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EXISTENCE OF NONOSCILLATORY SOLUTIONS
TO THIRD ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS
WITH DELAY AND ADVANCED ARGUMENTS

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Abstract. In this paper, we present several sufficient conditions for the existence of nonoscillatory solutions to the following third order neutral type difference equation

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0, \quad n \geq n_0$$

via Banach contraction principle. Examples are provided to illustrate the main results. The results obtained in this paper extend and complement some of the existing results.

Keywords: third order; nonoscillation; delay and advanced arguments; neutral difference equation

MSC 2020: 39A10

1. INTRODUCTION

This paper deals with the existence of nonoscillatory solutions of third order neutral type difference equations of the form

$$(1.1) \quad \Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0, \quad n \geq n_0$$

where n_0 , a nonnegative integer, is subject to the following conditions:

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- (H₁) $\{a_n\}$ and $\{b_n\}$ are real sequences;
 (H₂) $\{p_n\}$ and $\{q_n\}$ are non-negative real sequences;
 (H₃) l and m are positive integers and k and r are non-negative integers.

Let $\theta = \max\{l, k\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1.1) for all $n \geq n_0$. A nontrivial solution of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise.

Recently, many researchers have been interested in investigating the existence of nonoscillatory solutions of neutral type difference equations; see for example [4]–[7], [9], [10], [13], [15], [16], and the references cited therein.

In [10], the authors discussed the existence of nonoscillatory solutions for the equation

$$\Delta^3(x_n + p_n x_{n-k}) + q_n f(x_{n-l}) = h_n.$$

They established sufficient conditions for the existence of nonoscillatory solutions depending on the different ranges of $\{p_n\}$.

In [7], the authors discussed the existence of nonoscillatory solutions of the third order difference equation

$$\Delta(a_n \Delta(b_n \Delta(x_n + p x_{n-m}))) + p_n f(x_{n-k}) - q_n f(x_{n-l}) = 0,$$

when $p \geq 0$ or $p \leq 0$.

In [16], the existence of nonoscillatory solutions of a higher order nonlinear neutral difference equation

$$\Delta^m(x(n) + p(n)x(\tau(n))) + f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = 0$$

was studied.

On the other hand, there has been great interest in studying the oscillatory behavior of third and higher order neutral type difference equations with delay and advanced terms; see for example [1], [2], [3], [8], [11], [12], [14], and the references cited therein. To the best of our knowledge, only a few results are available for third order nonlinear difference equations with delay and advanced terms. This is due mainly to the technical difficulties arising in their analysis.

In view of the above observation, in this paper we obtain some new sufficient conditions for the existence of nonoscillatory solutions for the equation (1.1) using the Banach fixed point theorem. Examples are provided to illustrate the main results.

2. EXISTENCE OF NONOSCILLATORY SOLUTIONS

In this section, we present some sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) using the Banach contraction principle. For convenience we use the following notation:

$$(n)^{(m)} = n(n-1)\dots(n-m+1) \quad \text{for any positive integer } m.$$

We begin with the following theorem.

Theorem 2.1 (Banach's contraction mapping principle). *A contraction mapping on a complete metric space has a unique fixed point.*

Theorem 2.2. *Assume that $0 \leq a_n \leq a < 1$, and $0 \leq b_n \leq b \leq 1 - a$ for all $n \geq n_0$. If*

$$(2.1) \quad \sum_{n=n_0}^{\infty} (n+2)^2 p_n < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} (n+2)^2 q_n < \infty,$$

then equation (1.1) has a bounded nonoscillatory solution.

