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Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 3, 865–880

Czechoslovak Mathematical Journal, 71 (146) (2021), 865–880

MORSE-BOTT FUNCTIONS WITH TWO CRITICAL VALUES ON A SURFACE

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Received March 25, 2020. Published online March 15, 2021.

Abstract. We study Morse-Bott functions with two critical values (equivalently, nonconstant without saddles) on closed surfaces. We show that only four surfaces admit such functions (though in higher dimensions, we construct many such manifolds, e.g. as fiber bundles over already constructed manifolds with the same property). We study properties of such functions. Namely, their Reeb graphs are path or cycle graphs; any path graph, and any cycle graph with an even number of vertices, is isomorphic to the Reeb graph of such a function. They have a specific number of center singularities and singular circles with nonorientable normal bundle, and an unlimited number (with some conditions) of singular circles with orientable normal bundle. They can, or cannot, be chosen as the height function associated with an immersion of the surface in the three-dimensional space, depending on the surface and the Reeb graph. In addition, for an arbitrary Morse-Bott function on a closed surface, we show that the Euler characteristic of the surface is determined by the isolated singularities and does not depend on the singular circles.

Keywords: Morse-Bott function; height function; surface; critical value; Reeb graph

MSC 2020: 58C05, 57K20, 05C38

1. Introduction

A smooth function on a compact manifold has at least two critical values, the minimum and the maximum. We study some properties of the Morse-Bott functions on a closed surface that have no other critical values, as well as a related class of functions: Morse-Bott functions with no saddle singularities, i.e., with all singular points being local extrema.

In the class of well-known Morse functions (smooth functions with nondegenerate singularities), the choice of such functions is very narrow. A function with two critical values is possible only on a sphere S^n, see [13], Theorem 4.1. It can be chosen as a height function of an embedding of the sphere in $\mathbb{R}^{n+1}$; moreover, in the

DOI: 10.21136/CMJ.2021.0125-20

two-dimensional case, any Morse function can be realized as a height function of an immersion of the surface in $\mathbb{R}^3$, see [9], Theorem 1. Any Morse function with two critical levels has exactly two critical points: a minimum and a maximum, both of the center type. All its level sets are connected, i.e., its Reeb graph (the space obtained by contracting connected components of the level sets to points) is a closed interval.

In contrast, the class of Morse-Bott functions (smooth functions whose critical set is a submanifold with the function being nondegenerate in the normal direction) offers much greater variety: for example, a Morse-Bott function with two critical values can have any number of connected components of the critical submanifold; such a function does not necessarily admit a representation as a height function, and its Reeb graph is not necessarily an interval.

We study Morse-Bott functions with two critical values on a closed surface. For a given closed surface, existence of such function with certain relevant properties is equivalent to existence of a nonconstant function without saddle singularities, i.e., with all critical points being local extrema, see Proposition 4.1.

The critical set of a nonconstant Morse-Bott function on a closed surface consists of a finite number of isolated singularities and singular circles; the latter can only be local extrema. An interesting fact is that in the case of surfaces, the Euler characteristic of the manifold depends only on isolated singularities but not on the critical circles even in the presence of critical circles with nonorientable normal bundle (see Proposition 3.1); contrary to a widespread misconception (found in, e.g. [2], [7], [8]), in arbitrary dimension the situation (see Proposition 2.1) is different, see [17].

Whereas a Morse function with two critical values is possible only on the sphere, the set of closed surfaces admitting Morse-Bott functions with two critical values consists of the sphere S^2, the projective plane $\mathbb{R}P^2$, the torus T^2, and the Klein bottle K^2. The Reeb graph of such a function is a path graph or a cycle graph, depending on the surface. Such a function can have arbitrarily many singular circles with orientable normal bundle, though in the case of the Reeb graph being a cycle graph their number is to be even and nonzero (see Theorem 4.1). Any path graph and any cycle graph with an even number of vertices is isomorphic to the Reeb graph of such a function (see Corollary 4.1).

Such a function can have from zero to two center singularities and from zero to two singular circles with nonorientable normal bundle in different combinations, depending, again, on the surface and the Reeb graph, and can, or cannot, be chosen as the height function of an immersion of the surface in $\mathbb{R}^3$, depending on the presence of singular circles with nonorientable normal bundle. The possible combinations of these parameters are summarized in Table 1.

Martínez-Alfaro et al. in [11] introduced the notion of topological conjugacy of Morse-Bott functions and used the Reeb graphs for their classification on surfaces.

We show that on a connected surface there are up to two classes of topological conjugacy of Morse-Bott functions with two connected components of the critical set, and specify such classes for each surface, see Proposition 5.2. Functions with two connected components of the critical set (not necessarily Morse-Bott), especially when these components are projective spaces, were studied by Duan and Rees, see [3].

