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SPANNING TREES WHOSE REDUCIBLE STEMS HAVE A FEW BRANCH VERTICES

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Abstract. Let T be a tree. Then a vertex of T with degree one is a leaf of T and a vertex of degree at least three is a branch vertex of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. For two distinct vertices u, v of T, let $P_T[u, v]$ denote the unique path in T connecting u and v. Let T be a tree with $B(T) \neq \emptyset$. For each leaf x of T, let y_x denote the nearest branch vertex to x. We delete $V(P_T[x, y_x]) \setminus \{y_x\}$ from T for all $x \in L(T)$. The resulting subtree of T is called the reducible stem of T and denoted by R_Stem(T). We give sharp sufficient conditions on the degree sum for a graph to have a spanning tree whose reducible stem has a few branch vertices.

Keywords: spanning tree; independence number; degree sum; reducible stem

MSC 2020: 05C05, 05C07, 05C69

1. Introduction

In this paper, we consider only finite simple graphs. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ (or $N(v)$ and $\deg(v)$ if there is no ambiguity) to denote the set of neighbors of v and the degree of v in G, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of X. Sometime, we denote it by $|G|$ instead of $|V(G)|$. We define $N_G(X) = \bigcup_{x\in X} N_G(x)$ and $\deg_G(X) = \sum_{x\in X} \deg_G(x)$. For $k \geqslant 1$, we put $N_k(X) = \{x \in V(G)\colon\ |N(x) \cap X| = k\}$. We use $G - X$ to denote the graph obtained from G by deleting the vertices in X together with their incident edges. We introduce $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding a new edge uv joining two non-adjacent vertices u and v of G. For two vertices u and v of G, the distance between u and v in G is denoted by $d_G(u,v)$. We use K_n to denote the complete graph on n vertices. We write $A := B$ to rename B as A.

For an integer $m \geqslant 2$, let $\alpha^m(G)$ denote the number defined by

$$\alpha^m(G) = \max\{|S|\colon\ S \subseteq V(G),\ d_G(x,y) \geqslant m \text{ for all distinct vertices } x, y \in S\}.$$

For an integer $p \geqslant 2$, we put

$$\sigma_p^m(G) = \min\{\deg_G(S)\colon\ S \subseteq V(G),\ |S| = p,\ d_G(x,y) \geqslant m \text{ for all distinct vertices } x, y \in S\}.$$

For convenience, we set $\sigma_p^m(G) = \infty$ if $\alpha^m(G) < p$. We note that $\alpha^2(G)$ is often written as $\alpha(G)$, which is the independence number of G, and $\sigma_p^2(G)$ is often written as $\sigma_p(G)$, which is the minimum degree sum of p independent vertices.

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. The subtree $T - L(T)$ of T is called the *stem* of T and is denoted by $\mathrm{Stem}(T)$. For two distinct vertices u, v of T, let $P_T[u,v]$ denote the unique path in T connecting u and v. We define that the *orientation* of $P_T[u,v]$ is from u to v. For each vertex $x \in V(P_T[u,v])$, we denote by x^+ and x^- the successor and predecessor of x in $P_T[u,v]$, respectively, if they exist. We refer to [4] for terminology and notation not defined here.

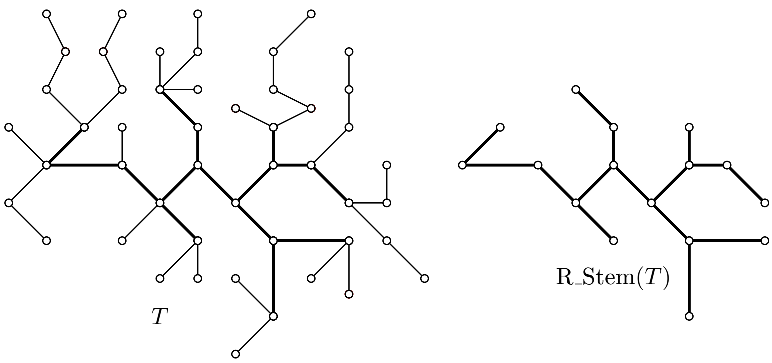

Figure 1. Tree T and R_Stem(T)