Proof. From condition (2.1), one can choose an integer $N > n_0$ so that

$$(2.2) \quad N \geq n_0 + \theta$$

sufficiently large such that

$$(2.3) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{M_2 - \alpha}{M_2}, \quad n \geq N,$$

and

$$(2.4) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - (a+b)M_2 - M_1}{M_2}, \quad n \geq N,$$

where M_1 and M_2 are positive constants such that

$$(a+b)M_2 + M_1 < M_2, \quad \text{and} \quad \alpha \in ((a+b)M_2 + M_1, M_2).$$

Let B be the set of all bounded real sequences $\{x_n\}$ defined for all $n \geq n_0$ with supremum norm $\|x\| = \sup_{n \geq n_0} |x_n|$. Clearly B is a Banach space. Set

$$S = \{x \in B: M_1 \leq x_n \leq M_2, n \geq n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B . Define an operator $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \quad + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have

$$(Tx)_n \leq \alpha + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} p_s x_{s-k}.$$

Since $x_{n-k} \in B$, we have $x_{n-k} \leq M_2$ and $s-n \leq s$ for all $s \geq n$, and $(s+2)^{(2)} \leq (s+2)^2$ and using these inequalities, we obtain

$$(Tx)_n \leq \alpha + M_2 \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \alpha + M_2 - \alpha \leq M_2,$$

and

$$\begin{aligned} (Tx)_n &\geq \alpha - a_n x_{n-l} - b_n x_{n+m} - \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} q_s x_{s+r} \\ &\geq \alpha - aM_2 - bM_2 - M_2 \sum_{s=n}^{\infty} (s+2)^2 q_s \geq M_1, \end{aligned}$$

where we have used (2.3) and (2.4). Thus $TS \subset S$.

Next, we show that T is a contraction mapping on S . Let $x, y \in S$, and $n \geq N$. Then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq a_n |x_{n-l} - y_{n-l}| + b_n |x_{n+m} - y_{n+m}| \\ &\quad + \sum_{s=n}^{\infty} \frac{(s+2)^2}{2} (p_s |x_{s-k} - y_{s-k}| + q_s |x_{s+r} - y_{s+r}|) \\ &\leq \|x - y\| \left(a + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) \\ &\leq \|x - y\| \left(a + b + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (a+b)M_2 - M_1}{M_2} \right) \\ &\leq \|x - y\| \left(\frac{M_2 - M_1}{M_2} \right) \leq \lambda_1 \|x - y\|, \end{aligned}$$

where $\lambda_1 = (1 - M_1/M_2)$. This implies that $\|Tx - Ty\| \leq \lambda_1 \|x - y\|$. Since $\lambda_1 = (1 - M_1/M_2) < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point, that is $T(x_n) = x_n$. Now for $n \geq n_0$,

$$x_n = \alpha - a_n x_{n-l} - b_n x_{n+m} + \sum_{s=n}^{\infty} \frac{(s-n+2)^2}{2} (p_s x_{s-k} - q_s x_{s+r}),$$

then

$$\Delta(x_n + a_n x_{n-l} + b_n x_{n+m}) = - \sum_{s=n}^{\infty} (s-n+1)(p_s x_{s-k} - q_s x_{s+r}),$$

and

$$\Delta^2(x_n + a_n x_{n-l} + b_n x_{n+m}) = \sum_{s=n}^{\infty} (p_s x_{s-k} - q_s x_{s+r}),$$

and hence

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0.$$

Thus $\{x_n\}$ is a positive and bounded solution of equation (1.1). The proof is now completed. \square

Theorem 2.3. *Assume that $0 \leq a_n \leq a < 1$, $a - 1 < b \leq b_n \leq 0$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), we can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$(2.5) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{(1+b)M_4 - \alpha}{M_4}, \quad n \geq N,$$

and

$$(2.6) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - aM_4 - M_3}{M_4}, \quad n \geq N,$$

where M_3 and M_4 are positive constants such that

$$M_3 + aM_4 < (1+b)M_4, \quad \text{and} \quad \alpha \in (M_3 + aM_4, (1+b)M_4).$$

Let B be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B: M_3 \leq x_n \leq M_4, n \geq n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B . Define an operator $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \quad + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.5) and (2.6)

$$(Tx)_n \leq \alpha - bM_4 + M_4 \sum_{s=n}^{\infty} (s+2)^2 p_s \leq M_4,$$

and

$$(Tx)_n \geq \alpha - aM_4 - M_4 \sum_{s=n}^{\infty} (s+2)^2 q_s \geq M_3.$$

This proves that $TS \subset S$.

