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UNIFORM REGULARITY FOR AN ISENTROPIC COMPRESSIBLE MHD-$P1$ APPROXIMATE MODEL ARISING IN RADIATION HYDRODYNAMICS

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Abstract. It is well known that people can derive the radiation MHD model from an MHD-$P1$ approximate model. As pointed out by F. Xie and C. Klingenberg (2018), the uniform regularity estimates play an important role in the convergence from an MHD-$P1$ approximate model to the radiation MHD model. The aim of this paper is to prove the uniform regularity of strong solutions to an isentropic compressible MHD-$P1$ approximate model arising in radiation hydrodynamics. Here we use the bilinear commutator and product estimates to obtain our result.

Keywords: uniform regularity; MHD-$P1$; compressible

MSC 2020: 35Q30, 35Q35, 35B25

1. Introduction

In this paper we consider the following isentropic compressible MHD-$P1$ approximate model, see [2], [3]:

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0, \tag{1.1}$$

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = (\sigma_a(\varrho) + \sigma_s(\varrho))I_1 + \operatorname{rot} b \times b, \tag{1.2}$$

$$\partial_t b + \operatorname{rot}(b \times u) - \eta \Delta b = 0, \quad \operatorname{div} b = 0, \tag{1.3}$$

$$\partial_t I_0 + \operatorname{div} I_1 = \sigma_a(\varrho)(B(\varrho) - I_0), \tag{1.4}$$

$$\partial_t I_1 + \nabla I_0 = -(\sigma_a(\varrho) + \sigma_s(\varrho))I_1 \quad \text{in } \mathbb{T}^3 \times (0, \infty), \tag{1.5}$$

$$(\varrho, u, b, I_0, I_1)(\cdot, 0) = (\varrho_0, u_0, b_0, I_{0,0}, I_{1,0})(\cdot) \quad \text{in } \mathbb{T}^3. \tag{1.6}$$

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Here ϱ, u, and $I := I_0 + I_1 \cdot \omega$ denote the density, velocity, and the radiation intensity of the fluid, respectively, and $\omega \in S^2$ is the direction vector. The viscosity coefficients of the fluid μ and λ satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geqslant 0$. Furthermore, $B(\varrho)$, $\sigma_a(\varrho)$ and $\sigma_s(\varrho)$ are smooth functions and the pressure $p := a\varrho^\gamma$ with positive constants a and $\gamma > 1$. The symbol $\eta > 0$ tends for the resistivity coefficient.

When $I_1 = 0$ and $b = 0$, (1.1) and (1.2) reduce to the well-known isentropic compressible Navier-Stokes system. There is an enormous amount of literature and results concerned with the compressible Navier-Stokes equations. In a series of seminal papers (see [7], [8], [9]) Feireisl proved the global existence of weak solutions, now called the *Lions-Feireisl theory*. Regarding to strong solutions, we refer to the monograph, see [16]. Recently, Gong, Li, Liu, Zhang in [10] and Huang in [12] proved the local well-posedness of strong solutions.

When $I_1 \neq 0$ and $b \neq 0$, this is the so called *compressible MHD-$P1$ approximate model*. Because of practicable applications, it was the subject of much theoretical research. Therefore, a number of mathematicians study the model from the mathematical point of view like existence, singular limit and blow-up. For example, Fan, Li, Nakamura in [5] (see also [3], [13]) showed non-relativistic and low Mach number limits of the problem. He, Fan, Zhou in [11] (see also [6]) proved the local well-posedness and blow-up criterion of strong solutions. Xie and Klingenberg in [19] studied the non-relativistic limit for the ideal problem ($\lambda = \mu = \eta = 0$).

Before stating our main results, we recall the local existence of smooth solutions to the problem (1.1)–(1.6). Since the system (1.1)–(1.6) is a parabolic-hyperbolic one, the results in [18] imply the following proposition.

