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ON DISTANCES AND METRICS IN DISCRETE ORDERED SETS

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Abstract. Discrete partially ordered sets can be turned into distance spaces in several ways. The distance functions may or may not satisfy the triangle inequality and restrictions of the distance to finite chains may or may not coincide with the natural, difference-of-height distance measured in a chain. It is shown that for semilattices the semimodularity ensures the good behaviour of the distances considered. The Jordan-Dedekind chain condition, which is weaker than semimodularity, is equivalent to the basic criterion that the graph-theoretic distance (realized by zig-zagging up and down freely in the poset to connect two points) is compatible with distances measured on chains by the relative height. Semimodularity is shown to be equivalent to the validity of the triangle inequality of a restricted graph-theoretic distance, called the up-down distance. The fact that the up-down distance corresponds to the computation of degrees of kinship in family trees leads to the observation that the less familiar canon-law method of computation corresponds also to a mathematically well behaved Chebyshev-type distance on discrete semilattices. For the Chebyshev distance also semimodularity is shown to imply the validity of the triangle inequality. The reverse implication fails, but assuming the validity of the triangle inequality, the semimodularity is shown to have a local characterization by a forbidden six-element subsemilattice. Like in the classical case of real spaces, the Chebyshev semilattice distance is shown to be the limit of a converging sequence of distances, all of them verifying the triangle inequality if the semilattice is semimodular.

Keywords: poset; semilattice; tree; semimodularity; chain condition; height; distance; metric; triangle inequality

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1. Proximity in trees

The degree of kinship between individuals has been considered relevant in ancient and contemporary societies alike, in the normative context of laws of inheritance, marriage prohibitions, rules against nepotism, and independence of judges or jurors, to name a few examples. Mathematically these degrees of kinship can be represented as distance functions $d(x, y)$ on a discrete partially ordered set (poset). The elements of the poset represent persons (human individuals) belonging to some population of individuals living at present or other times (possibly all humanity, or all females, or everyone who has or had an aristocratic title) and the partial order is defined as the reflexive-transitive closure of the child-parent relation. Remaining within the realm of biological child-parent relationships, and within some framework of currently accepted biological facts and beliefs, this closure will indeed be a partial order. If, in addition, only male and only female individuals, and only son-father and daughter-mother relations, respectively, are considered, then the partial order will consist of trees, or it will be a single tree if any two individuals in the population have a common ancestor within the population. Trees are the classical model of dynastic succession in patrilineal (male line) regimes, while they are obviously meaningless in the context of marriage prohibitions. The legally problematic situation of a couple in an avunculate marriage and their child is described by a 5-element partial order, for example the historically somewhat known marriage between

(a) the princess Pauline Sándor de Slavnicza, daughter of
(b) Leontine von Metternich, who was a daughter of
(c) Klemens Wenzel von Metternich, the Austrian chancellor,
(d) and the prince Richard von Metternich, son of the chancellor,
(e) of which a child named Sophie was born.

This poset is indeed a five-element lattice with the minimum (e), maximum (c) and maximal chains $e - a - b - c$ and $e - d - c$.

In Roman law, according to a method now referred to as the "civil-law method", the degree of kinship between two individuals, say Ego and Alter, was computed by determining their nearest common ancestor X (which can be Ego or Alter if these two are in a direct line related), and then adding the number $h(E, X)$ of generations from X to Ego and the number $h(A, X)$ of generations from X to Alter. According to another ancient method, adopted in Europe in the Middle Ages and called the "canon-law method", the degree of kinship is computed as the greater of the numbers $h(E, X)$ and $h(A, X)$ (for a historical account see Bouchard [1], Garner's legal dictionary [6] or Burtsell's article in the Catholic Encyclopedia [2]).

In the poset model, assuming the existence of common upper bounds for any pair of elements E, A, the civil law degree of kinship between E and A corresponds to the smallest number of the form $h(E, X) + h(A, X)$ where X is a common upper bound of E and A and $h(E, X)$ is the length of the shortest maximal chain from E to X (and similarly for $h(A, X)$). We call this the *top-down distance*, since it is computed by going up from E to X and then down to A. In contrast, the canon law degree of kinship between E and A corresponds to the smallest number of the form $\max(h(E, X), h(A, X))$, which, in the context of upper semilattices, we call (for reasons to be seen later) the "Chebyshev distance".