For a leaf x of T, let y_x denote the nearest branch vertex to x. For each leaf x of T, we remove the path $P_T[x,y_x)$ from T, where $P_T[x,y_x)$ denotes the path connecting x to y_x in T but not containing y_x. Moreover, the path $P_T[x,y_x)$ is called the *leaf-branch path of* T *incident to* x and denoted by $\mathrm{lb}P_T(x)$. The resulting subtree of T

is called the *reducible stem* of T and denoted by R_Stem(T) (see Figure 1 for an example of T and R_Stem(T)). Then $\mathrm{R\_Stem}(T) = T - \bigcup_{x\in L(T)} V(\mathrm{lb}P_T(x))$. A leaf of R_Stem($T$) is also called a *peripheral branch vertex* of T, see [6], [13].

There are several sufficient conditions (such as the independence number conditions and the degree sum conditions) for a graph G to have a spanning tree with a bounded number of leaves or branch vertices (see the survey paper [15] and the references cited therein for details). Win in [17] obtained the following theorem, which confirms a conjecture of Las Vergnas (see [12]), and Broersma and Tuinstra in [1] gave the following sufficient condition for a graph to have a spanning tree with at most k leaves.

Theorem 1.1 ([17]). *Let $l \geqslant 1$ and $k \geqslant 2$ be integers and let G be an l-connected graph. If $\alpha(G) \leqslant k + l - 1$, then G has a spanning tree with at most k leaves.*

Theorem 1.2 ([1]). *Let G be a connected graph and let $k \geqslant 2$ be an integer. If $\sigma_2(G) \geqslant |G| - k + 1$, then G has a spanning tree with at most k leaves.*

Recently, many researchers studied spanning trees in connected graphs whose stems have a bounded number of leaves or branch vertices, see [8], [9], [16] and [18] for more details. We introduce here some results on spanning trees whose stems have a few leaves or branch vertices.

Theorem 1.3 ([16]). *Let G be a connected graph and let $k \geqslant 2$ be an integer. If $\sigma_3(G) \geqslant |G| - 2k + 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 1.4 ([8]). *Let G be a connected graph and let $k \geqslant 2$ be an integer. If either $\alpha^4(G) \leqslant k$ or $\sigma_{k+1}(G) \geqslant |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 1.5 ([18]). *Let G be a connected graph and $k \geqslant 0$ be an integer. If one of the conditions*
(a) $\alpha^4(G) \leqslant k + 2$,
(b) $\sigma^4_{k+3}(G) \geqslant |G| - 2k - 3$
holds, then G has a spanning tree whose stem has at most k branch vertices.

Furthermore, by considering the graph G restricted in some special graph classes, many analogous researches have been introduced, see [2], [3], [5], [7], [10], [11] and [14] for example.

Recently, Ha, Hanh and Loan in [6] have introduced a new concept of spanning trees and gave a sufficient condition for a graph to have a spanning tree possessing such a property. Namely, they obtained the following theorem.

Theorem 1.6 ([6]). *Let G be a connected graph and let $k \geqslant 2$ be an integer. If one of the conditions*

(i) $\alpha(G) \leqslant 2k+2$,

(ii) $\sigma^4_{k+1}(G) \geqslant \lfloor \frac{1}{2}(|G|-k) \rfloor$

holds, then G has a spanning tree with at most k peripheral branch vertices. Here, the notation $\lfloor r \rfloor$ stands for the floor, i.e., the largest integer not exceeding the real number r.

In this paper, we would like to study sufficient conditions for a graph to have a spanning tree T such that R_Stem(T) has a bounded number of branch vertices. In particular, we prove the following theorem.

Theorem 1.7. *Let G be a connected graph and let $k \geqslant 2$ be an integer. If the condition*

$$\sigma^4_{k+3}(G) \geqslant \left\lfloor \frac{|G|-2k-2}{2} \right\rfloor$$

holds, then G has a spanning tree T whose reducible stem has at most k branch vertices.