Next we show that T is a contraction mapping. Let $x, y \in S$ and $n \geq N$. Then

$$|(Tx)_n - (Ty)_n| \leq \|x - y\| \left(a - b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) \leq \lambda_2 \|x - y\|,$$

where $\lambda_2 = (1 - M_3/M_4)$. This implies that $\|Tx - Ty\| \leq \lambda_2 \|x - y\|$. Since $\lambda_2 < 1$, T is a contraction mapping on S . Hence by Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. \square

Theorem 2.4. *Assume that $1 < a \leq a_n \leq d < \infty$, and $0 \leq b_n \leq b < a - 1$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ so that

$$(2.7) \quad N + l \geq n_0 + k$$

sufficiently large such that

$$(2.8) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{aM_6 - \alpha}{M_6}, \quad n \geq N,$$

and

$$(2.9) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - dM_5 - (1+b)M_6}{M_6}, \quad n \geq N,$$

where M_5 and M_6 are positive constants such that

$$dM_5 + (1+b)M_6 < aM_6, \quad \text{and} \quad \alpha \in (dM_5 + (1+b)M_6, aM_6).$$

Let B be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B: M_5 \leq x_n \leq M_6, n \geq n_0\}.$$

Obviously S is a bounded, closed and convex subset of B . Define a mapping $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \frac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_n x_{n+l+m} \right. \\ \quad \left. + \sum_{s=n+l}^{\infty} \frac{(s+2-n-l)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly, Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.8), and (2.9), respectively, that

$$(Tx)_n \leq \frac{1}{a_{n+l}} \left(\alpha + M_6 \sum_{s=n}^{\infty} (s+2)^2 p_s \right) \leq M_6,$$

and

$$(Tx)_n \geq \frac{1}{a_{n+l}} \left(\alpha - M_6 - bM_6 - M_6 \sum_{s=n}^{\infty} (s+2)^2 q_s \right) \geq M_5.$$

Thus $TS \subset S$. Next we show that T is a contraction mapping on S . If $x, y \in S$ and $n \geq N$, then

$$|(Tx)_n - (Ty)_n| \leq \frac{1}{a} \|x - y\| \left(1 + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) \leq \lambda_3 \|x - y\|,$$

where $\lambda_3 = (1 - dM_5/M_6)$. This implies that $\|Tx - Ty\| \leq \lambda_3 \|x - y\|$. Since $\lambda_3 < 1$, T is a contraction mapping on S . Therefore by Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. \square

Theorem 2.5. Assume that $1 < a \leq a_n \leq d < \infty$, $1 - a < b \leq b_n \leq 0$ for all $n \geq n_0$. If condition (2.1) holds then equation (1.1) has a bounded nonoscillatory solution.

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$(2.10) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{(a+b)M_8 - \alpha}{M_8}, \quad n \geq N,$$

and

$$(2.11) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - dM_7 - M_8}{M_8}, \quad n \geq N,$$

where M_7 and M_8 are positive constants such that

$$dM_7 + M_8 < (a+b)M_8, \quad \text{and} \quad \alpha \in (dM_7 + M_8, (a+b)M_8).$$

Let B be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B: M_7 \leq x_n \leq M_8, n \geq n_0\}.$$

Clearly S is a bounded, closed and convex subset of B . Define a mapping $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \frac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_{n+l} x_{n+l+m} \right. \\ \quad \left. + \sum_{s=n+l}^{\infty} \frac{(s-n-l+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

It is clearly that Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.10) and (2.11), respectively, that

$$(Tx)_n \leq \frac{1}{a} \left(\alpha - bM_8 + M_8 \sum_{s=n}^{\infty} (s+2)^2 p_s \right) \leq M_8,$$

and

$$(Tx)_n \geq \frac{1}{d} \left(\alpha - M_8 - M_8 \sum_{s=n}^{\infty} (s+2)^2 q_s \right) \geq M_7.$$