Abstracting from any possible applications or social context, we formulate both the "civil-law" and "canon-law" methods of kinship degree computation in the general abstract framework of partially ordered sets with a connected Hasse diagram, examine the relationship between these and some other distance functions, and address the question of validity of the triangle inequality.

In the mathematical development, non-symmetric distance functions also arise. This points to questions beyond the purpose of this note, but we bear in mind that while both the civil law and canon law degrees of kinship are symmetric (the distance functions modeling them are symmetric functions of two variables), the parentela systems of fundamental importance in some civil codes (e.g. in Switzerland and Hungary), designed to measure (in intestate succession) how close is the heir to the decedent, are by nature asymmetric (the nephew is in the second parentela of the uncle, but the uncle is in the third parentela of the nephew). Parentela systems could be formalized similarly to the non-symmetric semilattice distance function appearing in Section 2 below.

2. Distances in discrete partially ordered sets

By a *distance function* on a set S we mean a symmetric map d from S^2 to the non-negative reals for which $d(x, y) = 0$ if and only if $x = y$. A distance function may or may not satisfy the *triangle inequality*

$$d(x, y) + d(y, z) \geqslant d(x, z), \tag{1}$$

while the term *metric* is used for a distance function that does. If we omit the symmetry requirement from the definition of distance function, then we get the broader concept of a *directed distance*. Such a directed distance concept appears for example in Chartrand, Johns, Tian and Winters [3] or Deza and Panteleeva [4].

In the sequel, a given partially ordered set, finite or infinite, is called *discrete*, if every maximal chain in every order-convex interval $[x, y]$ is finite. This is a stronger condition than the requirement that the order relation be generated as the transitive-reflexive closure of its covering relation, which is a broader definition of discreteness adopted for example in [5]. However, discrete posets in the more restrictive sense presently understood have the convenient property that the order induced on any of their subsets is also discrete.

A distance function on a discrete poset is called *chain-compatible* if its restriction to any maximal chain coincides with this natural chain distance. This is a rather strong requirement, such distance functions may not always exist.

A poset is said to be *upper semimodular* (or simply *semimodular*) if whenever for a pair of distinct elements x, y there is an element z covered by both x and y, there also exists an element w covering both x and y (see Monjardet [9] or Haskins and Gudder [7]). For lattices this means just lattice semimodularity, but the extension obviously includes trees as well. *Lower semimodular* posets are defined dually.

In a discrete poset, if two elements are comparable, say $x \leqslant y$, then by the *height* of y above x, denoted indifferently by $h(x, y)$ or $h(y, x)$ we mean the number that equals the least cardinality of a finite maximal chain in $[x, y]$ minus 1.

The covering relation of any partial order defines a simple directed graph with an arrow from element x to element y if and only if x is *covered* by y, in symbols $x \prec y$. Forgetting the orientation of the arrows, we obtain a simple undirected graph called the poset's *Hasse diagram*. If the Hasse diagram is connected, then we call the poset *connected*. Between any two elements of a connected poset, we use the term *zigzag distance* for their graphic distance measured in the Hasse diagram of the poset. Zigzag distance satisfies the triangle inequality (1) (this is so in fact in non-discrete connected posets as well).

Recall that a poset has the *upper* (or *lower*) *filtering property* if any two elements have a common upper (or lower, respectively) bound. In a discrete poset with the upper (lower) filtering property, the *up-down* (*down-up*, respectively) *distance* of elements x and y is defined as the smallest number of the form $h(x, u) + h(y, u)$ (or the form $h(u, x) + h(u, y)$, respectively), where u is a common upper (lower) bound of x and y. These notions are dual, trees and other join semilattices have the upper filtering property, and lattices have both filtering properties. Obviously, on any discrete chain, the up-down, down-up and zigzag distance functions coincide and yield what is conceivably the most natural notion of distance on a chain. On any discrete join-semilattice, define the *"Chebyshev" distance function* $d(x, y) = \max[h(x, x \vee y), h(y, x \vee y)]$. Generally this distance need not satisfy the triangle inequality. Note also that the Chebyshev distance, like the zigzag distance, is always less than or equal to the up-down distance.

A sublattice K of a lattice L is said to be *cover-preserving*, if for any $a, b \in K$, a is covered by b in K if and only if a is covered by b in L, too.