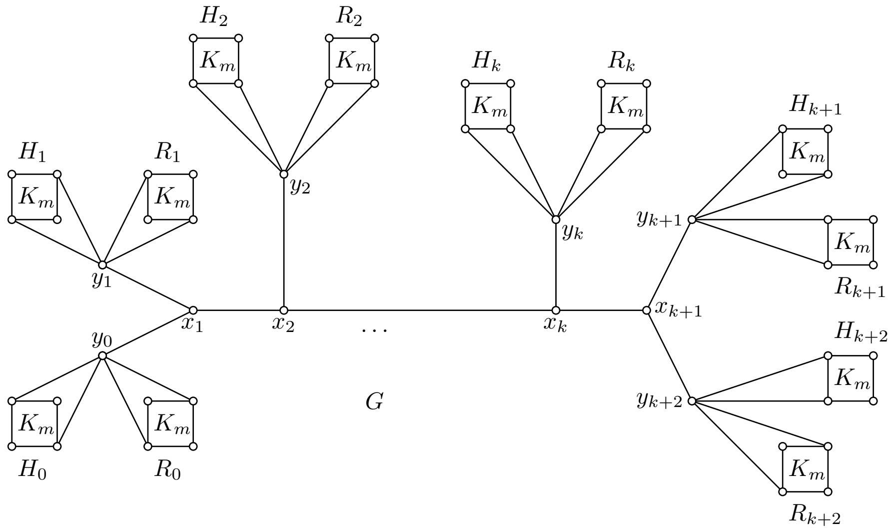

Figure 2. Graph G

To show that our result is sharp, we will give the following example. Let $k \geqslant 2$ and $m \geqslant 1$ be integers, and let $R_0, R_1, \ldots, R_{k+2}$ and $H_0, H_1, \ldots, H_{k+2}$ be $2k+6$ disjoint copies of the complete graph K_m of order m. Let $y_0, y_1, \ldots, y_{k+2}$, $x_1, x_2, \ldots, x_{k+1}$ be the $2k+4$ vertices not contained in $R_0 \cup R_1 \cup \ldots \cup R_{k+2} \cup H_0 \cup H_1 \cup \ldots \cup H_{k+2}$. Join y_i to all the vertices of $R_i \cup H_i$ for every $0 \leqslant i \leqslant k+2$. Add the two edges x_1y_0,

$x_{k+1}y_{k+2}$ and join x_i to y_i for each $1 \leqslant i \leqslant k+1$. Let G denote the resulting graph, see Figure 2. Then $\alpha^4(G) = k+3$. Moreover, we also obtain

$$\sigma^4_{k+3}(G) = \sum_{i=1}^{k+3} \deg_G(a_i) = (k+3)m = \left\lfloor \frac{|G| - 2k - 4}{2} \right\rfloor,$$

where a_i is any vertex of H_i for each $0 \leqslant i \leqslant k+2$.

But G has no spanning tree whose reducible stem has at most k branch vertices. Then, our main result is sharp.

2. Proof of Theorem 1.7

Firstly, we recall the following useful lemma.

Lemma 2.1. *Let T be a tree. Then the number of leaves in T is*

$$|L(T)| = \sum_{x \in B(T)} (\deg_T(x) - 2) + 2.$$

P r o o f of Theorem 1.7. Suppose to the contrary that there does not exist a spanning tree T of G such that $|B(\mathrm{R\_Stem}(T))| \leqslant k$. Then every spanning tree T of G satisfies $|B(\mathrm{R\_Stem}(T))| \geqslant k+1$.

Choose a maximal tree T of G that satisfies

(C0) $|B(\mathrm{R\_Stem}(T))| = k+1$,
(C1) $|L(\mathrm{R\_Stem}(T))|$ is as small as possible subject to (C0),
(C2) $|L(T)|$ is as small as possible subject to (C1),
(C3) $|\mathrm{R\_Stem}(T)|$ is as small as possible subject to (C2).