Proposition 1.1 ([18]). *Let*

$$\varrho_0, u_0, b_0, I_{0,0}, I_{1,0} \in H^3 \quad \text{and} \quad \frac{1}{C_0} \leqslant \varrho_0 \tag{1.7}$$

for a positive constant C_0. Then the problem (1.1)–(1.6) *has a unique smooth solution* $(\varrho, u, b, I_0, I_1)$ *satisfying*

$$\varrho, I_0, I_1 \in C^l([0,T]; H^{3-l}), \quad u, b \in C^l([0,T]; H^{3-2l}), \quad l = 0,1 \quad \text{and} \quad \frac{1}{C} \leqslant \varrho \tag{1.8}$$

for some $0 < T \leqslant \infty$.

It is well known that we can derive the radiation MHD model from the MHD-$P1$ approximate model. As pointed out in [19], the uniform regularity estimates play the important role in the convergence from the MHD-$P1$ approximate model to the radiation MHD model. The aim of this paper is to prove uniform regularity estimates in (λ, μ, η). We prove the following theorem.

Theorem 1.1. *Let $0 < \mu < 1$, $0 < \lambda + \mu < 1$, $0 < \eta < 1$, $0 < 1/C_0 \leqslant \varrho_0$, $\varrho_0, u_0, b_0, I_{0,0}, I_{1,0} \in H^3(\mathbb{T}^3)$. Let $(\varrho, u, b, I_0, I_1)$ be the unique local smooth solutions to the problem* (1.1)–(1.6). *Then*

$$\|(\varrho, u, b, I_0, I_1)(\cdot, t)\|_{H^3} \leqslant C \quad \text{in } [0, T_0] \tag{1.9}$$

holds true for some positive constants C and T_0 ($\leqslant T$) independent of λ, μ and η.

We put

$$M(t) := 1 + \sup_{0 \leqslant \tau \leqslant t} \Big\{ \|(\varrho, u, b, I_0, I_1, p)(\cdot, \tau)\|_{H^3} + \|\partial_t u(\cdot, \tau)\|_{L^2} + \Big\|\frac{1}{\varrho}(\cdot, \tau)\Big\|_{L^\infty} \Big\}. \tag{1.10}$$

Under the above definition, we can prove the following theorem.

Theorem 1.2. *For any $t \in [0, T_0]$, we have that*

$$M(t) \leqslant C_0(M_0) \exp(tC(M(t))) \tag{1.11}$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

It follows from (1.11) that (see [1], [4], [15])

$$M(t) \leqslant C. \tag{1.12}$$

Therefore, we only need to prove Theorem 1.2.

In the following proofs, we use the bilinear commutator and product estimates due to Kato, Ponce, see [14]:

$$\|D^s(fg) - fD^s g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}} \|D^{s-1} g\|_{L^{q_1}} + \|g\|_{L^{p_2}} \|D^s f\|_{L^{q_2}}), \tag{1.13}$$

$$\|D^s(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}} \|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}) \tag{1.14}$$

with $D = (-\Delta)^{1/2}$, $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

2. Proof of Theorem 1.2

First, testing (1.1) by ϱ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{dt}}\int \varrho^q \,\mathrm{d}x = \Big(1-\frac{1}{q}\Big)\int \varrho^q \operatorname{div} u \,\mathrm{d}x \leqslant \|\operatorname{div} u\|_{L^\infty}\int \varrho^q \,\mathrm{d}x$$

and thus

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\varrho\|_{L^q} \leqslant \|\operatorname{div} u\|_{L^\infty}\|\varrho\|_{L^q},$$

which gives

$$\|\varrho\|_{L^q} \leqslant \|\varrho_0\|_{L^q}\exp\bigg(\int_0^t \|\operatorname{div} u\|_{L^\infty}\,\mathrm{d}\tau\bigg). \tag{2.1}$$

Taking $q \to \infty$, we get

$$\|\varrho\|_{L^\infty} \leqslant \|\varrho_0\|_{L^\infty}\exp(tC(M)). \tag{2.2}$$

It follows from (1.1) that

$$\partial_t\frac{1}{\varrho} + u\cdot\nabla\frac{1}{\varrho} - \frac{1}{\varrho}\operatorname{div} u = 0. \tag{2.3}$$