Our main results allow for various generalizations. We show that they can be extended to a wider class of functions, namely, topological Morse-Bott functions, see Corollary 5.1. Another research direction is the study of higher-dimensional cases. While for two-dimensional case, only four closed surfaces admit Morse-Bott functions with two critical values, higher dimensions, apart from very similar examples (see Example 6.1), offer much greater variety, which explodes with dimension, see Example 6.2. In particular, we show that if a manifold admits such functions, so do fiber bundles over it (see Lemma 6.1); this allows us to construct many new such manifolds out of a few basic examples. We will study a classification of n-manifolds, $n \geqslant 3$, admitting a Morse-Bott function with two critical values in a future work.

This paper is organized as follows. In Section 2, we introduce the notation we use, clarify the definitions, and give some known facts. In Section 3, we show that the Euler characteristic of a surface does not depend on singular circles of an Morse-Bott function, including those with nonorientable normal bundle. In Section 4, we give our main result: the set of surfaces that admit Morse-Bott functions with two critical values, along with some properties of such functions. In Section 5, we generalize our results to topological Morse-Bott functions and give a classification of Morse-Bott functions with two connected components of the critical set up to topological congugacy. Finally, in Section 6 we show that in higher dimensions many more manifolds admit Morse-Bott functions with two critical values.

2. Definitions and useful facts

We use the following notation for specific manifolds: S^n for an n-dimensional sphere, D^n for a closed disk (ball), $\mathbb{R}P^n$ for a projective space, T^n for a torus, and K^2 for the Klein bottle.

2.1. Morse-Bott functions. A *Morse-Bott function* $f\colon M \to \mathbb{R}$ is a smooth function on a smooth manifold M, whose critical set $\mathrm{Crit}(f)$ is a closed submanifold[1] with the Hessian being nondegenerate in the normal direction. A Morse function is a Morse-Bott function with zero-dimensional critical manifold.

[1] We assume that connected components of a submanifold can have different dimensions: for example, a submanifold can consist of a point and a circle.

Theorem 2.1 (Morse-Bott Lemma [1], Theorem 2)**.** *Let* $f\colon M \to \mathbb{R}$, *with* $\dim M = n$, *be a Morse-Bott function,* N_j *a connected component of* $\mathrm{Crit}(f)$ *of codimension* c, *and* $x \in N_j$. *Then in some neighborhood* U *of* x *there is a coordinate system*

$$(\underbrace{x_1, \ldots, x_k, x_{k+1}, \ldots, x_c}_{\text{normal to } N_j}, \underbrace{x_{c+1}, \ldots, x_n}_{\text{in } N_j})$$

such that

$$f\big|_U = f(N_j) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{c} x_i^2. \tag{2.1}$$

The number k in (2.1) does not depend on the choice of point $x \in N_j$ and is called the *index* $i(N_j)$ (sometimes called *Morse index*) of the connected component of the critical submanifold. In particular, all points in a connected component of $\mathrm{Crit}(f)$ are, or are not, local extrema (and if so, of the same type), so one can refer to the whole connected component of $\mathrm{Crit}(f)$ as a local minimum or maximum.

On a closed surface, connected components of the critical submanifold of a Morse-Bott function are points p_j and circles S^1_j. An isolated critical point p_j can be a center (minimum or maximum) or a saddle point, their indices being $i(p_j^{\min}) = 0$, $i(p_j^{\text{saddle}}) = 1$, and $i(p_j^{\max}) = 2$. A critical circle S^1_j can only be a minimum or a maximum; $i(S_j^{\min}) = 0$ and $i(S_j^{\max}) = 1$; its normal bundle can be orientable or nonorientable. In the case of orientable normal bundle, a critical circle S^1_j has a product neighborhood; otherwise, a tubular neighborhood of the critical circle is a Möbius strip.

Proposition 2.1 ([14], Corollary 2.6.6)**.** *Let* f *be a Morse-Bott function on a closed manifold* M *such that for every connected component* N_j *of* $\mathrm{Crit}(f)$, *the negative normal bundle* $E^-(N_j)$ *is orientable. Then for the Euler characteristics* $\chi(M)$ *it holds*

$$\chi(M) = \sum_{N_j} (-1)^{i(N_j)} \chi(N_j),$$

where $i(N_j)$ *is the index, i.e., the rank of the negative normal bundle* $E^-(N_j)$.

Contrary to a widespread misconception, exemplified by [2], [7], [8], the condition for orientability of the normal bundle in this statement is, generally, important [17]. However, below (see Proposition 3.1) we show that in the two-dimensional case this condition is indeed irrelevant.

2.2. Graphs. We consider finite graphs that allow loop edges and multiple edges (a loop edge is an edge incident to only one vertex; it is counted twice in the degree of this vertex). Such graphs can be represented as one-dimensional CW complexes. Two graphs are *isomorphic* when there exists a homeomorphism of the CW complexes that maps cells to cells, i.e., in combinatorial terms, when there is a correspondence between their vertices and edges, preserving incidence between vertices and edges.

The *cycle rank* $b_1(G)$ of a graph G is the first Betti number of the graph considered as a one-dimensional CW complex; in computational geometry this value is called the *number of loops* (not to be confused with loop edges).