Our first observation, which follows easily from the definitions, is the following:

Proposition 2.1. *Suppose that a set is partially ordered by a discrete tree order (i.e. all order-convex intervals* $[x, z]$ *are finite, and each pair* x, y *of incomparable elements has a least common upper bound* $x \vee y$ *but has no common lower bound). Then the distance function* $d(x, y)$, *which assigns to the elements* x, y *the greater of* $(\mathrm{Card}[x, x \vee y]) - 1$ *and* $(\mathrm{Card}[y, x \vee y]) - 1$, *satisfies the triangle inequality* $d(x, y) \leqslant d(x, v) + d(v, y)$.

The next proposition shows that the existence of a chain-compatible distance function on a poset is a rather strong requirement:

Proposition 2.2. *For any discrete, connected partially ordered set satisfying either one of the upper or lower filtering properties, the following conditions are equivalent:*

(i) *there is a chain-compatible distance function on the poset,*
(ii) *the zigzag distance on the poset is chain-compatible,*
(iii) *the poset satisfies the Jordan-Dedekind chain condition (in any given interval* $[x, y]$ *all maximal chains have the same number of elements).*

P r o o f. As each of the conditions (i)–(iii) is self-dual, we may suppose, without loss of generality, that the poset satisfies the upper filtering condition.

Obviously condition (ii) implies (i), and (i) implies (iii). To show that (iii) implies (ii), assume (iii) and suppose that there are elements $x < y$ for which the zigzag distance $d(x, y)$ is less than $h(x, y)$: this will lead to a contradiction. For each such pair of elements $x < y$ there is a smallest positive integer $n = n(x, y)$ with the property that there is a sequence of elements $x = x_0, \ldots, y = x_n$, with x_i being comparable to x_{i+1} for $0 \leqslant i \leqslant n-1$, and such that $h(x, y) > h(x_0, x_1) + \ldots + h(x_{n-1}, x_n)$. Choose $x < y$ so that $n = n(x, y)$ is minimal. Then $n \geqslant 3$, $x < x_1$, $x_1 > x_2$ and $x_{n-1} < y$. Let u be a common upper bound of x_1 and y. We must have, as $x_{n-1} < u$ and n is minimal,

$$h(x_1, u) \leqslant h(x_1, x_2) + \ldots + h(x_{n-1}, u) = h(x_1, x_2) + \ldots + h(x_{n-1}, y) + h(y, u)$$

and

$$\begin{aligned} h(x, y) + h(y, u) = h(x, u) &= h(x, x_1) + h(x_1, u) \\ &\leqslant h(x, x_1) + h(x_1, x_2) + \ldots + h(x_{n-1}, y) + h(y, u), \end{aligned}$$

and further

$$h(x, y) \leqslant h(x, x_1) + h(x_1, x_2) + \ldots + h(x_{n-1}, y).$$

□

For join semilattices the following is not difficult to verify, it is also a consequence of broader statements appearing in Haskins and Gudders [7] in the context of semimodular posets in general (see also Lemma 3.6 in the paper of Kharat, Waphare and Thakare [8]).

Proposition 2.3. *The Jordan-Dedekind chain condition is satisfied in every discrete, semimodular join semilattice.*

Making use of this, again as in the case of lattices, we can see that for discrete join semilattices, semimodularity is equivalent to the condition that whenever elements x, y have an element z as a common lower bound, we should have $h(x, x \vee y) \leqslant h(z, y)$.

Semimodularity is not necessary for the Jordan-Dedekind condition to hold. Semimodularity can be characterized by a condition similar to the triangle inequality as follows:

Proposition 2.4. *A discrete join semilattice is semimodular if and only if we have for all elements x, y, z the inequality*

$$h(x, x \vee y) + h(y, y \vee z) \geqslant h(x, x \vee z). \tag{2}$$

P r o o f. Assume semimodularity. As x is a common lower bound of $x \vee y$, and $x \vee z$ and $x \vee y \vee z$ is their join, we have

$$h(x, x \vee y) \geqslant h(x \vee z, x \vee y \vee z)$$

and similarly we have

$$h(y, y \vee z) \geqslant h(x \vee y, x \vee y \vee z).$$

Therefore

$$h(x, x \vee y) + h(y, y \vee z) \geqslant [h(x, x \vee y) - h(x \vee z, x \vee y \vee z)] + h(x \vee y, x \vee y \vee z).$$

But the right-hand side of this latter inequality equals $h(x, x \vee z)$.