Claim 2.2. There does not exist a tree S in G such that $V(S) = V(T)$ and $|B(\mathrm{R\_Stem}(S))| \leqslant k$.

P r o o f. Indeed, assume that there exists a tree S in G such that $V(S) = V(T)$ and $|B(\mathrm{R\_Stem}(S))| \leqslant k$. Since $|B(\mathrm{R\_Stem}(S))| \leqslant k$, S is not a spanning tree of G. Then there exists $u \in V(G) - V(S)$ such that u is adjacent to a vertex $v \in S$. Let S_1 be a tree obtained from S by adding the edge uv. Then S_1 is a tree in G such that $|V(S_1)| = |V(T)| + 1$ and $|B(\mathrm{R\_Stem}(S_1))| \leqslant k+1$.

If $|B(\mathrm{R\_Stem}(S_1))| = k+1$, then S_1 contradicts the maximality of T (since $|V(S_1)| = |V(S)| + 1 = |V(T)| + 1 > |V(T)|$). So we may assume that

$$|B(\mathrm{R\_Stem}(S_1))| \leqslant k.$$

By repeating this process, we can recursively construct a set of trees $\{S_i\colon\ i \geqslant 1\}$ in G such that S_i satisfies that $|B(\text{R\_Stem}(S_i))| \leqslant k$ and $|V(S_{i+1})| = |V(S_i)| + 1$ for each $i \geqslant 1$. Since G has no spanning tree T with at most k branch vertices of R_Stem(T) and $|V(G)|$ is finite, the process must terminate after a finite number of steps, i.e., there exists some $h \geqslant 1$ such that S_{h+1} is a tree in G with $|B(\text{R\_Stem}(S_{h+1}))| = k+1$. But this contradicts the maximality of T. So the claim holds. □

Let $B(\text{R\_Stem}(T)) = \{x_1, x_2, \ldots, x_{k+1}\}$ and $L(\text{R\_Stem}(T)) = \{y_1, y_2, y_3, \ldots, y_l\}$. Then $l \geqslant k+3$ by Lemma 2.1. By the definition of the leaf of R_Stem(T), we have the following claim.

Claim 2.3. For each y_i, $1 \leqslant i \leqslant l$, there exist at least two leaves T which are connected to y_i by paths in T. Namely, T has at least two leaf-branch paths connecting y_i to leaves of T.

Claim 2.4. For each y_i, $1 \leqslant i \leqslant l$, there exist $a_i, b_i \in L(T)$ such that $\text{lb}P_T(a_i)$ and $\text{lb}P_T(b_i)$ connect a_i and b_i to y_i, respectively, and

$$N_G(a_i) \cap (V(\text{R\_Stem}(T)) - \{y_i\}) = \emptyset \quad \text{and} \quad N_G(b_i) \cap (V(\text{R\_Stem}(T)) - \{y_i\}) = \emptyset.$$

P r o o f. Assume that there exists y_s, $1 \leqslant s \leqslant l$ for which the claim does not hold. Then each leaf-branch path $P_T[z_j, y_s]$, $1 \leqslant j \leqslant m$, except at most one such a path, satisfies $N_G(z_j) \cap (V(\text{R\_Stem}(T)) - \{y_s\}) \neq \emptyset$. For each z_j, $1 \leqslant j \leqslant m$, take a vertex $t_j \in N_G(z_j) \cap (V(\text{R\_Stem}(T)) - \{y_s\})$ and let $v_j = N_T(y_s) \cap V(P_T[z_j, y_s])$. Then $T' := T + \{z_j t_j\colon\ 1 \leqslant j \leqslant m\} - \{y_s v_j\colon\ 1 \leqslant j \leqslant m\}$ satisfies $V(T') = V(T)$, $|L(\text{R\_Stem}(T'))| \leqslant |L(\text{R\_Stem}(T))|$, $|L(T')| = |L(T)|$ and $|\text{R\_Stem}(T')| < |\text{R\_Stem}(T)|$, since y_s is not a vertex of R_Stem(T'). This gives a contradiction. Therefore, Claim 2.4 holds. □

Set $U = \{a_i, b_i\colon\ 1 \leqslant i \leqslant l\}$.

