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OSCILLATORY AND NON-OSCILLATORY CRITERIA FOR LINEAR FOUR-DIMENSIONAL HAMILTONIAN SYSTEMS

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Abstract. The Riccati equation method is used to study the oscillatory and non-oscillatory behavior of solutions of linear four-dimensional Hamiltonian systems. One oscillatory and three non-oscillatory criteria are proved. Examples of the obtained results are compared with some well known ones.

Keywords: Riccati equation; oscillation; non-oscillation; conjoined (prepared, preferred) solution; Liouville's formula

MSC 2020: 34C10

1. Introduction

Let $A(t) \equiv (a_{jk}(t))^2_{j,k=1}$, $B(t) \equiv (b_{jk}(t))^2_{j,k=1}$, $C(t) \equiv (c_{jk}(t))^2_{j,k=1}$, $t \geqslant t_0$, be complex-valued continuous matrix functions on $[t_0, \infty)$ and let $B(t)$ and $C(t)$ be Hermitian, i.e. $B(t) = B^*(t)$, $C(t) = C^*(t)$, $t \geqslant t_0$. Consider the four-dimensional Hamiltonian system

$$\begin{cases} \varphi' = A(t)\varphi + B(t)\psi, \\ \psi' = C(t)\varphi - A^*(t)\psi, \quad t \geqslant t_0. \end{cases} \tag{1.1}$$

Here $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$ are unknown continuously differentiable vector functions on $[t_0, \infty)$. Along with the system (1.1) consider the linear system of matrix equations

$$\begin{cases} \Phi' = A(t)\Phi + B(t)\Psi, \\ \Psi' = C(t)\Phi - A^*(t)\Psi, \quad t \geqslant t_0, \end{cases} \tag{1.2}$$

where $\Phi(t)$ and $\Psi(t)$ are unknown continuously differentiable matrix functions of dimension 2×2 on $[t_0, \infty)$.

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Definition 1.1. A solution $(\Phi(t), \Psi(t))$ of the system (1.2) is called *conjoined* (or *prepared, preferred*) if $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \geqslant t_0$.

Definition 1.2. A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called *oscillatory* if $\det \Phi(t)$ has arbitrarily large zeros.

Definition 1.3. The system (1.1) is called *oscillatory* if all conjoined solutions of the system (1.2) are oscillatory, otherwise it is called *non-oscillatory.*

The study of the oscillatory and non-oscillatory behavior of Hamiltonian systems (in particular of the system (1.1)) is an important problem of qualitative theory of differential equations and many works are devoted to it (see, e.g., [1], [4], [11]–[14], [16], [18]–[20] and works cited therein). For any Hermitian matrix H, we denote by $H \geqslant 0$, $H > 0$, its nonnegative (positive) definiteness. In the works [1], [4], [12]–[14], [16], [18]–[20], the oscillatory behavior of general Hamiltonian systems is studied under the condition that the coefficient corresponding to $B(t)$ is assumed to be positive definite. In this paper we study the oscillatory and non-oscillatory behavior of the system (1.1) in the case where the assumption $B(t) > 0$, $t \geqslant t_0$, may be violated.

2. Auxiliary propositions

Let $f(t)$, $g(t)$, $h(t)$, $h_1(t)$ be real-valued continuous functions on $[t_0, \infty)$. Consider the Riccati equations

$$y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \geqslant t_0, \tag{2.1}$$

$$y' + f(t)y^2 + g(t)y + h_1(t) = 0, \quad t \geqslant t_0. \tag{2.2}$$

Theorem 2.1. *Let equation* (2.2) *have a real-valued solution* $y_1(t)$ *on* $[t_1, t_2)$ $(t_0 \leqslant t_1 < t_2 \leqslant \infty)$, *and let* $f(t) \geqslant 0$, $h(t) \leqslant h_1(t)$, $t \in [t_1, t_2)$. *Then for each* $y_{(0)} \geqslant y_1(t_0)$ *equation* (2.1) *has a solution* $y_0(t)$ *on* $[t_1, t_2)$ *with* $y_0(t_0) = y_{(0)}$, *and* $y_0(t) \geqslant y_1(t)$, $t \in [t_1, t_2)$.

Proof. A proof for a more general theorem is presented in [6] (see also [7]). □

Denote

$$I_{g,h}(\xi; t) \equiv \int_\xi^t \exp\biggl(-\int_\tau^t g(s)\,\mathrm{d}s\biggr) h(\tau)\,\mathrm{d}\tau, \quad t \geqslant \xi \geqslant t_0.$$

Let $t_0 < \tau_0 \leqslant \infty$ and let $t_0 < t_1 < \ldots$ be a finite or infinite sequence such that $t_k \in [t_0, \tau_0]$, $k = 1, 2, \ldots$ We assume that if $\{t_k\}$ is finite then the maximum of t_k is equal to τ_0 and if $\{t_k\}$ is infinite then $\lim\limits_{k\to\infty} t_k = \tau_0$.

Theorem 2.2. *Let $f(t) \geqslant 0$, $t \in [t_0, \tau_0)$, and*

$$\int_{t_k}^{t} \exp\Big(\int_{t_k}^{\tau} (g(s) - I_{g,h}(t_k; s))\,\mathrm{d}s\Big) h(\tau)\,\mathrm{d}\tau \leqslant 0, \quad t \in [t_k, t_{k+1}),\ k = 0, 1, \ldots$$

Then for every $y_{(0)} \geqslant 0$ equation (2.1) *has a solution $y_0(t)$ on $[t_0, \tau_0)$ satisfying the initial condition $y_0(t_0) = y_{(0)}$ and $y_0(t) \geqslant 0$, $t \in [t_0, \tau_0)$.*

P r o o f. See the proof in [7]. □

Consider the matrix Riccati equation

$$Z' + ZB(t)Z + A^*(t)Z + ZA(t) - C(t) = 0, \quad t \geqslant t_0. \tag{2.3}$$

The solutions $Z(t)$ of this equation existing on an interval $[t_1, t_2)$ ($t_0 \leqslant t_1 < t_2 \leqslant \infty$) are connected with solutions $(\varphi(t), \Psi(t))$ of the system (1.2) by the following relations (see [11]):

$$\Phi'(t) = (A(t) + B(t)Z(t))\Phi(t), \quad \Phi(t_1) \neq 0,\ \Psi(t) = Z(t)\Phi(t),\ t \in [t_1, t_2). \tag{2.4}$$

Let $Z_0(t)$ be a solution to equation (2.3) on $[t_1, t_2)$.