Conversely, if semimodularity fails, there are elements x, y, z such that both x and z cover y but x is not covered by $x \vee z$. Then

$$h(x, x \vee y) = h(x, x) = 0, \quad h(y, y \vee z) = h(y, z) = 1, \quad h(x, x \vee z) \geqslant 2$$

and (2) fails. □

A further characterization of semimodularity can be given in terms of the up-down distance. In any discrete poset with the upper filtering property, the up-down distance is always greater than or equal to the zigzag distance, and if it satisfies the triangle inequality, then it must be identical to the zigzag distance, as explained at the beginning of the proof of the next proposition. We note that this result can be derived also from Monjardet (see [9], Theorem 8).

Proposition 2.5. *The following conditions are equivalent for any discrete join semilattice L:*

(i) *L is semimodular,*
(ii) *the up-down distance on L satisfies the triangle inequality,*
(iii) *the up-down distance on L coincides with the zigzag distance.*

P r o o f. First of all, in any discrete poset with the upper filtering property, clearly the up-down distance between any two elements is at least equal to their zigzag distance. Also (iii) implies (ii) trivially. Conversely, assume that (ii) holds. An inductive argument on the zigzag distance $d(x, y)$ between elements x, y shows that every such pair of elements x, y has a common upper bound z such that $d(x, y) = h(x, z) + h(y, z)$. Thus (ii) and (iii) are equivalent.

If L is not semimodular, then the element z is covered by both x and y for some elements x, y, z, but the join $x \vee y$ does not cover x, i.e. $h(x, x \vee y) \geqslant 2$. Then $3 \leqslant d(x, y)$ and the triangle inequality $d(x, y) \leqslant d(x, z) + d(z, y) = 2$ fails for the up-down distance.

Conversely, assume that L is semimodular. If the triangle inequality failed for the up-down distance, for some elements x, y, z we would have

$$h(x, x \vee y) + h(y, x \vee y) + h(y, y \vee z) + h(z, y \vee z) < h(x, x \vee z) + h(z, x \vee z).$$

But this is impossible, since by Proposition 2.4 we must have

$$h(x, x \vee y) + h(y, y \vee z) \geqslant h(x, x \vee z)$$

and

$$h(z, z \vee y) + h(y, y \vee x) \geqslant h(z, z \vee x).$$

□

As we mentioned earlier, generally the Chebyshev distance need not satisfy the triangle inequality. However, it does satisfy the triangle inequality in a large class of semilattices, including trees.

Proposition 2.6. *On any discrete, semimodular join semilattice, the Chebyshev distance satisfies the triangle inequality.*

P r o o f. Assume that the triangle inequality fails in some semilattice, denote the Chebyshev distance by d, and let x, y, z be elements such that $d(x,y) + d(y,z) < d(x,z)$. Let a, b, c, d and f, e denote the heights $h(x, x \vee y)$, $h(y, x \vee y)$, $h(y, y \vee z)$, $h(z, y \vee z)$ and $h(x, x \vee z)$, $h(z, x \vee z)$, respectively, in that order. Without loss of generality $f \geqslant e$, and then f must be (strictly) greater than each one of the numbers $a+c$, $a+d$, $b+c$, $b+d$. Denote by g, h, i the heights of $x \vee y \vee z$ above $x \vee y$, $x \vee z$, $y \vee z$, respectively. By the Jordan-Dedekind condition, $f + h = a + g$. From this and from $f > a + c$ it follows that

$$a + c + h < a + g$$

which implies $c < g$. This contradicts semimodularity because $c = h(y, x \vee z)$ and $g = h[x \vee y, (x \vee y) \vee (y \vee z)]$. □

In contrast to the equivalence of (i) and (ii) in Proposition 2.5, semimodularity is only sufficient but not necessary for the triangle inequality to hold for the Chebyshev distance in a discrete join semilattice, as the example of a six-element poset N_6 displayed in Figure 1 shows.

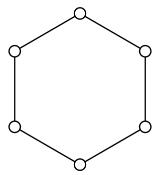

Figure 1. Poset N_6.