A *trivial* graph has one vertex and no edges. A *path graph* P_n is a finite tree with two of its n vertices being of degree 1 (and all other, if any, of degree 2); note that we consider a path graph to be connected and nontrivial. A *cycle graph* C_n is a finite connected graph with all its n vertices being of degree 2; again, we consider a cycle graph to be nontrivial, so $b_1(C_n) = 1$.

2.3. Reeb graph. For a continuous function $f\colon M \to \mathbb{R}$ on a manifold M, consider the quotient space $M/{\sim}$ endowed with the quotient topology, where the equivalence relation $x \sim y$ holds whenever x and y belong to the same *contour* (connected component of a level set) of f. For a closed manifold M and a smooth function $f\colon M \to \mathbb{R}$ with a finite number of critical values, Saeki (see [18], Theorem 3.1) showed that this quotient space is homeomorphic to a finite graph (allowing multiple edges) (V, E), represented as a one-dimensional CW complex, where the set of vertices V corresponds to the set of the *critical contours* (contours containing a critical point). We will call this graph the *Reeb graph* R_f of the function f.

The quotient map $\varphi\colon M \to R_f$, called the *Reeb quotient map*, induces a continuous function $F = f \circ \varphi^{-1}\colon R_f \to \mathbb{R}$; this function is an embedding on the edges of the graph. The graph can be endowed with an orientation according to the increasing direction of the function F.

For a Morse-Bott function on a compact manifold, since its critical set has a finite number of connected components, the corresponding Reeb graph is a finite graph.

2.4. Corank of the fundamental group. The *corank* of a finitely generated group G is the maximum rank of a free homomorphic image of G. For a path-connected topological space X, consider the fundamental group $\pi_1(X)$. If it is finitely generated, as in the case of compact manifolds, then $\operatorname{corank}\pi_1(X)$ is finite. Obviously, $\operatorname{corank}\pi_1(X) \leqslant b_1(X)$, the first Betti number. For a surface M of genus g, it holds

$$\operatorname{corank}(\pi_1(M)) = \begin{cases} g & \text{if } M \text{ is orientable, see [10],} \\ \left\lfloor \dfrac{g}{2} \right\rfloor & \text{otherwise, see [5], equation (4.1).} \end{cases} \tag{2.2}$$

For a connected locally path-connected topological space X and a continuous function $f\colon X \to \mathbb{R}$ whose Reeb graph R_f is a finite topological graph, it holds [6], Theorem 3.1

$$b_1(R_f) \leqslant \operatorname{corank}(\pi_1(X)), \tag{2.3}$$

where $b_1(R_f)$ is the cycle rank. For connected smooth closed manifolds, this inequality is tight, see [6], Proposition 3.9.

3. Euler characteristic of a surface with a Morse-Bott function

We show that in the case of closed surfaces, the condition of orientability of negative normal bundles in Proposition 2.1 is not needed:

Proposition 3.1. *Let $f\colon M \to \mathbb{R}$ be a Morse-Bott function on a closed surface M. Then for the Euler characteristic of M it holds*

$$\chi(M) = |\{p_j^{\text{center}}\}| - |\{p_j^{\text{saddle}}\}|, \tag{3.1}$$

where $\{p_j^{\text{center}}\}$ is the set of all center singularities of f, and $\{p_j^{\text{saddle}}\}$ is the set of all isolated saddle singularities of f.

Note that for a closed surface, $\chi(M)$ does not depend on the number $|\{S_j^1\}|$ of critical circles (irrespective of the orientability of their normal bundles).

P r o o f. If all normal bundles of all critical circles $S_j = S_j^1$ of f are orientable, then, since $\chi(S_j) = 0$ and $\chi(p_j) = 1$, Proposition 2.1 gives

$$\chi(M) = \sum_{p_j} (-1)^{i(p_j)} \chi(p_j) = |\{p_j^{\text{center}}\}| - |\{p_j^{\text{saddle}}\}|.$$

Now, denote by $\widetilde{S}_j$ all critical circles $\widetilde{S}_j$ of f with nonorientable normal bundle. For each $\widetilde{S}_j$, a small f-saturated tubular neighborhood T_j is a Möbius strip, its boundary $\partial T_j = S^1$ being a circle on which f is regular and constant. Replace each T_j with a disk D_j and extend f on D_j as a Morse function with one center. This gives a surface M' with a Morse-Bott function f' having $|\{p_j^{\text{center}}\}| + |\{\widetilde{S}_j\}|$ centers, all its critical circles having orientable normal bundles. By the above, we obtain $\chi(M') = |\{p_j^{\text{center}}\}| + |\{\widetilde{S}_j\}| - |\{p_j^{\text{saddle}}\}|$. However, since M is M' with $|\{\widetilde{S}_j\}|$ Möbius strips glued, we have $\chi(M) = \chi(M') - |\{\widetilde{S}_j\}|$, which again gives (3.1). □