Definition 2.1. We say that $[t_1, t_2)$ is the *maximum existence interval for* $Z_0(t)$ if $Z_0(t)$ cannot be continued to the right of t_2 as a solution of equation (2.3).

Lemma 2.1. *Let $Z_0(t)$ be a solution of equation* (2.3) *on $[t_1, t_2)$ and let $t_2 < \infty$. Then $[t_1, t_2)$ cannot be the maximum existence interval for $Z_0(t)$ provided the function $G(t) \equiv \int_{t_1}^{t} \operatorname{tr}(B(\tau)Z_0(\tau))\,\mathrm{d}\tau$, $t \in [t_1, t_2)$, is bounded from below on $[t_1, t_2)$.*

P r o o f. The proof is similar to that of Lemma 2.1 in [11]. □

Assume $B(t) = \operatorname{diag}\{b_1(t), b_2(t)\}$, $t \geqslant t_0$. Then it is not difficult to verify that for Hermitian unknowns $Z = \begin{pmatrix} z_{11} & z_{12} \\ \overline{z}_{12} & z_{22} \end{pmatrix}$, equation (2.3) is equivalent to the following nonlinear system:

$$\begin{cases} z_{11}' + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} + b_2(t)|z_{12}|^2 \\ \qquad + a_{21}(t)z_{12} + \bar{a}_{21}(t)\overline{z}_{12} - c_{11}(t) = 0, \\ z_{12}' + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))z_{12} \\ \qquad + a_{12}(t)z_{11} + a_{21}(t)z_{22} - c_{12}(t) = 0, \\ z_{22}' + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} + b_1(t)|z_{12}|^2 \\ \qquad + \bar{a}_{12}(t)z_{12} + a_{12}(t)\overline{z}_{12} - c_{22}(t) = 0, \end{cases} \quad t \geqslant t_0. \tag{2.5}$$

If $b_2(t) \neq 0$, $t \geqslant t_0$, then it is not difficult to verify that the first equation of the system (2.5) can be rewritten in the form

$$
\begin{aligned}
&z'_{11} + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} \\
&\qquad + b_2(t)\Big|z_{12} + \frac{\bar{a}_{21}(t)}{b_2(t)}\Big|^2 - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0, \quad t \geqslant t_0,
\end{aligned}
\tag{2.6}
$$

and if, in addition, $\bar{a}_{21}(t)/b_2(t)$ is continuously differentiable on $[t_0, \infty)$, then by the substitution

$$
z_{12} = y - \frac{\bar{a}_{21}(t)}{b_2(t)}, \quad t \geqslant t_0,
\tag{2.7}
$$

in the first and second equations of the system (2.5), we get the subsystem

$$
\begin{cases}
z'_{11} + b_1(t)z_{11}^2 + 2\operatorname{Re} a_{11}(t)z_{11} + b_2(t)|y|^2 - \dfrac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0, \\
y' + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))y \\
\qquad + \Big(a_{12}(t) - \dfrac{b_1(t)}{b_2(t)}\bar{a}_{21}(t)\Big)z_{11} - \Big(\dfrac{\bar{a}_{21}(t)}{b_2(t)}\Big)' \\
\qquad - \dfrac{\bar{a}_{21}(t)}{b_2(t)}(\bar{a}_{11}(t) + a_{22}(t)) - c_{12}(t) = 0, \qquad t \geqslant t_0.
\end{cases}
\tag{2.8}
$$

Analogously, if $b_1(t) \neq 0$, $t \geqslant t_0$, then the third equation of the system (2.5) can be rewritten in the form

$$
\begin{aligned}
&z'_{22} + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} \\
&\qquad + b_1(t)\Big|z_{12} + \frac{a_{12}(t)}{b_1(t)}\Big|^2 - \frac{|a_{12}(t)|^2}{b_1(t)} - c_{22}(t) = 0, \quad t \geqslant t_0,
\end{aligned}
\tag{2.9}
$$

and if, in addition, $a_{12}(t)/b_1(t)$ is continuously differentiable on $[t_0, \infty)$, then by the substitution

$$
z_{12} = v - \frac{a_{12}(t)}{b_1(t)}, \quad t \geqslant t_0,
\tag{2.10}
$$

in the second and third equations of the system (2.5) we obtain the subsystem

$$
\begin{cases}
z'_{22} + b_2(t)z_{22}^2 + 2\operatorname{Re} a_{22}(t)z_{22} \\
\qquad + b_1(t)|v|^2 - \dfrac{|a_{12}(t)|^2}{b_1(t)} - c_{22}(t) = 0, \\
v' + (b_1(t)z_{11} + b_2(t)z_{22} + \bar{a}_{11}(t) + a_{22}(t))v \\
\qquad + \Big(\bar{a}_{21}(t) - \dfrac{b_2(t)}{b_1(t)}a_{12}(t)\Big)z_{22} - \Big(\dfrac{a_{12}(t)}{b_1(t)}\Big)' \\
\qquad - \dfrac{a_{12}(t)}{b_1(t)}(\bar{a}_{11}(t) + a_{22}(t)) - c_{12}(t) = 0, \qquad t \geqslant t_0.
\end{cases}
\tag{2.11}
$$