Finally, by analogy with classical l_p distances, for any real $p \geqslant 1$ consider the distance function d_p on any discrete join semilattice, given by

$$d_p(x,y) = [h(x, x \vee y)^p + h(y, x \vee y)^p]^{1/p}. \tag{3}$$

Obviously d_1 is the up-down distance, and as expected the Chebyshev distance is the limit of the d_p distances as p tends to infinity:

$$\lim_{p \to \infty} d_p(x,y) = \max[h(x, x \vee y) + h(y, x \vee y)].$$

In fact, again as expected, the d_p distance on any discrete semimodular join semilattice (including all discrete semimodular lattices and trees) satisfies the triangle inequality. In contrast to the Chebyshev distance, semimodularity is characterized by the triangle inequality for any of the d_p distances on a discrete join semilattice, generalizing the equivalence of (i) and (ii) in Proposition 2.5.

Proposition 2.7. *Let $p \geqslant 1$. A discrete join semilattice L is semimodular if and only if the d_p distance function* (3) *on L satisfies the triangle inequality.*

P r o o f. Assume semimodularity. Proposition 2.4 allows to deduce the triangle inequality from Minkowski's inequality (on which the triangle inequality is based in classical l_p spaces). In fact we only need the following specialized two-dimensional case of Minkowski's inequality: if a_1, a_2, b_1, b_2 are non-negative real numbers and $1 \leqslant p < \infty$, then

$$(a_1^p + a_2^p)^{1/p} + (b_1^p + b_2^p)^{1/p} \geqslant [(a_1 + b_1)^p + (a_2 + b_2)^p]^{1/p}. \tag{4}$$

To establish the triangle inequality for the distance d_p in L as defined by (3), we need to show that for all semilattice elements x, y, z,

$$d_p(x, y) + d_p(y, z) \geqslant d_p(x, z). \tag{5}$$

Letting $a_1 = h(x, x \vee y)$, $a_2 = h(y, x \vee y)$, $b_1 = h(y, y \vee z)$, $b_2 = h(z, y \vee z)$, the left-hand side of (5) is equal to the left-hand side of (4), while the right-hand side of (4) is

$$\{[h(x, x \vee y) + h(y, y \vee z)]^p + [h(z, y \vee z) + h(y, x \vee y)]^p\}^{1/p}. \tag{6}$$

Now by Proposition 2.4

$$[h(x, x \vee y) + h(y, y \vee z)]^p \geqslant h(x, x \vee z)^p,$$
$$[h(z, y \vee z) + h(y, x \vee y)]^p \geqslant h(z, z \vee x)^p$$

and thus (6) is at least $d_p(x, y)$, completing the proof of (5).

Conversely, if semimodularity fails, then for some elements x, y covering an element z, the join $x \vee y$ does not cover x and thus

$$h(x, x \vee y)^p \geqslant 2^p, \quad d_p(x, y)^p > 2^p, \quad d_p(x, y) > 2,$$

but $d_p(x, z) = d_p(y, z) = 1$ and therefore $d_p(x, z) + d_p(z, y) \geqslant d_p(x, y)$ fails. □

While, in contrast with the d_p distances, the equivalence of semimodularity with the validity of the triangle inequality fails for the Chebyshev distance, Proposition 2.6 above can still be complemented by the following statement. The Jordan-Dedekind condition is assumed, as otherwise chain-compatibility cannot hold for any distance by Proposition 2.2 above.

Proposition 2.8. *Let L be a discrete join semilattice in which the Chebyshev distance is chain-compatible and satisfies the triangle inequality. Then the following conditions are equivalent:*

(i) *L is semimodular,*
(ii) *L does not contain N_6 as a join-subsemilattice with height 3 in L.*

P r o o f. (i) obviously implies (ii). Suppose (i) does not hold. Observe that the Jordan-Dedekind condition holds in L by Proposition 2.2. Since L is not semimodular, there are elements a, b covering an element x such that $y = a \vee b$ does not cover the elements a, b. By the triangle inequality for the Chebyshev distance, y must be at height 2 above a and b, because otherwise the triangle inquality is violated by $d(a, b) > 2 = d(a, x) + d(x, b)$.