This implies that $TS \subset S$. Further, if $x, y \in S$ and $n \geq N$, then

$$|(Tx)_n - (Ty)_n| \leq \frac{1}{a} \|x - y\| \left(1 - b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) = \lambda_4 \|x - y\|,$$

where $\lambda_4 = (1 - dM_7/M_8)$. This implies that $\|Tx - Ty\| \leq \lambda_4 \|x - y\|$. Since $\lambda_4 < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point, that is $T(x_n) = x_n$. Now for $n \geq n_0$,

$$x_n = \frac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_{n+l} x_{n+l+m} + \sum_{s=n+l}^{\infty} \frac{(s-n-l+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}) \right\},$$

or

$$a_{n+l} x_n = \alpha - x_{n+l} - b_{n+l} x_{n+l+m} + \sum_{s=n+l}^{\infty} \frac{(s-n-l+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}),$$

and by replacing n by $n-l$, we have

$$x_n + a_n x_{n-l} + b_n x_{n+m} = \alpha + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}),$$

then arguing as in the proof of Theorem 2.2, we obtain

$$\Delta^3(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0.$$

Thus, $\{x_n\}$ is a positive and bounded solution of equation (1.1). The proof is now completed. \square

Theorem 2.6. *Assume that $-1 < a \leq a_n \leq 0$, $0 \leq b_n \leq b \leq 1 + a$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$(2.12) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{(1+a)M_{10} - \alpha}{M_{10}}, \quad n \geq N,$$

and

$$(2.13) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - bM_{10} - M_9}{M_{10}}, \quad n \geq N,$$

where M_9 and M_{10} are positive constants such that

$$M_9 + bM_{10} < (1 + a)M_{10}, \quad \text{and} \quad \alpha \in (M_9 + bM_{10}, (1 + a)M_{10}).$$

Let B be the Banach space as defined in Theorem 2.1. Set

$$S = \{x \in B: M_9 \leq x_n \leq M_{10}, n \geq n_0\}.$$

Clearly S is a bounded, closed and convex subset of B . Define a mapping $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} \\ \quad + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Obviously, Tx is continuous. For $n \geq N$ and $x \in S$, from (2.12) and (2.13) it follows that

$$(Tx)_n \leq \alpha - aM_{10} + M_{10} \sum_{s=n}^{\infty} (s+2)^2 p_s \leq M_{10},$$

and

$$(Tx)_n \geq \alpha - bM_{10} - M_{10} \sum_{s=n}^{\infty} (s+2)^2 q_s \geq M_9.$$

Thus, $TS \subset S$. Next we show that T is a contraction mapping on S . If $x, y \in S$, and $n \geq N$, then

$$|(Tx)_n - (Ty)_n| \leq \|x - y\| \left(-a + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) = \lambda_5 \|x - y\|,$$

where $\lambda_5 = (1 - M_9/M_{10})$. This implies that $\|Tx - Ty\| \leq \lambda_5 \|x - y\|$. Since $\lambda_5 < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. \square

Theorem 2.7. *Assume that $-1 < a \leq a_n \leq 0$, $-1 - a < b \leq b_n \leq 0$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$(2.14) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{(1+a+b)M_{12} - \alpha}{M_{12}}, \quad n \geq N,$$

and

$$(2.15) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{\alpha - M_{11}}{M_{12}}, \quad n \geq N,$$

where M_{11} and M_{12} are positive constants such that

$$M_{11} < (1 + a + b)M_{12}, \quad \text{and} \quad \alpha \in (M_{11}, (1 + a + b)M_{12}).$$

Let B be the Banach space as defined in Theorem 2.1. Set

$$S = \{x \in B: M_{11} \leq x_n \leq M_{12}, n \geq n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B . Define an operator $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-1} - b_n x_{n+m} \\ \quad + \sum_{s=n}^{\infty} \frac{(s-n+2)^{(2)}}{2} (p_s x_{s-k} - q_s x_{s+r}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, from (2.14) and (2.15), it follows that

$$(Tx)_n \leq \alpha - aM_{12} - bM_{12} + M_{12} \sum_{s=n}^{\infty} (s+2)^2 p_s \leq M_{12},$$

and

$$(Tx)_n \geq \alpha - M_{12} \sum_{s=n}^{\infty} (s+2)^2 q_s \geq M_{11}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \geq N$, then we have

$$|(Tx)_n - (Ty)_n| \leq \|x - y\| \left(-a - b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) = \lambda_6 \|x - y\|,$$

where $\lambda_6 = (1 - M_{11}/M_{12})$. This implies that $\|Tx - Ty\| \leq \lambda_6 \|x - y\|$. Since $\lambda_6 < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. \square