Testing (2.3) by $(1/\varrho)^{q-1}$, we find that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{dt}}\int\Big(\frac{1}{\varrho}\Big)^q \mathrm{d}x = \Big(1+\frac{1}{q}\Big)\int\Big(\frac{1}{\varrho}\Big)^q \operatorname{div} u\,\mathrm{d}x \leqslant \Big(1+\frac{1}{q}\Big)\Big\|\frac{1}{\varrho}\Big\|_{L^q}^q\|\operatorname{div} u\|_{L^\infty}$$

and, therefore,

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\|\frac{1}{\varrho}\Big\|_{L^q} \leqslant \Big(1+\frac{1}{q}\Big)\Big\|\frac{1}{\varrho}\Big\|_{L^q}\|\operatorname{div} u\|_{L^\infty},$$

which gives

$$\Big\|\frac{1}{\varrho}\Big\|_{L^q} \leqslant \Big\|\frac{1}{\varrho_0}\Big\|_{L^q}\exp\bigg(\Big(1+\frac{1}{q}\Big)\int_0^t\|\operatorname{div} u\|_{L^\infty}\,\mathrm{d}\tau\bigg)$$

and we have

$$\Big\|\frac{1}{\varrho}\Big\|_{L^\infty} \leqslant \Big\|\frac{1}{\varrho_0}\Big\|_{L^\infty}\exp(tC(M)) \tag{2.4}$$

by letting $q \to \infty$.

Equations (2.2) and (2.4) give

$$\|p\|_{L^\infty} + \Big\|\frac{1}{p}\Big\|_{L^\infty} \leqslant C_0(M_0)\exp(tC(M)). \tag{2.5}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int |u|^2\,\mathrm{d}x = 2\int u\partial_t u\,\mathrm{d}x \leqslant 2\|u\|_{L^2}\|\partial_t u\|_{L^2} \leqslant C(M),$$

which implies

$$\|u\|_{L^2} \leqslant C_0(M_0)\exp(tC(M)). \tag{2.6}$$

Testing (1.3) by b, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int |b|^2\,\mathrm{d}x + \eta\int |\nabla b|^2\,\mathrm{d}x = \int \operatorname{rot}(u\times b)b\,\mathrm{d}x = \int (u\times b)\operatorname{rot} b\,\mathrm{d}x$$
$$\leqslant \|u\|_{L^\infty}\|b\|_{L^2}\|\operatorname{rot} b\|_{L^2} \leqslant C(M),$$

which implies

$$\|b\|_{L^2}^2 + \eta\int_0^t\int |\nabla b|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C_0(M_0)\exp(tC(M)). \tag{2.7}$$

Testing (1.4) and (1.5) by I_0 and I_1, respectively, and summing up the result, we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (I_0^2 + |I_1|^2)\,\mathrm{d}x = \int \sigma_a(\varrho)(B(\varrho) - I_0)I_0\,\mathrm{d}x - \int (\sigma_a + \sigma_s)I_1^2\,\mathrm{d}x$$
$$\leqslant \|\sigma_a\|_{L^\infty}\|B - I_0\|_{L^2}\|I_0\|_{L^2} + \|\sigma_a + \sigma_s\|_{L^\infty}\|I_1\|_{L^2}^2 \leqslant C(M),$$

which yields

$$\|I_0\|_{L^2} + \|I_1\|_{L^2} \leqslant C_0(M_0)\exp(tC(M)). \tag{2.8}$$

It is obvious that

$$\frac{1}{\gamma p}\partial_t p + \frac{1}{\gamma p}u\cdot\nabla p + \operatorname{div} u = 0. \tag{2.9}$$