Claim 2.5. U is an independent set in G.

P r o o f. Suppose that there exist two vertices $u, v \in U$ such that $uv \in E(G)$. Without lost of generality, we assume that $v = a_i$ for some $i \in \{1, 2, \ldots, l\}$. Set $v_i \in N_T(y_i) \cap V(\text{lb}P_T(a_i))$. Consider the tree $T' := T + ua_i - v_i y_i$. Then the number of vertices of T' remains unchanged, i.e., equal to that of T, $|B(\text{R\_Stem}(T'))| \leqslant |B(\text{R\_Stem}(T))|$, $|L(\text{R\_Stem}(T'))| \leqslant |L(\text{R\_Stem}(T))|$ and $|L(T')| < |L(T)|$. This contradicts either Claim 2.2 or the condition (C1) or the condition (C2). The proof of Claim 2.5 is completed. □

Claim 2.6. For each $i, j \in \{1, 2, \ldots, k+1\}$ with $i \neq j$, it follows that $N_G(a_i) \cap \text{lb}P_T(a_j) = \emptyset$ and $N_G(a_i) \cap \text{lb}P_T(b_j) = \emptyset$.

P r o o f. As a_i and b_i play the same role, we only need to prove $N_G(a_i) \cap \mathrm{lb}P_T(a_j) = \emptyset$. Suppose the assertion of the claim is false. Then there exists a vertex $x \in N_G(a_i) \cap \mathrm{lb}P_T(a_j)$. Set $T' := T + xa_i$. Then T' is a subgraph of G including a unique cycle C, which contains both y_i and y_j.

Since $|B(\mathrm{R\_Stem}(T))| \geqslant 1$. Then there exists a branch vertex u of $\mathrm{R\_Stem}(T)$ contained in C. Let e be an edge of C incident with u. By removing the edge e from T' we obtain a tree T'' of G satisfying $V(T'') = V(T)$, $|B(\mathrm{R\_Stem}(T''))| \leqslant |B(\mathrm{R\_Stem}(T))|$ and $|L(\mathrm{R\_Stem}(T''))| < |L(\mathrm{R\_Stem}(T))|$, since y_i and y_j are not leaves of $\mathrm{R\_Stem}(T'')$. This contradicts either Claim 2.2 or the condition (C1). So Claim 2.6 is proved. □

Claim 2.7. For each $1 \leqslant i \neq j \leqslant l$, $d_G(s_i, s_j) \geqslant 4$ for $s_i \in \{a_i, b_i\}$ and $s_j \in \{a_j, b_j\}$.

P r o o f. By the symmetry of a_i and b_i, it suffices to show that $d_G(a_i, a_j) \geqslant 4$. Let $P[a_i, a_j]$ be a shortest path connecting a_i and a_j in G. Assume that all the vertices of $P[a_i, a_j]$ are contained in $(V(G) - \mathrm{R\_Stem}(T)) \cup \{y_i, y_j\}$.

Let t_i be the vertex of $\mathrm{lb}P_T(a_i) \cap P[a_i, a_j]$ closest to y_i, and t_j be the vertex of $\mathrm{lb}P_T(a_j) \cap P[a_i, a_j]$ closest to y_j. Then $P[a_i, a_j] = P_G[a_i, t_i] \cup P_G[t_i, t_j] \cup P_G[t_j, a_j]$, where $P_G[t_i, t_j]$ passes only through vertices contained in $(V(G - \mathrm{R\_Stem}(T))) \cup \{y_i, y_j\}$.