4. Surfaces admitting Morse-Bott functions with two critical values

The class of functions we are interested in can be described in various ways:

Proposition 4.1. *Let M be a closed surface. Denote by $\mathcal{F}$ the set of Morse-Bott functions $f\colon M \to \mathbb{R}$, with a given*[2] *Reeb graph, such that there exists (does not exist) an immersion of M in $\mathbb{R}^3$ with f being its associated height function.*

Then the following conditions are equivalent:

(i) *$\mathcal{F}$ includes a function with two critical values,*

(ii) *$\mathcal{F}$ includes a nonconstant function with no saddle singularities,*

(iii) *$\mathcal{F}$ includes a nonconstant function with all critical points being local extrema.*

The parentheses here mean two different versions of the statement.

Proof. (i) ⇒ (ii): Since a saddle singularity is not an extremum, a function with such singularity has at least three critical values.

(ii) ⇒ (iii): Critical points that are not saddles either are centers or belong to critical circles. Since M is a surface, they are local extrema.

(iii) ⇒ (i): Given a Morse-Bott function f with all its singular points or circles being local extrema, by suitable distortion of the function in a small saturated neighborhood of its critical set one can obtain a Morse-Bott function f' with all its local maxima at the same high enough level, and all its local minima at the same low enough level. This can be done in such a way that all level sets of f' be level sets of f and vice versa.

Indeed, let s be a center or circle singularity of f that is not a global extremum; without loss of generality assume it to be a local maximum. Consider a small connected f-saturated neighborhood U of s containing no other singularities; thus $f(s) = \max f(U)$ and $f(\partial U) = \inf f(U)$. Denote $I = f(U)$, a half-open interval. Consider a smooth function $g\colon I \to \mathbb{R}$ such that $g \equiv 1$ near the left end of I and $g \equiv \big(\max_M f\big)/f(s) > 1$ near its right end, monotonously increasing in between. Denote $g_s\colon M \to \mathbb{R}$ such that $g_s = g \circ f$ on U and $g_s \equiv 1$ on $M \setminus U$. This is a smooth function constant on level sets of f, the product $f'_s = g_s f$ being a Morse-Bott function with the same decomposition of M into level sets, the same critical set, and $f'_s(s) = \max_M f$. Repeating this operation for all center or circle singularities of f, we obtain the desired function f' with all local maximum values being $\max_M f$ (similarly, all local minimum values being $\min_M f$).

Note that if one of the two functions can be represented as the height function of a suitable immersion of M in $\mathbb{R}^3$, then so can be the other. □

[2] We say that functions share the same Reeb graph when they define the same decomposition of M into contours and their critical sets coincide.

In the class of Morse functions, a function with two critical values (i.e., nonconstant without saddles, or nonconstant with all critical points being local extrema) is possible only on S^2; such a function is the height function associated with an immersion (actually, embedding) of S^2 in $\mathbb{R}^3$, its Reeb graph is a path graph, and all singularities have orientable normal bundle. In contrast, in the class of Morse-Bott functions there are more options:

Theorem 4.1. *Let M be a connected closed surface. Then there exists a Morse-Bott function $f\colon M \to \mathbb{R}$ with exactly two critical values (equivalently, nonconstant without saddle singularities, or nonconstant with all critical points being local extrema) if and only if M is S^2, $\mathbb{R}P^2$, T^2, or K^2.*

The function f can be chosen with the Reeb graph isomorphic to a given graph G (with possible loop edges and multiple edges) if and only if

▷ *G is a path cycle graph P_n and M is S^2, $\mathbb{R}P^2$, or K^2, or*

▷ *G is a cycle graph C_n with an even n, and M is T^2 or K^2.*

Such function f can be chosen as the height function associated with an immersion of M in $\mathbb{R}^3$ if and only if G is a cycle graph or M is S^2.

The function f has center singularities and singular circles with nonorientable normal bundle only when R_f is a path graph: on S^2 (two centers), on $\mathbb{R}P^2$ (one center and one circle), and on K^2 (two circles). There exist such functions with any number of singular circles with orientable normal bundle when R_f is a path graph, and with any even number (at least two) of such circles when R_f is a cycle graph.

The cases listed in the theorem are summarized in Table 1. Note that we consider path and cycle graphs to be finite and nontrivial.

1 Reeb graph	2 S^1 orient. bundle no.	3 S^1 orient. bundle min.	4 S^1 nonor. bundle	5 Centers	6 Height function	7 Surface
path	any	0	0	2	yes	S^2
			1	1	no	$\mathbb{R}P^2$
			2	0	no	K^2
cycle	even	2	0	0	yes	T^2, K^2

Table 1. All cases of a Morse-Bott function f with two critical values on closed surfaces, according to Theorem 4.1. Columns 2 and 3 indicate the possible number and the minimum number of singular circles S^1 of f with orientable normal bundle; there are functions with any number of such circles satisfying these restrictions. Column 4 indicates the number of singular circles of f with nonorientable normal bundle. Column 5 indicates the number of isolated singularities of f (they are of the center type). Column 6 indicates whether the function can be chosen as the height function associated with an immersion of the surface in $\mathbb{R}^3$.