Applying D^3 to (2.9), testing by $D^3 p$, and using (2.9), (1.13) and (1.14), we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int \frac{1}{\gamma p}(D^3 p)^2\,\mathrm{d}x + \int D^3 p D^3 \operatorname{div} u\,\mathrm{d}x \\
&\quad= \frac{1}{2}\int (D^3 p)^2\Big[\operatorname{div}\Big(\frac{u}{\gamma p}\Big) - \frac{1}{\gamma p^2}\partial_t p\Big]\,\mathrm{d}x \\
&\qquad - \int \Big(D^3\Big(\frac{1}{\gamma p}\partial_t p\Big) - \frac{1}{\gamma p}D^3\partial_t p\Big)D^3 p\,\mathrm{d}x \\
&\qquad - \int \Big(D^3\Big(\frac{u}{\gamma p}\cdot\nabla p\Big) - \frac{u}{\gamma p}\cdot\nabla D^3 p\Big)D^3 p\,\mathrm{d}x \\
&\quad\leqslant C\|D^3 p\|_{L^2}^2\Big\|\operatorname{div}\Big(\frac{u}{\gamma p}\Big) - \frac{1}{\gamma p^2}\partial_t p\Big\|_{L^\infty} + C\|\partial_t p\|_{L^\infty}\Big\|D^3\Big(\frac{1}{\gamma p}\Big)\Big\|_{L^2}\|D^3 p\|_{L^2} \\
&\qquad + C\Big\|\nabla\frac{1}{\gamma p}\Big\|_{L^\infty}\|D^2\partial_t p\|_{L^2}\|D^3 p\|_{L^2} + C\|\nabla p\|_{L^\infty}\Big\|D^3\Big(\frac{u}{\gamma p}\Big)\Big\|_{L^2}\|D^3 p\|_{L^2} \\
&\qquad + C\Big\|\nabla\frac{u}{\gamma p}\Big\|_{L^\infty}\|D^3 p\|_{L^2}^2
\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\leqslant C(M) + C(M)\|\partial_t p\|_{L^\infty} + C(M)\|D^2\partial_t p\|_{L^2}\\
&\leqslant C(M) + C(M)\|u\cdot\nabla p + \gamma p\operatorname{div} u\|_{L^\infty} + C(M)\|D^2(u\cdot\nabla p + \gamma p \operatorname{div} u)\|_{L^2}\\
&\leqslant C(M).
\end{aligned}
$$

Here we have used the estimate [17]

$$
\Big\|D^3\frac{1}{p}\Big\|_{L^2} \leqslant C(M)\|D^3 p\|_{L^2} \leqslant C(M). \tag{2.11}
$$

It is obvious that

$$
\int_0^t\int |\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant t\sup\int|\partial_t u|^2\,\mathrm{d}x \leqslant tC(M). \tag{2.12}
$$

Applying D^2 to (1.2), testing by $D^2\partial_t u$, and using (1.13) and (1.14), we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int|D^3u|^2\,\mathrm{d}x + \frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int(D^2\operatorname{div}u)^2\,\mathrm{d}x + \int\varrho|D^2\partial_t u|^2\,\mathrm{d}x\\
&\qquad = -\int D^2\nabla p\cdot D^2\partial_t u\,\mathrm{d}x - \int D^2(\varrho u\cdot\nabla u)\cdot D^2\partial_t u\,\mathrm{d}x\\
&\qquad\quad - \int[D^2(\varrho\partial_t u) - \varrho D^2\partial_t u]D^2\partial_t u\,\mathrm{d}x\\
&\qquad\quad + \int D^2((\sigma_a(\varrho)+\sigma_s(\varrho))I_1)D^2\partial_t u\,\mathrm{d}x + \int D^2(\operatorname{rot} b\times b)D^2\partial_t u\,\mathrm{d}x\\
&\qquad \leqslant C\|D^3p\|_{L^2}\|D^2\partial_t u\|_{L^2} + C\|\varrho\|_{H^2}\|u\|_{H^3}^2\|D^2\partial_t u\|_{L^2}\\
&\qquad\quad + C(\|\nabla\varrho\|_{L^\infty}\|D\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}\|D^2\varrho\|_{L^2})\|D^2\partial_t u\|_{L^2}\\
&\qquad\quad + \|D^2(\varrho\nabla\varphi)\|_{L^2}\|D^2\partial_t u\|_{L^2} + \|D^2((\sigma_a+\sigma_s)I_1)\|_{L^2}\|D^2\partial_t u\|_{L^2}\\
&\qquad\quad + \|D^2(\operatorname{rot} b\times b)\|_{L^2}\|D^2\partial_t u\|_{L^2}\\
&\qquad \leqslant C(M)\|D^2\partial_t u\|_{L^2} + C(M)(\|D\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty})\|D^2\partial_t u\|_{L^2}\\
&\qquad \leqslant C(M)\|D^2\partial_t u\|_{L^2} + C(M)(\|\partial_t u\|_{L^2}^{1/2}\|D^2\partial_t u\|_{L^2}^{1/2} + \|\partial_t u\|_{L^2}\\
&\qquad\quad + \|\partial_t u\|_{L^2}^{1/4}\|D^2\partial_t u\|_{L^2}^{3/4})\|D^2\partial_t u\|_{L^2}\\
&\qquad \leqslant C(M)\|D^2\partial_t u\|_{L^2} + C(M)(\|D^2\partial_t u\|_{L^2}^{1/2} + \|D^2\partial_t u\|_{L^2}^{3/4})\|D^2\partial_t u\|_{L^2}\\
&\qquad \leqslant \frac{1}{2}\int\varrho|D^2\partial_t u|^2\,\mathrm{d}x + C(M),
\end{aligned}
$$