Theorem 2.8. *Assume that $-\infty < d \leq a_n \leq a < -1$, $0 \leq b_n \leq b < -a - 1$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$(2.16) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{dM_{13} + \alpha}{M_{14}}, \quad n \geq N,$$

and

$$(2.17) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{(-a-1-b)M_{14} - \alpha}{M_{14}}, \quad n \geq N,$$

where M_{13} and M_{14} are positive constants such that

$$-dM_{13} < (-a-1-b)M_{14}, \quad \text{and} \quad \alpha \in (-dM_{13}, (-a-1-b)M_{14}).$$

Let B be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_{13} \leq x_n \leq M_{14}, n \geq n_0\}.$$

Clearly S is a bounded, closed and convex subset of B . Define a mapping $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} -\frac{1}{a_{n+l}} \left\{ \alpha + x_{n+l} + b_{n+l}x_{n+l+m} \right. \\ \quad \left. - \sum_{s=n+l}^{\infty} \frac{(s-n-l+2)^{(2)}}{2} (p_s x_{n-k} - q_s x_{s+r}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, from (2.16) and (2.17) we see that

$$(Tx)_n \leq -\frac{1}{a} \left(\alpha + M_{14} + bM_{14} - M_{14} \sum_{s=n}^{\infty} (s+2)^2 p_s \right) \leq M_{14},$$

and

$$(Tx)_n \geq -\frac{1}{d} \left(\alpha + M_{14} \sum_{s=n}^{\infty} (s+2)^2 q_s \right) \geq M_{13}.$$

Thus $TS \subset S$. If $x, y \in S$ and $n \geq N$, then we have

$$|(Tx)_n - (Ty)_n| \leq \frac{1}{a} \|x - y\| \left(1 + b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) = \lambda_7 \|x - y\|,$$

where $\lambda_7 = (1 - M_{13}/M_{14})$. This implies that $\|Tx - Ty\| \leq \lambda_7 \|x - y\|$. Since $\lambda_7 < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. \square

Theorem 2.9. Assume that $-\infty < d \leq a_n \leq a < -1$, $a + 1 < b \leq b_n \leq 0$ for all $n \geq n_0$. If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$(2.18) \quad \sum_{s=n}^{\infty} (s+2)^2 p_s \leq \frac{dM_{15} + bM_{16} + \alpha}{M_{16}}, \quad n \geq N,$$

and

$$(2.19) \quad \sum_{s=n}^{\infty} (s+2)^2 q_s \leq \frac{(-a-1)M_{16} - \alpha}{M_{16}}, \quad n \geq N,$$

where M_{15} and M_{16} are positive constants such that

$$-dM_{15} - bM_{16} < (-a-1)M_{16}, \quad \text{and} \quad \alpha \in (-dM_{15} - bM_{16}, (-a-1)M_{16}).$$

Let B be the Banach space as defined in Theorem 2.2. Set

$$S = \{x \in B: M_{15} \leq x_n \leq M_{16}, n \geq n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B . Define a mapping $T: S \rightarrow B$ as follows:

$$(Tx)_n = \begin{cases} -\frac{1}{a_{n+l}} \left\{ \alpha + x_{n+l} + b_{n+l} x_{n+l+m} \right. \\ \left. - \sum_{s=n+l}^{\infty} \frac{(s-n-l+2)^{(2)}}{2} (p_s x_{n-k} - q_s x_{s+r}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.18) and (2.19) that

$$(Tx)_n \leq -\frac{1}{a} \left(\alpha + M_{16} - M_{16} \sum_{s=n}^{\infty} (s+2)^2 p_s \right) \leq M_{16},$$

and

$$(Tx)_n \geq -\frac{1}{d} \left(\alpha + bM_{16} + M_{16} \sum_{s=n}^{\infty} (s+2)^2 q_s \right) \geq M_{15}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \geq N$, then

$$|(Tx)_n - (Ty)_n| \leq \frac{1}{a} \|x - y\| \left(1 - b + \sum_{s=n}^{\infty} (s+2)^2 (p_s + q_s) \right) = \lambda_8 \|x - y\|,$$

where $\lambda_8 = (1 - dM_{15}/M_{16})$. This implies that $\|Tx - Ty\| \leq \lambda_8 \|x - y\|$. Since $\lambda_8 < 1$, T is a contraction mapping on S . By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. \square