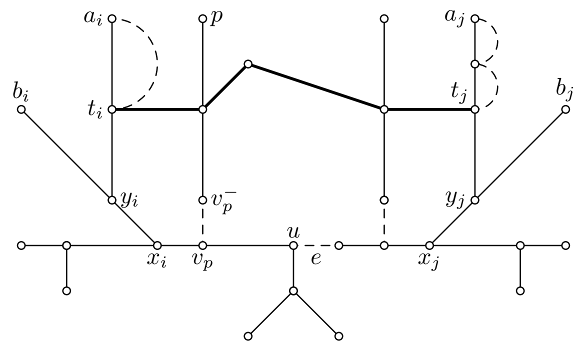

Figure 3. Tree T''

For each leaf-branch path $\mathrm{lb}P_T(p)$ of T such that $\mathrm{lb}P_T(p) \cap P[t_i, t_j] \neq \emptyset$, remove the edge of $\mathrm{lb}P_T(p)$ incident to $\mathrm{R\_Stem}(T)$ and add $P[t_i, t_j]$. Then the resulting subgraph T' of G includes a unique cycle C, which contains two vertices y_i and y_j. Because $|B(\mathrm{R\_Stem}(T))| \geqslant 1$, there exists a branch vertex u of $\mathrm{R\_Stem}(T)$ contained in C. Let e be an edge in C incident with u. Denote by T'' the tree obtained from T by removing the edge e, see Figure 3 for an example. Then $V(T) \subseteq V(T') = V(T'')$, $|B(\mathrm{R\_Stem}(T''))| \leqslant |B(\mathrm{R\_Stem}(T))|$ and

$|L(\text{R\_Stem}(T''))| < |L(\text{R\_Stem}(T))|$, where y_i and y_j are not leaves of $\text{R\_Stem}(T'')$. This contradicts either the maximality of T or Claim 2.2 or the condition (C1). Therefore, $P[a_i, a_j] \cap (\text{R\_Stem}(T) - \{y_i, y_j\}) \neq \emptyset$. Set $v \in P[a_i, a_j] \cap (\text{R\_Stem}(T) - \{y_i, y_j\})$. Hence, by combining with Claim 2.4, we obtain

$$d_G(a_i, a_j) = d_{P[a_i,a_j]}(a_i, a_j) = d_{P[a_i,a_j]}(a_i, v) + d_{P[a_i,a_j]}(v, a_j) \geqslant 2 + 2 = 4.$$

This completes the proof of Claim 2.7. □

By Claim 2.7 we obtain that $\alpha^4(G) \geqslant l \geqslant k + 3$.

Claim 2.8. $\sum\limits_{y \in U} |N_G(y) \cap \text{lb}P_T(p)| \leqslant |\text{lb}P_T(p)| - 1$ for every $p \in L(T) - U$.

Proof. Let $p \in L(T) - U$ and let v_p be the nearest branch vertex of T to p. Then $P_T[p, v_p) \cap B(T) = \emptyset$.

Subclaim 2.8.1. $\{p, v_p^-\} \cap N_G(U) = \emptyset$.

Proof. Indeed, to the contrary, without loss of generality, assume that $q \in N_G(a_i)$ for some $a_i \in U$ and $q \in \{p, v_p^-\}$. We consider the tree $T' := T + a_i q - v_p v_p^-$. Hence, T' is a tree with $|V(T')| = |V(T)|$, $|B(\text{R\_Stem}(T'))| = k + 1$, $|L(\text{R\_Stem}(T'))| = |L(\text{R\_Stem}(T))|$ and $|L(T')| < |L(T)|$. This contradicts the condition (C2). Therefore, $\{p, v_p^-\} \cap N_G(U) = \emptyset$. □

Subclaim 2.8.2. If every $x \in \text{lb}P_T(p)$ then x is adjacent to at most 2 vertices in U.