If $(z_{11}(t), y(t))$ is a solution of the subsystem (2.8) on $[t_0, t_1)$ $(t_0 < t_1 \leqslant \infty)$ with $y(t_0) = 0$ and $(z_{22}(t), v(t))$ is a solution of the subsystem (2.11) on $[t_0, t_1)$ with $v(t_0) = 0$, then by Cauchy formula from the second equation of the subsystem (2.8) and from the second equation of the subsystem (2.11), we have, respectively,

$$\begin{aligned} y(t) = & -\exp\Big(-\int_{t_0}^t b_1(\tau)z_{11}(\tau)\,\mathrm{d}\tau\Big)\int_{t_0}^t \Big(\exp\Big(\int_{t_0}^{\tau} b_1(s)z_{11}(s)\,\mathrm{d}s\Big)\Big)' \\ & \times \Big(\frac{a_{12}(\tau)}{b_1(\tau)} - \frac{\bar a_{21}(\tau)}{b_2(\tau)}\Big)\exp\Big(-\int_{\tau}^t (b_2(s)z_{22}(s) + \bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ & + \int_{t_0}^t \exp\Big(-\int_{\tau}^t (b_1(s)z_{11}(s) + b_2(s)z_{22}(s) + \bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{\bar a_{21}(\tau)}{b_2(\tau)}\Big)' + \frac{\bar a_{21}(\tau)}{b_2(\tau)}(\bar a_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\,\mathrm{d}\tau, \\ v(t) = & -\exp\Big(-\int_{t_0}^t b_2(\tau)z_{22}(\tau)\,\mathrm{d}\tau\Big)\int_{t_0}^t \Big(\exp\Big(\int_{t_0}^{\tau} b_2(s)z_{22}(s)\,\mathrm{d}s\Big)\Big)' \\ & \times \Big(\frac{\bar a_{21}(\tau)}{b_2(\tau)} - \frac{a_{12}(\tau)}{b_1(\tau)}\Big)\exp\Big(-\int_{\tau}^t (b_1(s)z_{11}(s) + \bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\,\mathrm{d}\tau \\ & + \int_{t_0}^t \exp\Big(-\int_{\tau}^t (b_1(s)z_{11}(s) + b_2(s)z_{22}(s) + \bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)}\Big)' + \frac{a_{12}(\tau)}{b_1(\tau)}(\bar a_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\,\mathrm{d}\tau, \quad t \in [t_0, t_1). \end{aligned}$$

From here it is easy to derive the following lemma.

Lemma 2.2. *Let $b_j(t) > 0$, $j = 1, 2$, let the functions $a_{12}(t)/b_1(t)$, $\bar a_{21}(t)/b_2(t)$ be continuously differentiable on $[t_0, t_1)$ $(t_0 < t_1 < \infty)$ and let $(z_{11}(t), y(t))$ and $(z_{22}(t), v(t))$ be solutions of the subsystems (2.8) and (2.11), respectively, on $[t_0, t_1)$ such that $z_{jj}(t) \geqslant 0$, $t \in [t_0, t_1)$, $j = 1, 2$, $y(t_0) = v(t_0) = 0$. Then*

$$\begin{aligned} |y(t)| \leqslant \mathfrak{M}(t) + \int_{t_0}^t \Big| & \exp\Big(-\int_{\tau}^t (\bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{\bar a_{21}(\tau)}{b_2(\tau)}\Big)' + \frac{\bar a_{21}(\tau)}{b_2(\tau)}(\bar a_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \\ |v(t)| \leqslant \mathfrak{M}(t) + \int_{t_0}^t \Big| & \exp\Big(-\int_{\tau}^t (\bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\ & \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)}\Big)' + \frac{a_{12}(\tau)}{b_1(\tau)}(\bar a_{11}(\tau) + a_{22}(\tau)) + c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \quad t \in [t_0, t_1), \end{aligned}$$

where

$$\mathfrak{M}(t) \equiv \max_{\tau \in [t_0, t]} \Big|\exp\Big(-\int_{\tau}^t (\bar a_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big)\Big(\frac{a_{12}(\tau)}{b_1(\tau)} - \frac{\bar a_{21}(\tau)}{b_2(\tau)}\Big)\Big|, \quad t \geqslant t_0.$$

Lemma 2.3. *For any two square matrices* $M_1 \equiv (m^1_{ij})^n_{ij=1}$, $M_2 \equiv (m^2_{ij})^n_{ij=1}$ *the equality*

$$\operatorname{tr}(M_1M_2) = \operatorname{tr}(M_2M_1)$$

is valid.

P r o o f. We have $\operatorname{tr}(M_1M_2) = \sum\limits_{j=1}^n\Bigl(\sum\limits_{k=1}^n m^1_{jk}m^2_{kj}\Bigr) = \sum\limits_{k=1}^n\Bigl(\sum\limits_{j=1}^n m^1_{jk}m^2_{kj}\Bigr) = \sum\limits_{k=1}^n\Bigl(\sum\limits_{j=1}^n m^2_{kj}m^1_{jk}\Bigr) = \operatorname{tr}(M_2M_1)$. The lemma is proved. □

3. Main results

Let $f_{jk}(t)$, $j,k = 1,2$, $t \geqslant t_0$, be real-valued continuous functions on $[t_0,\infty)$. Consider the linear system of equations

$$\begin{cases} \varphi_1' = f_{11}(t)\varphi_1 + f_{12}(t)\psi_1, \\ \psi_1' = f_{21}(t)\varphi_1 + f_{22}(t)\psi_1, \quad t \geqslant t_0, \end{cases} \tag{3.1}$$

and the Riccati equation

$$y' + f_{12}(t)y^2 + (f_{11}(t) - f_{22}(t))y - f_{12}(t) = 0, \quad t \geqslant t_0. \tag{3.2}$$

All solutions $y(t)$ of the last equation, existing on some interval $[t_1, t_2)$ ($t_0 \leqslant t_1 < t_2 \leqslant \infty$), are connected with solutions $(\varphi_1(t), \psi_1(t))$ of the system (3.1) by the following relations (see [8]):

$$\begin{aligned} &\varphi_1(t) = \varphi_1(t_1)\exp\biggl(\int_{t_1}^t (f_{12}(\tau)y(\tau) + f_{11}(\tau))\,\mathrm{d}\tau\biggr), \quad \varphi_1(t_1) \neq 0, \\ &\psi_1(t) = y(t)\varphi_1(t), \quad t \in [t_1, t_2). \end{aligned} \tag{3.3}$$

Definition 3.1. The system (3.1) is called *oscillatory* if for its every solution $(\varphi_1(t), \psi_1(t))$ the function $\varphi_1(t)$ has arbitrarily large zeros.