which gives

$$
\int_0^t\int|D^2\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C_0(M_0)\exp(tC(M)). \tag{2.13}
$$

Applying D^3 to (1.2), testing by D^3u, and using (1.1), (1.13) and (1.14), we have

$$
\begin{aligned}
(2.14)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int \varrho|D^3u|^2\,\mathrm{d}x + \mu\int |D^4u|^2\,\mathrm{d}x + (\lambda+\mu)\int (D^3\operatorname{div} u)^2\,\mathrm{d}x \\
& \quad + \int D^3\nabla p\cdot D^3u\,\mathrm{d}x + \int (b\times D^3\operatorname{rot} b)D^3u\,\mathrm{d}x \\
& = \int D^3((\sigma_a+\sigma_s)I_1)D^3u\,\mathrm{d}x - \int (D^3(\varrho\partial_t u) - \varrho D^3\partial_t u)D^3u\,\mathrm{d}x \\
& \quad - \int (D^3(\varrho u\cdot\nabla u) - \varrho u\cdot\nabla D^3u)D^3u\,\mathrm{d}x \\
& \quad - \int (D^3(b\times\operatorname{rot} b) - b\times D^3\operatorname{rot} b)D^3u\,\mathrm{d}x \\
& \leqslant C(\|\nabla\varrho\|_{L^\infty}\|D^2\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}\|D^3\varrho\|_{L^2})\|D^3u\|_{L^2} \\
& \quad + C(\|\nabla u\|_{L^\infty}\|D^3(\varrho u)\|_{L^2} + \|\nabla(\varrho u)\|_{L^\infty}\|D^3u\|_{L^2})\|D^3u\|_{L^2} \\
& \quad + C\|D^3((\sigma_a+\sigma_s)I_1)\|_{L^2}\|D^3u\|_{L^2} + C\|\nabla b\|_{L^\infty}\|D^3b\|_{L^2}\|D^3u\|_{L^2} \\
& \leqslant C(M) + C(M)(\|D^2\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}) \\
& \leqslant C(M) + \|D^2\partial_t u\|_{L^2}^2.
\end{aligned}
$$

Applying D^3 to (1.3), testing by D^3b, using (1.13) and (1.14), we have

$$
\begin{aligned}
(2.15)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int |D^3b|^2\,\mathrm{d}x + \eta\int |D^4b|^2\,\mathrm{d}x + \int (b\times D^3u)D^3\operatorname{rot} b\,\mathrm{d}x \\
& = -\int (D^3(b\times u) - D^3b\times u - b\times D^3u)D^3\operatorname{rot} b\,\mathrm{d}x \\
& \quad - \int (D^3b\times u)D^3\operatorname{rot} b\,\mathrm{d}x \\
& = -\int \operatorname{rot}(D^3(b\times u) - D^3b\times u - b\times D^3u)D^3b\,\mathrm{d}x \\
& \quad + \int (D^3b\times D^3\operatorname{rot} b)u\,\mathrm{d}x \\
& = -\int \operatorname{rot}(D^3(b\times u) - D^3b\times u - b\times D^3u)D^3b\,\mathrm{d}x \\
& \quad + \int \Big[\frac{1}{2}\nabla|D^3b|^2 - (D^3b\cdot\nabla)D^3b\Big]u\,\mathrm{d}x \\
& = -\int \operatorname{rot}(D^3(b\times u) - D^3b\times u - b\times D^3u)D^3b\,\mathrm{d}x \\
& \quad - \frac{1}{2}\int |D^3b|^2\operatorname{div} u\,\mathrm{d}x + \int D^3b\otimes D^3b : \nabla u\,\mathrm{d}x \\
& \leqslant \|\operatorname{rot}(D^3(b\times u) - D^3b\times u - b\times D^3u)\|_{L^2}\|D^3b\|_{L^2} \\
& \quad + \frac{1}{2}\|D^3b\|_{L^2}^2\|\operatorname{div} u\|_{L^\infty} + \|D^3b\|_{L^2}^2\|\nabla u\|_{L^\infty} \\
& \leqslant C(M).
\end{aligned}
$$