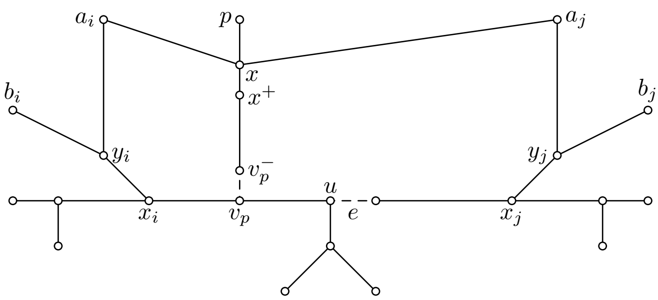

Figure 4. Tree T''

Proof. Indeed, we first prove that if $x \in N_G(a_i) \cap \text{lb}P_T(p)$, then $x \notin N_G(a_j) \cup N_G(b_j)$ for all $1 \leqslant i \neq j \leqslant l$. In particular, $N_3(U) \cap \text{lb}P_T(p) = \emptyset$. To the contrary, without loss of generality, assume that there exist $1 \leqslant i \neq j \leqslant k + 1$ such that $x \in N_G(a_i) \cap \text{lb}P_T(p)$ and $x \in N_G(a_j)$. Set $T' := T + \{xa_i, xa_j\} - v_p v_p^-$. Then T' is a subgraph of G including a unique cycle C, which contains two vertices, y_i and y_j. Since $|B(\mathrm{R\_Stem}(T))| \geqslant 1$, there exists a branch vertex in $\mathrm{R\_Stem}(T)$ contained in C. Let e be an edge of C incident with u. By removing the edge e we obtain a tree T'' of G, see Figure 4 for an example. Then $|V(T'')| = |V(T)|$, $|B(\mathrm{R\_Stem}(T''))| \leqslant |B(\mathrm{R\_Stem}(T))|$ and $|L(\mathrm{R\_Stem}(T''))| < |L(\mathrm{R\_Stem}(T))|$, where y_i and y_j are not leaves of $\mathrm{R\_Stem}(T'')$. This contradicts either Claim 2.2 or the condition (C1). Therefore, we obtain $|U \cap N_G(x)| \leqslant 2$. □

For convenience, let $a_{l+j} := b_j$ for all $1 \leqslant j \leqslant l$, and thus $U = \{a_1, a_2, \ldots, a_{2l}\}$.

Subclaim 2.8.3. *For each $i \in \{1, 2, \ldots, 2l\}$, if $x \in N_G(a_i) \cap \mathrm{lb}P_T(p)$ then $x^+ \notin N_G(U - \{a_i\}) \cap \mathrm{lb}P_T(p)$.*

Proof. Suppose that there exists $x^+ \in N_G(z) \cap \mathrm{lb}P_T(p)$ with $z \in U - \{a_i\}$. Let $T' := T + \{xa_i, x^+z\} - \{xx^+, v_pv_p^-\}$. Then, T' is a tree with $|V(T')| = |V(T)|$, $|B(\mathrm{R\_Stem}(T'))| = k+1$, $|L(\mathrm{R\_Stem}(T'))| = |L(\mathrm{R\_Stem}(T))|$ and $|L(T')| < |L(T)|$. This contradicts the condition (C2). □

Now, by Subclaims 2.8.2 and 2.8.3 we conclude that $\{p\}$, $N_G(a_i) \cap \mathrm{lb}P_T(p)$, $(N_G(U - \{a_i\}) \cap \mathrm{lb}P_T(p))^+$ and $(N_2(U) - N(a_i)) \cap \mathrm{lb}P_T(p)$ are pairwise disjoint subsets in $\mathrm{lb}P_T(p)$ for each $1 \leqslant i \leqslant 2l$. Note that if $z \in (N_2(U) - N(a_i)) \cap \mathrm{lb}P_T(p)$, then $z \in N_G(a_k) \cap N_G(b_k)$ for some $1 \leqslant k \neq i \leqslant l$ and $z^- \notin N(U)$. Recall that $N_3(U) \cap \mathrm{lb}P_T(p) = \emptyset$ by Subclaim 2.8.2. Then by combining with Subclaim 2.8.1 we obtain

$$\begin{aligned}|\mathrm{lb}P_T(p)| - 1 &\geqslant |N(a_i) \cap \mathrm{lb}P_T(p)| + |(N(U - \{a_i\}) \cap \mathrm{lb}P_T(p))^+| \\ &\quad + |(N_2(U) - N(a_i)) \cap \mathrm{lb}P_T(p)| \\ &= |N(a_i) \cap \mathrm{lb}P_T(p)| + |N(U - \{a_i\}) \cap \mathrm{lb}P_T(p)| \\ &\quad + |(N_2(U) - N(a_i)) \cap \mathrm{lb}P_T(p)| \\ &= \sum_{y \in U} |N_G(y) \cap \mathrm{lb}P_T(p)|.\end{aligned}$$