R e m a r k 3.1. Some explicit oscillatory criteria for the system (3.1) are proved in [10] and [11].

3.1. The case where $B(t)$ is a diagonal matrix. In this subsection we will assume that $B(t) = \operatorname{diag}\{b_1(t), b_2(t)\}$. Denote:

$$\chi_j(t) \equiv \begin{cases} c_{jj}(t) & \text{if } b_{3-j}(t) = 0, \\ c_{jj}(t) + \dfrac{|a_{3-j,j}(t)|^2}{b_{3-j}(t)} & \text{if } b_{3-j}(t) \neq 0, \end{cases} \qquad t \geqslant t_0,\ j = 1,2.$$

Theorem 3.1. *Assume $b_j(t) \geqslant 0$, $t \geqslant t_0$, and if $b_j(t) = 0$ then $a_{3-j,j}(t) = 0$, $j = 1, 2$, $t \geqslant t_0$. Under these restrictions, the system* (1.1) *is oscillatory provided one of the systems*

$$\begin{cases} \varphi_1' = 2\operatorname{Re}(a_{jj}(t))\varphi_1 + b_j(t)\psi_1, \\ \psi_1' = -\chi_j(t)\varphi_1, & t \geqslant t_0, \end{cases} \tag{3.4$_j$}$$

$j = 1, 2$, is oscillatory.

P r o o f. Suppose the system (1.1) is not oscillatory. Then for some conjoined solution $(\Phi(t), \Psi(t))$ of the system (1.2), there exists $t_1 \geqslant t_0$ such that $\det\Phi(t) \neq 0$, $t \geqslant t_1$. Due to (2.4), it follows that $Z(t) \equiv \Psi(t)\Phi^{-1}(t)$, $t \geqslant t_1$, is a Hermitian solution to equation (2.3) on $[t_1, \infty)$. Let $Z(t) = \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ \overline{z}_{12}(t) & z_{22}(t) \end{pmatrix}$, $t \geqslant t_1$. Consider the Riccati equations

$$\begin{aligned} y' + b_1(t)y^2 + 2(\operatorname{Re} a_{11}(t))y + b_2(t)|z_{12}(t)|^2 \\ + a_{21}(t)z_{12}(t) + \bar{a}_{21}(t)\overline{z}_{12}(t) - c_{11}(t) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} y' + b_2(t)y^2 + 2(\operatorname{Re} a_{22}(t))y + b_1(t)|z_{12}(t)|^2 \\ + \bar{a}_{12}(t)z_{12}(t) + a_{12}(t)\overline{z}_{12}(t) - c_{22}(t) = 0, \end{aligned} \tag{3.6}$$

$$y' + b_j(t)y^2 + 2(\operatorname{Re} a_{jj}(t)y + \chi_j(t) = 0, \quad j = 1, 2, \ t \geqslant t_1. \tag{3.7$_j$}$$

By (2.6) and (2.9), from the conditions of the theorem it follows that

$$\begin{aligned} \chi_1(t) &\leqslant b_2(t)|z_{12}(t)|^2 + a_{21}(t)z_{12}(t) + \bar{a}_{21}(t)\overline{z}_{12}(t) - c_{11}(t), \quad t \geqslant t_1, \\ \chi_2(t) &\leqslant b_1(t)|z_{12}(t)|^2 + \bar{a}_{12}(t)z_{12}(t) + a_{12}(t)\overline{z}_{12}(t) - c_{22}(t), \quad t \geqslant t_1. \end{aligned}$$

Using Theorem 2.1 to the pairs of equations (3.5), (3.7$_1$) and (3.6), (3.7$_2$) we conclude that the equations (3.7$_j$), $j = 1, 2$, have solutions on $[t_1, \infty)$. By (3.1)–(3.3), it follows that the systems (3.4$_j$), $j = 1, 2$, are not oscillatory, which contradicts the condition of the theorem. The obtained contradiction completes the proof of the theorem. □

Denote $I_j(\xi; t) \equiv \int_\xi^t \exp\big(-\int_\tau^t 2(\operatorname{Re} a_{jj}(s))\,\mathrm{d}s\big)\chi_j(\tau)\,\mathrm{d}\tau$, $t \geqslant \xi \geqslant t_0$, $j = 1, 2$.

Theorem 3.2. *Assume $b_1(t) \geqslant 0$ ($\leqslant 0$), $b_2(t) \leqslant 0$ ($\geqslant 0$), and if $b_j(t) = 0$ then $a_{j,3-j}(t) = 0$, $j = 1, 2$, $t \geqslant t_0$; in addition, assume there exist infinitely large sequences $\xi_{j,0} = t_0 < \xi_{j,1} < \ldots < \xi_{j,m} < \ldots$, $j = 1, 2$, such that*

$$(-1)^j \int_{\xi_{j,m}}^t \exp\biggl(\int_{\xi_{j,m}}^\tau (2\operatorname{Re} a_{jj}(s) - (-1)^j I_j(\xi_{j,m}, s))\,\mathrm{d}s\biggr)\chi_j(\tau)\,\mathrm{d}\tau \geqslant 0 \ (\leqslant 0), \tag{1$_j$}$$

$t \in [\xi_{j,m}, \xi_{j,m+1})$, $m = 1, 2, 3, \ldots$, $j = 1, 2$. Then the system (1.1) *is non-oscillatory.*