P r o o f. The equivalence of the definitions of the class of the functions of interest is given by Proposition 4.1. Now, in one direction, let such a function f exist. Since it is nonconstant, its Reeb graph R_f is not trivial.

Consider the restrictions stated in the columns 1, 5 and 7 of Table 1.

Since f has no saddles $p_j^{\rm saddle}$, i.e., the set of its isolated singularities is $\{p_j\} = \{p_j^{\rm center}\}$, Proposition 3.1 gives

$$b_1(M) = b_2(M) + 1 - |\{p_j\}|. \tag{4.1}$$

If M is orientable, then $b_2(M) = 1$ and (4.1) implies either the genus $g(M) = 0$ with $|\{p_j\}| = 2$, or $g(M) = 1$ with $|\{p_j\}| = 0$. By (2.3) and (2.2), the only option for S^2 is a path graph. For T^2, the graph could be path or cycle; however, a path graph is ruled out by the fact that, given $|\{p_j\}| = 0$, the function f on T^2 has only singular circles S_j^1; since their normal bundles are orientable (an orientable surface cannot have submanifolds with nonorientable normal bundle), the Reeb graph R_f has only vertices of degree 2.

If M is nonorientable, then $b_2(M) = 0$ and $b_1(M) \geqslant 1$, so (4.1) gives $g(M) = 1$ with $|\{p_j\}| = 1$ or $g(M) = 2$ with $|\{p_j\}| = 0$. Again, (2.3) implies that G is a path graph for $\mathbb{R}P^2$, or path or cycle graph for K^2.

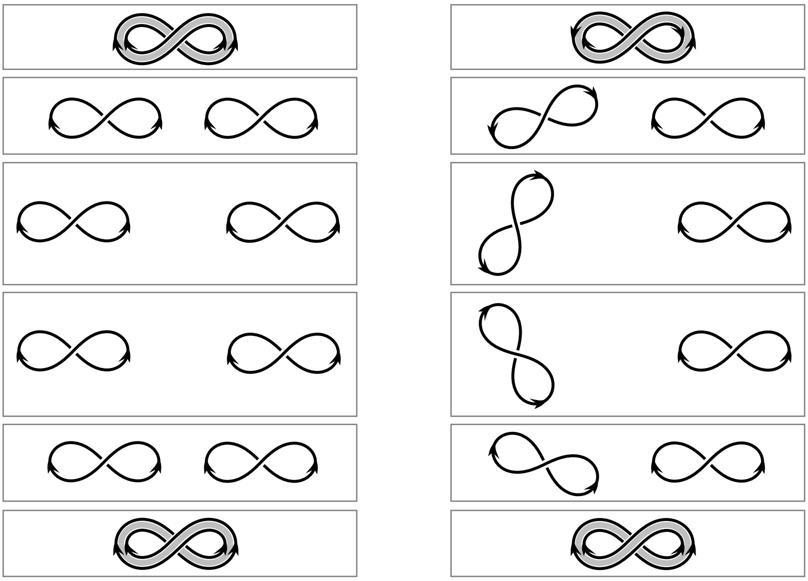
Let us show that in the case of G being a path graph and M being nonorientable, the function f cannot be the height function associated with an immersion of M to $\mathbb{R}^3$ (column 6 of the table). Since in this case we have $|\{p_j\}| \leqslant 1$, at least one of the extrema is to be a singular circle S^1. For the corresponding vertex of R_f to have the degree one, its normal bundle has to be nonorientable, a small tubular neighborhood of S^1 being a Möbius strip. Suppose such an immersion exists. Since $df = 0$ on S^1, by the implicit function theorem the projection to the horizontal plane gives an immersion of this Möbius strip in $\mathbb{R}^2$. It is, however, impossible to immerse a nonorientable manifold into an orientable one of the same dimension; a contradiction.

We have seen that in the case of R_f being a cycle graph, the only type of singularities is singular circles with orientable normal bundle. Since minima and maxima go in alternating order along the cycle graph, the number of such singular circles in this case is even (column 2 of the table). Since these are the only singularities, their number is positive (column 3).

Finally, singular circles with nonorientable normal bundle (Möbius strip) represent vertices of degree 1 of the Reeb graph. Thus, they are possible only for R_f being a path graph, and their number is the complement to 2 of the number of center singularities, which also correspond to the vertices of degree 1 (column 4).

In the opposite direction, we only need to give examples of the five combinations of the type of G and the type of M with the function f being the height function associated with an immersion of M in $\mathbb{R}^3$ whenever possible. We will first describe examples with the minimum number of singular circles (column 3).

The orientable case is easy: the desired function is the height function on a unit sphere S^2 or on a T^2 embedded as a doughnut lying flat on the table.