Here we have used the fact that

$$a\cdot\nabla a + a\times \operatorname{rot} a = \frac{1}{2}\nabla|a|^2. \tag{2.16}$$

Summing up (2.10), (2.14) and (2.15), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{1}{\gamma p}(D^3p)^2 + \varrho|D^3u|^2 + |D^3b|^2\Big)\,\mathrm{d}x + \mu\int|D^4u|^2\,\mathrm{d}x \\ + (\lambda+\mu)\int(D^3\operatorname{div}u)^2\,\mathrm{d}x + \int(D^3\nabla p\cdot D^3u + D^3pD^3\operatorname{div}u)\,\mathrm{d}x \\ + \int((b\times D^3\operatorname{rot}b)\cdot D^3u + (b\times D^3u)\cdot D^3\operatorname{rot}b)\,\mathrm{d}x \\ \leqslant C(M) + \|D^2\partial_t u\|_{L^2}^2.\end{aligned} \tag{2.17}$$

Noting that the last two terms of the LHS of (2.17) are zero and using (2.13), we arrive at

$$\|D^3(p,u,b)\|_{L^2}\leqslant C_0(M_0)\exp(tC(M)). \tag{2.18}$$

Applying D^3 to (1.4) and (1.5), testing by D^3I_0 and D^3I_1, respectively, summing up the results, and using (1.13) and (1.14), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int((D^3I_0)^2 + |D^3I_1|^2)\,\mathrm{d}x \\ &= \int D^3(\sigma_a(\varrho)(B(\varrho)-I_0))D^3I_0\,\mathrm{d}x - \int D^3((\sigma_a+\sigma_s)I_1)D^3I_1\,\mathrm{d}x \\ &\leqslant \|D^3(\sigma_a(B-I_0))\|_{L^2}\|D^3I_0\|_{L^2} + \|D^3((\sigma_a+\sigma_s)I_1)\|_{L^2}\|D^3I_1\|_{L^2}\leqslant C(M),\end{aligned}$$

which leads to

$$\|D^3I_0\|_{L^2} + \|D^3I_1\|_{L^2}\leqslant C_0(M_0)\exp(tC(M)). \tag{2.19}$$

On the other hand, it follows from (1.2) that

$$\begin{aligned}\|\partial_t u\|_{L^2} &= \Big\|\frac{1}{\varrho}((\sigma_a+\sigma_s)I_1 + \operatorname{rot}b\times b + \mu\Delta u + (\lambda+\mu)\nabla\operatorname{div}u - \nabla p - \varrho u\cdot\nabla u)\Big\|_{L^2} \\ &\leqslant C_0(M_0)\exp(tC(M)).\end{aligned} \tag{2.20}$$

Using the estimate [17]

$$\|D^3\varrho\|_{L^2}\leqslant C(1+\|p\|_{L^\infty})^3\|f\|_{W^{3,\infty}(I)}\|D^3p\|_{L^2} \tag{2.21}$$

with $\varrho = f(p) := (p/a)^{1/\gamma}$ and

$$I \subset \Big(\frac{1}{C_0(M_0)}\exp(-tC(M)), C_0(M_0)\exp(tC(M))\Big),$$

we have

$$\|D^3\varrho\|_{L^2} \leqslant C_0(M_0)\exp(tC(M)). \tag{2.22}$$

Combining (2.4), (2.5), (2.6), (2.7), (2.8), (2.18), (2.19), (2.20) and (2.22), we conclude that (1.11) holds true.

This completes the proof. □

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