P r o o f. Let us prove the theorem only for the case $b_1(t) \geqslant 0$, $b_2(t) \leqslant 0$, $t \geqslant t_0$. The case $b_1(t) \leqslant 0$, $b_2(t) \geqslant 0$, $t \geqslant t_0$, can be proved analogously. Let $(\Phi(t), \Psi(t))$ be a conjoined solution of the system (1.2) with $\Phi(t_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let $[t_0, T)$ be the maximum interval such that $\det \Phi(t) \neq 0$, $t \in [t_0, T)$. Then by (2.4) the matrix function $Z(t) \equiv \Psi(t)\varphi^{-1}(t)$, $t \in [t_0, T)$, is a Hermitian solution to equation (2.3) on $[t_0, T)$. By (2.5), (2.7), (2.8), (2.10), (2.11) it follows that the subsystems (2.8) and (2.11) have solutions $(z_{11}(t), y(t))$ and $(z_{22}(t), v(t))$, respectively, on $[t_0, T)$ with $z_{11}(t_0) = 1$, $z_{22}(t_0) = -1$. We wish to show that

$$z_{11}(t) \geqslant 0, \quad t \in [t_0, T). \tag{3.8}$$

Consider the Riccati equations

$$z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + b_2(t)|y(t)|^2 + \chi_1(t) = 0, \quad t \in [t_0, T), \tag{3.9}$$

$$z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + \chi_1(t) = 0, \quad t \in [t_0, T). \tag{3.10}$$

By Theorem 2.2, it follows that the last equation has a nonnegative solution on $[t_0, T)$. Then using Theorem 2.1 to the pair of equations (3.9), (3.10) we conclude that equation (3.9) has a nonnegative solution $z_0(t)$ on $[t_0, T)$ with $z_0(t_0) = 0$. Then, since $z_{11}(t)$ is a solution to equation (3.9) on $[t_0, T)$ and $z_{11}(t_0) = 1$, we have (3.8). To show that

$$z_{22}(t) \leqslant 0, \quad t \in [t_0, T), \tag{3.11}$$

consider the Riccati equations

$$z' - b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z - \chi_2(t) = 0, \quad t \in [t_0, T), \tag{3.12}$$

$$z' - b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z - b_1(t)|v(t)|^2 - \chi_2(t) = 0, \quad t \in [t_0, T). \tag{3.13}$$

By Theorem 2.2 it follows that equation (3.12) has a nonnegative solution $z_1(t)$ on $[t_0, T)$ with $z_1(t_0) = 0$. Then using Theorem 2.1 to the pair of equations (3.12) and (3.13) we derive that equation (3.13) has a nonnegative solution $z_2(t)$ on $[t_0, T)$ with $z_2(t_0) = 0$. Hence, since obviously $-z_{22}(t)$ is a solution of equation (3.13) on $[t_0, T)$ and $-z_{11}(t_0) = 1$, we have (3.11). Since $b_1(t) \geqslant 0$, $b_2(t) \leqslant 0$, $t \in [t_0, T)$, from (3.8) and (3.11) it follows that

$$\int_{t_0}^{t} (b_1(\tau)z_{11}(\tau) + b_2(\tau)z_{22}(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T). \tag{3.14}$$

To complete the proof of the theorem, it remains to show that $T=\infty$. Suppose $T<\infty$. Then, by virtue of Lemma 2.1, from (3.14) it follows that $[t_0,T)$ is not the maximum existence interval for $Z(t)$. By (2.4), it follows that $\det\Phi(t)\neq 0$, $t\in[t_0,T_1)$ for some $T_1>T$. We have obtained a contradiction, which completes the proof of the theorem. □

R e m a r k 3.2. The conditions (1_j), $j=1,2$, are satisfied if in particular $(-1)^j\chi_j(t)\geqslant 0$ $(\leqslant 0)$, $t\geqslant t_0$.

Denote:

$$\begin{aligned}
\chi_3(t)\equiv b_2(t)\Big(&\mathfrak{M}(t)+\int_{t_0}^t\Big|\exp\Big(-\int_\tau^t(\bar a_{11}(s)+a_{22}(s))\,\mathrm{d}s\Big)\\
&\times\Big(\Big(\frac{\bar a_{21}(t)}{b_2(t)}\Big)'+\frac{\bar a_{21}(\tau)}{b_2(\tau)}(\bar a_{11}(\tau)+a_{22}(\tau))+c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau\Big)^2\\
&-\frac{|a_{21}(t)|^2}{b_2(t)}-c_{11}(t),\\
\chi_4(t)\equiv b_1(t)\Big(&\mathfrak{M}(t)+\int_{t_0}^t\Big|\exp\Big(-\int_\tau^t(\bar a_{11}(s)+a_{22}(s))\,\mathrm{d}s\Big)\\
&\times\Big(\Big(\frac{a_{12}(t)}{b_1(t)}\Big)'+\frac{a_{12}(\tau)}{b_1(\tau)}(\bar a_{11}(\tau)+a_{22}(\tau)\Big)+c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau\Big)^2\\
&-\frac{|a_{12}(t)|^2}{b_1(t)}-c_{22}(t),\\
I_{j+2}(\xi;t)\equiv&\int_\xi^t\exp\Big(-\int_\tau^t 2(\operatorname{Re}a_{jj}(s))\,\mathrm{d}s\Big)\chi_{j+2}(\tau)\,\mathrm{d}\tau,\quad t\geqslant\xi\geqslant t_0,\ j=1,2.
\end{aligned}$$

Theorem 3.3. *Let the following conditions be satisfied:*

(1) $b_j(t)>0, t\geqslant t_0$, $j=1,2$;

(2) *the functions* $a_{12}(t)/b_1(t)$ *and* $\bar a_{21}(t)/b_2(t)$ *are continuously differentiable on* $[t_0,\infty)$*;*

(3) *there exist infinitely large sequences* $\xi_{j,0}=t_0<\xi_{j,1}<\ldots<\xi_{j,m}<\ldots$, $j=1,2$, *such that*

$$\int_{\xi_{j,m}}^t\exp\Big(\int_{\xi_{j,m}}^\tau(2\operatorname{Re}a_{jj}(s)-I_{j+2}(\xi_{j,m},s))\,\mathrm{d}s\Big)\chi_{j+2}(\tau)\,\mathrm{d}\tau\leqslant 0,\quad t\in[\xi_{j,m},\xi_{j,m+1}),$$

$m=1,2,3,\ldots$, $j=1,2$.