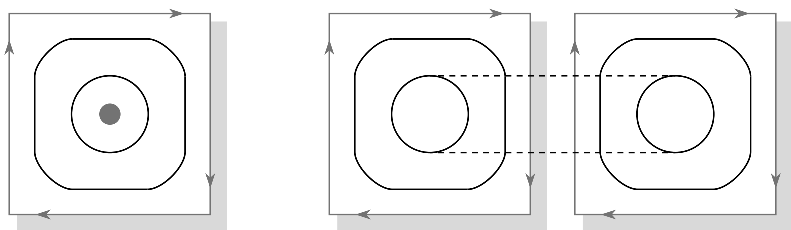


(a) (b)

Figure 1. (a) An immersion of a torus T^2 in $\mathbb{R}^3$ with coordinates (x, y, z) with the associated height function f being of Morse-Bott type and having no isolated singularities. The rectangles represent horizontal planes $\{z = \text{const.}\}$ at different levels, with the images (8-shaped) of the corresponding level sets of f shown. In thick lines, the images of the singular circles (minimum and maximum) of f are shown, along with a nearby level set each. Arrows show the evolution of orientation on the connected components of the level sets; it is seen that the orientation is consistent, implying that the obtained circle bundle over S^1 is orientable, i.e., T^2. (b) An immersion of the Klein bottle K^2 with the same properties. The twist (left) results in inconsistent orientation, as the evolution of the arrows, seen clockwise starting from the top, shows; therefore, the obtained circle bundle over S^1 is nonorientable, i.e., K^2.

An immersion of Klein bottle K^2 in $\mathbb{R}^3$ with the associated height function f of Morse-Bott type having only circle singularities with R_f being a path graph was found by Panov (see [15]); we briefly describe it here for completeness. The above example for T^2 (doughnut lying flat on the table) can also be implemented via an immersion of T^2 with the images of the connected components of the level sets of the height function f being 8-shaped, as in Figure 1 (a). Now, twist one of the tubes, as in Figure 1 (b). The obtained surface is a circle bundle over S^1; since the fiber changes its orientation, this is K^2.

For G being a path graph and $M = \mathbb{R}P^2$, the desired function f cannot be obtained as the height function associated with an immersion in $\mathbb{R}^3$. To construct the function f, represent $\mathbb{R}P^2$ as a closed unit disk D with the opposite points of the boundary ∂D identified, and consider f as the distance function from the center of D, suitably smoothed near its extrema. This is a Morse-Bott function with one center singularity (minimum) at the center of D and one singular circle (maximum) at the boundary ∂D; see Figure 2 (a).



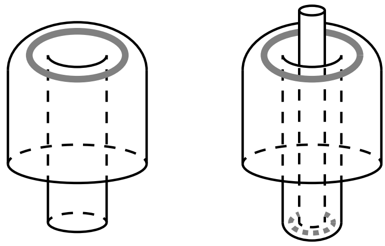
(a) (b)

Figure 2. The two nonorientable surfaces M admitting a Morse-Bott function f with two critical values, the Reeb graph R_f being a path graph. The critical set is shown in thick lines and a thick point; other level sets are shown as rounded thin lines. (a) $M = \mathbb{R}P^2$, the projective plane, shown as its fundamental square with the sides identified according to the arrows. The function f has one singular point and one singular circle. (b) $M = K^2$, the Klein bottle, shown as the connected sum $K^2 = \mathbb{R}P^2 \# \mathbb{R}P^2$, with two singular circles.

Finally, a desired function on K^2 with R_f being a path graph can be obtained as a connected sum of two copies of $\mathbb{R}P^2$ with the function described above. Namely, remove a small f-saturated neighborhood of the singular point in each copy and glue them together by the resulting boundary. Mirror the function from one of the copies to the other and suitably smooth it near the place of gluing. The obtained Morse-Bott function f on K^2 has two singular circles (a maximum and a minimum) and no isolated singularities, all its level sets being connected; see Figure 2 (b).

The only thing missing now is the column 2 of the table: once we have an example of a function with a minimum number of singular circles with orientable normal bundle, we can add more such circles. Consider a connected component c of a regular level of f. This is a circle; some its saturated neighborhood C is a cylinder, its fibers being level sets of $f\big|_C$, see [4], Lemma 3.1.

If the Reeb graph R_f is a path graph, then c is homologically trivial, i.e., $M \setminus c = M_1 \cup M_2$ is not connected. Assume, without loss of generality, $f(c) = 0$. Define $g\big|_{c \cup M_1} = f$ and $g\big|_{M_2} = -f$, and smooth it near c; see Figure 3 (a). We obtained a Morse-Bott function g having one more singular circle with orientable normal bundle than f, preserving other relevant properties, including the possibility of defining it as the height function of an immersion of M in $\mathbb{R}^2$ (the M_2 part of M is now immersed upside-down). Repeating this operation, we can obtain any number of such circles.



