in the internal feedback. *Appl. Math. Comput.* *217* (2010), 2857–2869. [zbl](#) [MR](#) [doi](#)
- [32] *N.-E. Tatar*: Stabilization of a laminated beam with interfacial slip by boundary controls. *Bound. Value Probl.* *2015* (2015), Article ID 169, 11 pages. [zbl](#) [MR](#) [doi](#)
- [33] *J.-M. Wang, G.-Q. Xu, S.-P. Yung*: Exponential stabilization of laminated beams with structural damping and boundary feedback controls. *SIAM J. Control Optim.* *44* (2005), 1575–1597. [zbl](#) [MR](#) [doi](#)
- [34] *G. Q. Xu, S. P. Yung, L. K. Li*: Stabilization of wave systems with input delay in the boundary control. *ESAIM, Control Optim. Calc. Var.* *12* (2006), 770–785. [zbl](#) [MR](#) [doi](#)
- [35] *Q. Zhang, M. Ran, D. Xu*: Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay. *Appl. Anal.* *96* (2017), 1867–1884. [zbl](#) [MR](#) [doi](#)
- [36] *E. Zuazua*: Uniform stabilization of the wave equation by nonlinear boundary feedback. *SIAM J. Control Optim.* *28* (1990), 466–477. [zbl](#) [MR](#) [doi](#)

Authors' addresses: *Kassimu Mpungu* (corresponding author), Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia (UTM), Johor Bahru 81310, Johor, Malaysia, e-mail: mpungu@graduate.utm.my; *Tijani A. Apalara*, Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin 31991, Saudi Arabia, e-mail: tijani@uhb.edu.sa; *Mukhiddin Muminov*, Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia (UTM), Johor Bahru 81310, Johor, Malaysia, e-mail: mukhiddin@utm.my.