Then the system (1.1) *is non-oscillatory.*

P r o o f. Let $Z(t) \equiv \begin{pmatrix} z_{11}(t) & z_{12}(t) \\ \overline{z}_{12}(t) & z_{22}(t) \end{pmatrix}$ be the Hermitian solution of equation (2.3) on $[t_0, T)$ satisfying the initial condition $Z(t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $[t_0, T)$ is the maximum existence interval for $Z(t)$. Due to (2.4), to prove the theorem it is enough to show that

$$T = \infty. \tag{3.15}$$

By (2.5), (2.7), (2.8), (2.10), (2.11), from conditions (1) and (2), it follows that $(z_{11}(t), z_{12}(t) + \bar{a}_{21}(t)/b_2(t))$ and $(z_{22}(t), z_{12}(t) + a_{12}(t)/b_1(t))$ are solutions of the subsystems (2.8) and (2.11), respectively, on $[t_0, T)$. To show that

$$z_{jj}(t) > 0, \quad t \in [t_0, T), \tag{3.16}$$

assume it is not so. Then there exists $T_1 \in (t_0, T)$ such that

$$z_{11}(t)z_{22}(t) > 0,\ t \in [t_0, T_1), \quad z_{11}(T_1)z_{22}(T_1) = 0. \tag{3.17}$$

Without loss of generality we may take that $a_{12}(t_0) = a_{21}(t_0) = 0$. Then by virtue of Lemma 2.2, from (3.17) it follows that

$$\begin{aligned}
&\Big|z_{12}(t) + \frac{\bar{a}_{21}(t)}{b_2(t)}\Big| \\
&\quad \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big|\exp\Big(-\int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\
&\qquad \times \Big(\Big(\frac{\bar{a}_{21}(\tau)}{b_2(\tau)}\Big)' + \frac{\bar{a}_{21}(\tau)}{b_2(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) - c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \\
&\Big|z_{12}(t) + \frac{a_{12}(t)}{b_1(t)}\Big| \\
&\quad \leqslant \mathfrak{M}(t) + \int_{t_0}^{t} \Big|\exp\Big(-\int_{\tau}^{t} (\bar{a}_{11}(s) + a_{22}(s))\,\mathrm{d}s\Big) \\
&\qquad \times \Big(\Big(\frac{a_{12}(\tau)}{b_1(\tau)}\Big)' + \frac{a_{12}(\tau)}{b_1(\tau)}(\bar{a}_{11}(\tau) + a_{22}(\tau)) - c_{12}(\tau)\Big)\Big|\,\mathrm{d}\tau, \quad t \in [t_0, T_1).
\end{aligned}$$

Hence

$$\begin{aligned}
&b_2(t)\Big|z_{12}(t) + \frac{\bar{a}_{21}(t)}{b_2(t)}\Big| - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) \leqslant \chi_3(t), \\
&b_1(t)\Big|z_{12}(t) + \frac{a_{12}(t)}{b_1(t)}\Big|^2 - \frac{|a_{12}(t)|^2}{b_2(t)} - c_{22}(t) \leqslant \chi_4(t), \quad t \in [t_0, T_1),
\end{aligned}$$

By virtue of Theorem 2.1 and Theorem 2.2 and from condition (3), it follows that the Riccati equations

$$(3.18)\quad z' + b_1(t)z^2 + 2(\operatorname{Re} a_{11}(t))z + b_2(t)\Big|z_{12}(t) + \frac{\bar a_{21}(t)}{b_2(t)}\Big| - \frac{|a_{21}(t)|^2}{b_2(t)} - c_{11}(t) = 0,$$

$$(3.19)\quad z' + b_2(t)z^2 + 2(\operatorname{Re} a_{22}(t))z + b_1(t)\Big|z_{12}(t) + \frac{a_{12}(t)}{b_1(t)}\Big|^2 - \frac{|a_{12}(t)|^2}{b_2(t)} - c_{22}(t) = 0,$$

$t \in [t_0, T_1)$, have nonnegative solutions $z_1(t)$ and $z_2(t)$, respectively, on $[t_0, T_1)$ with $z_1(t_0) = z_2(t_0) = 0$. Obviously $z_{11}(t)$ and $z_{22}(t)$ are solutions of equation (3.18) and (3.19), respectively, on $[t_0, T_1]$. Therefore, since $z_{jj}(t_0) = 1 > z_j(t_0) = 0$, $j = 1, 2$, due to the uniqueness theorem $z_{jj}(t) > 0$, $t \in [t_0, T_1]$, $j = 1, 2$, which contradicts (3.17). The obtained contradiction proves (3.16). From (3.16) and condition (1) it follows that

$$(3.20)\qquad \int_{t_0}^{t} (b_1(\tau)z_{11}(\tau) + b_2(\tau)z_{22}(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T).$$

Suppose $T < \infty$. Then by Lemma 2.1, from (3.20) it follows that $[t_0, T)$ is not the maximum existence interval for $Z(t)$, which contradicts our assumption. The obtained contradiction proves (3.15). The theorem is proved. □

R e m a r k 3.3. Condition (3) of Theorem 3.3 is satisfied if in particular $\chi_j(t) \leqslant 0$, $t \geqslant t_0$, $j = 1, 2$.

3.2. The case where $B(t)$ is nonnegative definite. In this subsection we will assume that $B(t)$ is nonnegative definite and $\sqrt{B(t)}$ is continuously differentiable on $[t_0, \infty)$. Consider the matrix equation

$$(3.21)\qquad \sqrt{B(t)}X\big(A(t)\sqrt{B(t)} - \sqrt{B(t)}'\big) = A(t)\sqrt{B(t)} - \sqrt{B(t)}', \quad t \geqslant t_0.$$

Obviously this equation has always a solution on $[a, b]$ $(\subset [t_0, \infty))$ when $B(t) > 0$, $t \in [a, b]$ $(X(t) = B^{-1}(t),\ t \in [a, b])$. It may have also a solution on $[a, b]$ in some cases when $B(t) \geqslant 0$, $t \in [a, b]$ (e.g., $A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ 0 & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} b_1(t) & 0 \\ 0 & 0 \end{pmatrix}$, $b_1(t) > 0$, $t \in [a, b]$). In this subsection we also will assume that equation (3.21) has always a solution on $[t_0, \infty)$. Let $F(t)$ be a solution of equation (3.21) on $[t_0, \infty)$. Denote

$$(3.22)\qquad \begin{aligned} P(t) &\equiv F(t)\big(A(t)\sqrt{B(t)} - \sqrt{B(t)}'\big) = (p_{jk}(t))_{j,k=1}^{2},\\ Q(t) &\equiv \sqrt{B(t)}C(t)\sqrt{B(t)} = (q_{jk}(t))_{j,k=1}^{2},\\ \widetilde\chi_j(t) &\equiv q_{jj}(t) + |p_{3-j,j}(t)|^2, \quad j = 1, 2,\ t \geqslant t_0. \end{aligned}$$

Corollary 3.1. *The system* (1.1) *is oscillatory provided one of the equations*

$$\varphi_1'' + 2(\operatorname{Re} p_{jj}(t))\varphi_1' + \widetilde{\chi}_j(t)\varphi_1 = 0, \quad j = 1, 2, \ t \geqslant t_0 \tag{3.23$_j$}$$

is oscillatory.