3. EXAMPLES

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the neutral difference equation of the form

$$(3.1) \quad \Delta^3 \left(x_n + \frac{1}{2}x_{n-1} + \frac{1}{3}x_{n+2} \right) + \left(\frac{9n+3}{n^2(n+1)(n+2)(n+3)} + \frac{(3n^2+21n+38)(n-1)}{n(n+2)(n+3)^2(n+4)(n+5)} \right) x_{n-1} - \frac{1}{(n+3)^4} x_{n+2} = 0, \quad n \geq 1.$$

Here $a_n = \frac{1}{2}$, $b_n = \frac{1}{3}$,

$$p_n = \frac{9n+3}{n^2(n+1)(n+2)(n+3)} + \frac{(3n^2+21n+38)(n-1)}{n(n+2)(n+3)^2(n+4)(n+5)}, \quad q_n = \frac{1}{(n+3)^4}.$$

One can easily verify that all conditions of Theorem 2.2 are satisfied, and hence equation (3.1) has a bounded nonoscillatory solution. In fact, $\{x_n\} = \{(n+1)/n\}$ is one such solution of equation (3.1).

Example 3.2. Consider a neutral difference equation of the form

$$(3.2) \quad \Delta^3 \left(x_n + \frac{1}{4}x_{n-3} - \left(\frac{3}{4} - \frac{1}{3^n} \right) x_{n+2} \right) + \frac{36}{27} \frac{1}{3^n} x_{n-2} - \frac{28}{27} \frac{1}{3^n} x_{n+1} = 0, \quad n \geq 1.$$

Here $a_n = \frac{1}{4}$, $b_n = -\left(\frac{3}{4} - 3^{-n}\right)$, $p_n = \frac{36}{27} 3^{-n}$, $q_n = \frac{28}{27} 3^{-n}$. A straight-forward verification shows that all conditions of Theorem 2.3 are satisfied, and hence equation (3.2) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{2 + (-1)^n\}$ is one such solution of equation (3.2).

Example 3.3. Consider a neutral difference equation of the form

$$(3.3) \quad \Delta^3 \left(x_n - \frac{1}{2} \left(\frac{3}{4} - \frac{1}{2^n} \right) x_{n-2} - \frac{1}{4} x_{n+2} \right) + \frac{217}{384} \frac{1}{2^n} x_{n-1} - \frac{55}{96} \frac{1}{2^n} x_{n+1} = 0, \quad n \geq 1.$$

Here $a_n = -\frac{1}{2} \left(\frac{3}{4} - 2^{-n} \right)$, $b_n = -\frac{1}{4}$, $p_n = \frac{217}{384} 2^{-n}$, $q_n = \frac{55}{96} 2^{-n}$. It is easy to verify that all conditions of Theorem 2.7 are satisfied. In fact $\{x_n\} = \{1 + 2^{-n}\}$ is a bounded nonoscillatory solution of equation (3.3).

Example 3.4. Consider a neutral difference equation of the form

$$(3.4) \quad \Delta^3 (x_n - 4x_{n-1} - 2x_{n+1}) + \frac{1}{2^{n+2}(2+2^n)} x_{n-1} - \frac{1}{2^n} x_{n+2} = 0, \quad n \geq 1.$$

Here $a_n = -4$, $b_n = -2$, $p_n = 1/(2^{n+2}(2 + 2^n))$, and $q_n = 2^{-n}$. One can easily verify that all conditions of Theorem 2.9 are valid. Hence equation (3.4) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{1 + 2^{-n}\}$ is one such solution of equation (3.4).

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