(a) (b)

Figure 3. (a) Adding a singular circle with orientable normal bundle by turning half of the surface upside-down. (b) Two such operations result in two such singular circles without affecting the function on the rest of the surface.

If R_f is a cycle graph, we can add such additional singular circles in pairs, without distorting the function outside a small cylinder C; see Figure 3 (b). □

Recall that a path graph is nontrivial.

Corollary 4.1. *A graph (admitting loop edges and multiple edges) is isomorphic to the Reeb graph of a Morse-Bott function on a closed surface with two critical values (equivalently, to the Reeb graph of a nonconstant Morse-Bott function on a surface without saddles) if and only if it is a (finite) path graph, or a (finite) cycle graph with an even number of vertices.*

5. A generalization: Topological Morse-Bott functions

Our results can be generalized to wider classes of functions. One possible simple generalization can proceed as follows.

Two smooth functions $f, g\colon M \to \mathbb{R}$ are *topologically equivalent* if there exist homeomorphisms $h\colon M \to M$ and $r\colon \mathbb{R} \to \mathbb{R}$ such that $g = r^{-1} \circ f \circ h$:

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle r} \\ M & \xrightarrow[g]{} & \mathbb{R}. \end{array}$$

Martínez-Alfaro et al. in [11], Definition 4 introduced a notion of (topological) conjugacy for Morse-Bott functions, which we can extend to arbitrary smooth functions:

Definition 5.1. Two topologically equivalent smooth functions $f, g\colon M \to \mathbb{R}$ are *topologically conjugate* if r preserves orientation and $\mathrm{Crit}(f) = h(\mathrm{Crit}(g))$.

Such functions have the same Reeb graph:

Proposition 5.1. *Let $f, g\colon M \to \mathbb{R}$ be topologically conjugate smooth functions with finite number of critical values on a closed manifold M. Then their Reeb graphs are isomorphic.*

Proof. By [18], Theorem 3.1, both Reeb graphs R_f and R_g have the structure of finite graphs. If $L = g^{-1}(a)$ is a level of g, then $h(L) = f^{-1}(r(a))$ is a level of f; moreover, h maps individual contours of g to contours of f. Denote by $\varphi_f\colon M \to R_f$ and $\varphi_g\colon M \to R_g$ the Reeb quotient maps; see Section 2.3. Obviously, $\varphi_f \circ h \circ \varphi_g^{-1}\colon R_g \to R_f$ is a homeomorphism of CW complexes that maps cells to cells; thus the corresponding graphs are isomorphic. □

By a *topological Morse-Bott function* we mean a smooth function topologically conjugate to a Morse-Bott function. Such functions are not necessarily Morse-Bott: for example, the Hessian on the critical set can be degenerate in the normal direction, and the critical set itself can even be not a submanifold but, e.g. have corners. For our main results, Proposition 5.1 implies:

Corollary 5.1. *Theorem* 4.1 *and Corollary* 4.1 *also hold for topological Morse-Bott functions.*

Morse-Bott functions with only two connected components of the critical set are minimal on the four surfaces on which they exist: S^2, $\mathbb{R}P^2$, T^2 and K^2. By [16], Table 1 on these surfaces all minimal functions with isolated singularities are topologically conjugate. In contrast, for minimal Morse-Bott functions on these surfaces there are more options:

Proposition 5.2. *On a connected surface, there are at most two classes of topological conjugacy of Morse-Bott functions whose critical set has two connected components:*

(i) *on S^2 and T^2, all such functions are topologically conjugate;*
(ii) *on $\mathbb{R}P^2$, the two classes A and B differ in the sign: $B = \{-f \mid f \in A\}$;*
(iii) *on K^2, the two classes differ in the Reeb graph: path vs. cycle.*

Proof. Theorem 4.1 lists all possible cases; see Table 1.

(i) On S^2 and T^2, all such functions define the same Reeb graph (P_2 or C_2, respectively), which admits only one acyclic orientation. By [11], Theorem 22, such functions are topologically conjugate.

(ii) In the case of $\mathbb{R}P^2$, the Reeb graph is again P_2, but its vertices correspond to different critical contours: one is a center, the other is a singular circle with nonorientable normal bundle. In terms of [11], Theorem 22, we have two different labeled digraphs, so [12], Theorem 5.7 gives two classes of topological conjugacy.

(iii) Finally, for K^2 there are two graphs, P_2 and C_2, in which both vertices are of the same type, so [12], Theorem 5.7 again gives two classes. □

6. Higher dimensions

The fact that only few surfaces admit Morse-Bott functions with two critical values is interesting in the context of that higher dimensions offer much greater variety of such manifolds. Below, we will give some examples; a more detailed study will be the topic of a separate paper.

Example 6.1. The following closed manifolds admit Morse-Bott functions with two critical values:

(i) S^n, a sphere $n \geqslant 1$;
(ii) $\mathbb{R}P^n$, a projective space, $n \geqslant 2$;
(iii) $L(p;q)$, a 3-dimensional lens space.