Proof. Multiply equation (2.3) on the left and on the right by $\sqrt{B(t)}$. Taking into account the equality $\big(\sqrt{B(t)}Z\sqrt{B(t)}\big)' = \sqrt{B(t)}Z'\sqrt{B(t)} + \sqrt{B(t)}'Z\sqrt{B(t)} + \sqrt{B(t)}Z\sqrt{B(t)}'$, $t \geqslant t_0$, we obtain

$$V' + V^2 + P^*(t)V + VP(t) - Q(t) = 0, \quad t \geqslant t_0, \tag{3.24}$$

where $V \equiv \sqrt{B(t)}Z\sqrt{B(t)}$. This equation corresponds to the following matrix Hamiltonian system

$$\begin{cases} \Phi' = P(t)\Phi + \Psi, \\ \Psi' = Q(t)\Phi - P^*(t)\Psi, \quad t \geqslant t_0. \end{cases} \tag{3.25}$$

Suppose the system (1.1) is not oscillatory. Then by (2.4), equation (2.3) has a Hermitian solution $Z(t)$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. Therefore, $V(t) \equiv \sqrt{B(t)}Z(t)\sqrt{B(t)}$, $t \geqslant t_1$, is a Hermitian solution of equation (3.24) on $[t_1, \infty)$ and hence the system (3.25) has a conjoined solution $(\Phi(t), \Psi(t))$ such that $\det \Phi(t) \neq 0$, $t \geqslant t$. It means that the Hamiltonian system

$$\begin{cases} \varphi' = P(t)\varphi + \psi, \\ \psi' = Q(t)\varphi - P^*(t)\psi, \quad t \geqslant t_0, \end{cases}$$

is not oscillatory. By Theorem 3.1, it follows that the scalar systems

$$\begin{cases} \varphi_1' = 2\operatorname{Re} p_{jj}(t)\varphi_1 + \psi_1, \\ \psi_1' = -\widetilde{\chi}_j(t)\varphi_1, \qquad t \geqslant t_0, \end{cases}$$

$j = 1, 2$, are not oscillatory. Therefore, the corresponding equations (3.23$_j$), $j = 1, 2$, are not oscillatory, which contradicts the conditions of the corollary. This completes the proof. □

Denote:

$$\widetilde{\mathfrak{M}}(t) \equiv \max_{\tau\in[t_0,t]}\left|\exp\left(-\int_\tau^t (\bar p_{11}(s)+p_{22}(s))\,\mathrm{d}s\right)(p_{12}(\tau)-\bar p_{21}(\tau))\right|;$$

$$\widetilde{\chi}_3(t) \equiv \left(\widetilde{\mathfrak{M}}(t) + \int_{t_0}^t \left|\exp\left(-\int_\tau^t (\bar p_{11}(s)+p_{22}(s))\,\mathrm{d}s\right)\right.\right.$$
$$\left.\left.\times\,(\overline{p}_{21}' + \bar p_{21}(\tau)(\bar p_{11}(\tau)+p_{22}(\tau)) + q_{12}(\tau))\right|\,\mathrm{d}\tau\right)^2 - |p_{21}(t)|^2 - q_{11}(t);$$

$$\widetilde{\chi}_4(t) \equiv \left(\widetilde{\mathfrak{M}}(t) + \int_{t_0}^t \left|\exp\left(-\int_\tau^t \bar p_{11}(s)+p_{22}(s)\,\mathrm{d}s\right)\right.\right.$$
$$\left.\left.\times\,(p_{12}'(t) + p_{12}(\tau)(\bar p_{11}(\tau)+p_{22}(\tau)) + q_{12}(\tau))\right|\,\mathrm{d}\tau\right)^2$$
$$- |p_{12}(t)|^2 - q_{22}(t), \quad t \geqslant t_0;$$

$$\tilde I_{j+2}(\xi,t) \equiv \int_\xi^t \exp\left(-\int_\tau^t 2(\operatorname{Re} p_{jj}(s))\,\mathrm{d}s\right)\widetilde{\chi}_{j+2}(\tau)\,\mathrm{d}\tau, \quad t\geqslant\xi\geqslant t_0,\ j=1,2.$$

Theorem 3.4. *Let the following conditions be satisfied:*

(1′) $B(t)\geqslant 0$, $t\geqslant t_0$;

(2′) *equation* (3.21) *has a solution* $F(t)$ *on* $[t_0,\infty)$*;*

(3′) *the functions* $p_{12}(t)$ *and* $p_{21}(t)$*, defined by* (3.22)*, are continuously differentiable on* $[t_0,\infty)$*;*

(4′) *there exist infinitely large sequences* $\xi_{j,0}=t_0<\xi_{j,1}<\ldots<\xi_{j,m}<\ldots$ *such that*

$$\int_{\xi_{j,m}}^t \exp\left(\int_{\xi_{j,m}}^\tau (2\operatorname{Re} a_{jj}(s) - \tilde I_{j+2}(\xi_{j,m},s))\,\mathrm{d}s\right)\widetilde{\chi}_{j+2}(\tau)\,\mathrm{d}\tau \leqslant 0, \quad t\in[\xi_{j,m},\ \xi_{j,m+1}),$$

$m=1,2,3,\ldots$, $j=1,2$.