Thus there are elements q and r in L covered by y such that a is covered by q and b is covered by r. Now x, a, b, q, r, y constitute an N_6 join-subsemilattice of height 3 in L. □

Observe that in any non-semimodular join-semilattice L, if the chain-compatible Chebyshev distance satisfies the triangle inequality, then the two atoms of any N_6 subsemilattice of height 3 must be join-irreducible elements in L: for if such an atom b covers any other element c than the null element x of N_6, then the Chebyshev distance between c and the other atom a of N_6 would be 3, however $d(c, x)+d(x, a) = 1+1 = 2$ in the Chebyshev metric.

In the remainder, we focus on (discrete) lattices. Let L be a lattice satisfying the Jordan-Dedekind chain condition, and $a, b, p, q \in L$ with $a \wedge b \prec a \prec p \prec a \vee b$ and $a \wedge b \prec b \prec q \prec a \vee b$. Since $p \vee q = a \vee b$, if $p \wedge q = a \wedge b$, then the elements $a \wedge b$, a, b, p, q, $a \vee b$ form a cover-preserving sublattice of L isomorphic to N_6. In the case $a \wedge b \neq p \wedge q$, the chains $a \wedge b < p \wedge q < p < a \vee b$ and $a \wedge b < p \wedge q < q < a \vee b$ must be the maximal chains of length 3, therefore we get $a \wedge b \prec p \wedge q \prec p, q$. Observe that this implies that $\{a \wedge b, a, b, p \wedge q, p, q, a \vee b\}$ forms a cover-preserving sublattice of L isomorphic to S_7^* on Figure 2. (Note that the lattice S_7^* admits N_6 as a join-subsemilattice.)

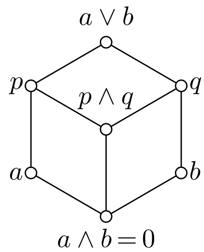

Figure 2. Lattice S_7^*.

Corollary 2.1. *Let L be an atomistic and discrete lattice in which the Chebyshev distance is chain-compatible and satisfies the triangle inequality. Then the following conditions are equivalent:*

(i) *L is semimodular,*

(ii) *L does not contain a cover-preserving sublattice with the bottom element 0 isomorphic to N_6 or S_7^*.*

P r o o f. (i)⇒(ii) is obvious.

(ii)⇒(i). Observe that (ii) implies that L does not contain N_6 as a height 3 join-subsemilattice, therefore, (i) follows from Proposition 2.8. Indeed, suppose that L contains a join-subsemilattice $\{a \wedge b, a, b, p, q, a \vee b\}$ given in the proof of Proposition 2.8. Since the only join-irreducible elements in an atomistic lattice are its atoms, the elements a, b must be atoms in L, and hence $a \wedge b = 0$. Now, either $\{0, a, b, p, q, a \vee b\}$ is a cover-preserving sublattice of L isomorphic to N_6, or $\{0, a, b, p \wedge q, p, q, a \vee b\}$ is a cover-preserving sublattice of L isomorphic to S_7^*. □

3. Conclusion

The investigation presented in this paper originated from the observation that in trees the triangle inequality is valid not only for the usual distance, but also for a Chebyshev-type distance function (see Proposition 2.1). The class of posets examined was enlarged from trees to semilattices, in the case of Proposition 2.2 in fact to posets with the filtering property, in which larger context of the Jordan-Dedekind chain condition was seen to be equivalent to chain-compatibility of the graph-theoretical zigzag distance. From this point on, semimodularity was in the focus of the statements made, starting with the observation stated in Proposition 2.3 that it implies the Jordan-Dedekind condition and the technical result that the directed, asymmetric distance function $h(x, x \vee y)$ also validates the triangle inequality in semimodular semilattices. Relying partly on this, we established Propositions 2.5–2.7, which state that semimodularity of a semilattice implies the validity of the triangle inequality of the following distance functions:

(i) the up-down distance (being a restricted graph-theoretical zigzag distance);

(ii) the Chebyshev distance (being a variant of the up-down distance where the heights of $x \vee y$ above x and y are not added, but their maximum is taken);

(iii) any of the d_p distances (that analogously to l_p spaces converge to the Chebyshev distance, d_1 being in fact the up-down distance).

Semimodularity, however, is a necessary condition for the triangle inequality only for the d_p distances, including the up-down distance (Propositions 2.5 and 2.7): it is only in the absence of a forbidden six-element subsemilattice that semimodularity is implied by the validity of the triangle inequality for the Chebyshev distance.

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