Indeed, (i) S^n admits a Morse function with two extrema.

(ii) Similarly to Figure 2 (a), represent $\mathbb{R}P^n = \mathbb{R}P^{n-1} \bigcup_\pi D^n$, where D^n is a unit ball and the attaching map is the projection $\pi\colon S^{n-1} \to \mathbb{R}P^{n-1}$. Consider the function $f_D\colon D^n \to \mathbb{R}$ that is the distance from the center $p \in D^n$, suitably smoothed near p, its center singularity (minimum), and the boundary $\partial D^n = S^{n-1}$, on which f_D is constant: $f_D(\partial D^n) \equiv 1$. Its extension on $\mathbb{R}P^{n-1}$ is constant, so the Morse-Bott function $f\colon \mathbb{R}P^{n-1} \bigcup_\pi D^n \to \mathbb{R}$ has two critical submanifolds: p and $\mathbb{R}P^{n-1}$.

(iii) Similarly, represent $L(p;q)$ as two solid tori $S^1 \times D^2$ glued by their boundary T^2. On each solid torus, consider a function $g = (1 - f_D) \circ \pi$, where π is the projection to the second factor and f_D is as above; this function is constant zero at the boundary T^2 and increases to a maximum on a singular circle $S^1 \times p$, where p is as above. Now, consider $f\colon L(p;q) \to \mathbb{R}$ with $f = g$ on one of the two solid tori and $f = -g$ on the other. This is a Morse-Bott function with two critical circles, one in each of the two solid tori.

Lemma 6.1. *Let M be a closed manifold, $\pi\colon M \to N$ be a fiber bundle over N, and $f\colon N \to \mathbb{R}$ be a Morse-Bott function with k critical values. Then so is the composition $g = \pi \circ f\colon M \to \mathbb{R}$.*

Proof. Since N is also a closed manifold and f is a Morse-Bott function, its critical set $\mathrm{Crit}(f) = \bigcup_j N_j$ is the finite union of closed submanifolds $N_j \subset N$ with nondegenerate Hessian in the normal direction. The bundle projection π is a submersion, so $\mathrm{Crit}(g) = \pi^{-1}(\mathrm{Crit}(f)) = \bigcup_j M_j$, where each $M_j = \pi^{-1}(N_j)$ is a closed submanifold of M.

Consider $x \in M_j$. Let $n = \dim N$ and $m = \dim \pi(M_j)$. By the Morse-Bott Lemma (see Theorem 2.1), in a neighborhood U of $\pi(x)$ in N there are coordinates

$$(\overbrace{\underbrace{x_1, \dots, x_{n-m}}_{\text{normal to } \pi(M_j)}, \underbrace{x_{n-m+1}, \dots, x_n}_{\pi(M_j)}}^{N})$$

such that $f\big|_U = f(\pi(x)) + \sum_{i=0}^{n-m} \pm x_i^2$. Denote by F the fiber, $l = \dim F$, and complete this coordinate system to a coordinate system

$$(\underbrace{\overbrace{\underbrace{x_1, \dots, x_{n-m}}_{\text{normal to } \pi(M_j)}}}_{\text{normal to } M_j} \underbrace{\overbrace{\underbrace{x_{n-m+1}, \dots, x_n}_{\pi(M_j)}}^{N}, \overbrace{x_{n+1}, \dots, x_{n+l}}^{F}}_{M_j})$$

in a neighborhood V of x in M with $\pi(V) = U$. In this coordinate system, we have the same expression $g = g(x) + \sum_{i=0}^{n-m} \pm x_i^2$, since g does not depend on the coordinates in the fiber F. We obtained that the Hessian of g is nondegenerate in the normal direction to M_j; thus g is a Morse-Bott function.

Finally, the critical levels and critical values of g correspond to those of f, thus g has the same number k of critical values. □

This lemma allows us to construct iteratively new closed manifolds with functions having two critical values, their variety increasing with dimension, e.g.:

Example 6.2. The following closed manifolds admit a Morse-Bott function with two critical values:

(i) manifolds from Theorem 4.1 and Example 6.1: S^n, $\mathbb{R}P^n$, T^2, K^2, $L(p;q)$;
(ii) connected sums $M_1 \# M_2$ by a center of such function, see Figure 2 (b);
(iii) similarly, connected sums along a manifold (when R_f is a path graph);
(iv) products $M^n = M^k \times M^{n-k}$, M^k being an already constructed manifold;
(v) fiber bundles over already constructed manifolds;
(vi) torus T^n and mapping tori (as fiber bundles over S^1);
(vii) compact nilmanifolds (as iterated torus bundles over a torus), e.g. the Heisenberg nilmanifold H^3 or the Kodaira-Thurston nilmanifold $H^3 \times S^1$.

Acknowledgements. We thank D. Panov for finding, by our request, the immersion (see [15]) of the Klein bottle, see Figure 1. We also thank the anonymous reviewer for suggesting us to consider topological Morse-Bott functions.

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