Then the system (1.1) *is non-oscillatory.*

Proof. Let $Z(t)\equiv\begin{pmatrix} z_{11}(t) & z_{12}(t)\\ \overline{z}_{12}(t) & z_{22}(t)\end{pmatrix}$ be the Hermitian solution of equation (2.3) satisfying the initial condition $Z(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and let $[t_0,T)$ be the maximum existence interval for $Z(t)$. Then $V(t)\equiv\sqrt{B(t)}Z(t)\sqrt{B(t)}$ is a solution of equation (3.24) on $[t_0,T)$. Without loss of generality, we may assume that $B(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Then, $V(t_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and similarly to the proof of Theorem 3.3, we can show that

$$\int_{t_0}^t \operatorname{tr} V(\tau)\,\mathrm{d}\tau \geqslant 0, \quad t\in[t_0,T). \tag{3.26}$$

By virtue of Lemma 2.3 we have $\operatorname{tr} V(t) = \operatorname{tr}(B(t)Z(t))$, $t \in [t_0, T)$. From this and from (3.26) it follows that

$$\int_{t_0}^{t} \operatorname{tr}(B(\tau)Z(\tau))\,\mathrm{d}\tau \geqslant 0, \quad t \in [t_0, T). \tag{3.27}$$

To complete the proof of the theorem, it remains to show that $T = \infty$. Suppose $T < \infty$. Then, by virtue of Lemma 2.2, from (3.27) it follows that $[t_0, T)$ is not the maximum existence interval for $Z(t)$, which contradicts our assumption. This contradiction shows that $T = \infty$, and the theorem is proved. □

Example 3.1. Consider the second-order vector equation

$$\varphi'' + K(t)\varphi = 0, \quad t \geqslant t_0, \tag{3.28}$$

where $K(t) \equiv \begin{pmatrix} \mu(t) & 10\mathrm{i} \\ -10\mathrm{i} & -t^2 \end{pmatrix}$, $\mu(t) \equiv p_1 \sin(\lambda_1 t + \theta_1) + p_2 \sin(\lambda_2 t + \theta_2)$, $t \geqslant t_0$, p_j, $\lambda_j \neq 0$, θ_j, $j = 1, 2$, are real constants such that λ_1 and λ_2 are rational independent. This equation is equivalent to the system (1.1) with $A(t) \equiv 0$, $B(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C(t) = -K(t)$, $t \geqslant t_0$. Hence, by Theorem 3.1, equation (3.28) is oscillatory provided the scalar system

$$\begin{cases} \varphi_1' = \psi_1, \\ \psi_1' = -\mu(t)\varphi_1, \quad t \geqslant t_0, \end{cases}$$

is oscillatory. This system is equivalent to the second-order scalar equation

$$\varphi_1'' + \mu(t)\varphi_1 = 0, \quad t \geqslant t_0,$$

which is oscillatory (see [9]). Therefore, equation (3.28) is oscillatory. It is not difficult to verify that the results in works [2], [3], [5], [15], [17] are not applicable to equation (3.28).

Example 3.2. Let

$$B(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad t \geqslant t_0. \tag{3.29}$$

Then $\sqrt{B(t)} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\sqrt{B(t)}' \equiv 0$, $t \geqslant t_0$, and $F(t) = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $t \geqslant t_0$, is a solution of equation (3.21) on $[t_0, \infty)$,

$$P(t) = \begin{pmatrix} a_{11}(t) + a_{12}(t) a_{11}(t) + a_{12}(t) \\ a_{21}(t) + a_{22}(t) a_{21}(t) + a_{22}(t) \end{pmatrix}, \tag{3.30}$$

$$Q(t) = (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))B(t), \quad t \geqslant t_0. \tag{3.31}$$

Assume

$$a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) \equiv 0, \quad t \geqslant t_0. \tag{3.32}$$

Then taking into account (3.30) and (3.31) we have $\widetilde{\chi}_1(t) = \widetilde{\chi}_2(t) = -c_{11}(t) - 2\operatorname{Re} c_{12}(t) - c_{22}(t)$, $t \geqslant t_0$. Therefore, by Corollary 3.1, (3.29), and (3.32), the system (1.1) is oscillatory provided the scalar equation

$$\varphi_1''(t) - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))\varphi_1(t) = 0, \quad t \geqslant t_0,$$

is oscillatory.

Assume now:

$$\begin{aligned} a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) &= \frac{\alpha}{t}, \\ c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t) &= \frac{\alpha - \alpha^2}{t^2}, \end{aligned} \tag{3.33}$$

$0 \leqslant \alpha \leqslant 1$, $t \geqslant 1$. Then taking into account (3.30) and (3.31), it is not difficult to verify that $\widetilde{\chi}_3(t) = \widetilde{\chi}_4(t) = (\alpha^2 - \alpha)/t^2 \leqslant 0$, $t \geqslant 1$. Hence, by Theorem 3.4, (3.29) and (3.33) the system (1.1) is non-oscillatory.

Now assume:

(α_1) $a_{11}(t) + a_{12}(t) = a_{21}(t) + a_{22}(t) > 0$, $t \geqslant t_0$;

(α_2) $a_{11}(t) + a_{12}(t)$ is increasing and continuously differentiable on $[t_0, \infty)$;

(α_3) $|(a_{11}(t) + a_{12}(t))' + c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t)|/(a_{11}(t) + a_{12}(t)) \leqslant \lambda = \text{const.}$, $t \geqslant t_0$.

Then taking into account (3.30) and (3.31) it is not difficult to verify that $\widetilde{\chi}_3(t) \leqslant \lambda - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))$, $\widetilde{\chi}_4(t) \leqslant \lambda - (c_{11}(t) + 2\operatorname{Re} c_{12}(t) + c_{22}(t))$, $t \geqslant t_0$. Therefore by virtue of Theorem 3.4, (3.29), and conditions (α_1)–(α_3), the system (1.1) is non-oscillatory.

Remark 3.4. Under the restriction (3.29), $\det B(t) \equiv 0$, $t \geqslant t_0$, the results of works [1], [4], [12]–[14], [16], [18]–[20] are not applicable to the system (1.1) with (3.29).

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