Claim 2.8 is proved. □

Claim 2.9. For each $1 \leqslant i \leqslant l$, it is $\sum\limits_{y \in U} |N_G(y) \cap \mathrm{lb}P_T(a_i)| \leqslant |\mathrm{lb}P_T(a_i)| - 1$ and $\sum\limits_{y \in U} |N_G(y) \cap \mathrm{lb}P_T(b_i)| \leqslant |\mathrm{lb}P_T(b_i)| - 1$.

Proof. As a_i and b_i play the same role, we only need to prove $\sum\limits_{y \in U} |N_G(y) \cap \mathrm{lb}P_T(a_i)| \leqslant |\mathrm{lb}P_T(a_i)| - 1$.

By Claim 2.6, we conclude that $N_G(U) \cap \mathrm{lb}P_T(a_i) = N_G(\{a_i, b_i\}) \cap \mathrm{lb}P_T(a_i)$.

Subclaim 2.9.1. *For* $y_i^- \in \mathrm{lb}P_T(a_i)$, $y_i^- \notin N_G(b_i)$.

Proof. Assume that y_i^- is adjacent to b_i in G. Consider the tree $T' = T + b_i y_i{}^- - y_i{}^- y_i$. Then T' is a tree of G such that $V(T') = V(T)$, $|B(\mathrm{R\_Stem}(T'))| \leqslant |B(\mathrm{R\_Stem}(T))|$, $|L(\mathrm{R\_Stem}(T'))| \leqslant |L(\mathrm{R\_Stem}(T))|$ and $|L(T')| < |L(T)|$. This contradicts either Claim 2.2 or the condition (C1) or the condition (C2). □

Subclaim 2.9.2. *If* $x \in N_G(a_i) \cap \mathrm{lb}P_T(a_i)$, *then* $x^- \notin N_G(b_i)$.

Proof. Suppose that there exists $x \in N_G(a_i) \cap \mathrm{lb}P_T(a_i)$ such that $x^- \in N_G(b_i) \cap \mathrm{lb}P_T(a_i)$. Set $T' := T + \{xa_i, b_i x^-\} - \{xx^-, y_i^- y_i\}$, where $y_i^- \in \mathrm{lb}P_T(a_i)$. Hence, T' is a tree of G such that $V(T') = V(T)$, $|B(\mathrm{R\_Stem}(T'))| \leqslant |B(\mathrm{R\_Stem}(T))|$, $|L(\mathrm{R\_Stem}(T'))| \leqslant |L(\mathrm{R\_Stem}(T))|$ and $|L(T')| < |L(T)|$. This contradicts either Claim 2.2 or the condition (C1) or the condition (C2). Subclaim 2.9.2 holds. □

By Subclaims 2.9.1 and 2.9.2 and Claim 2.6 we conclude that $\{a_i\}$, $N_G(a_i) \cap \mathrm{lb}P_T(a_i)$ and $(N_G(b_i) \cap \mathrm{lb}P_T(a_i))^+$ are pairwise disjoint subsets in $\mathrm{lb}P_T(a_i)$. Then

$$\begin{aligned}\sum_{y\in U}|N_G(y) \cap \mathrm{lb}P_T(a_i)| &= |N_G(a_i) \cap \mathrm{lb}P_T(a_i)| + |N_G(b_i) \cap \mathrm{lb}P_T(a_i)| \\ &= |N_G(a_i) \cap \mathrm{lb}P_T(a_i)| + |(N_G(b_i) \cap \mathrm{lb}P_T(a_i))^+| \\ &\leqslant |\mathrm{lb}P_T(a_i)| - 1.\end{aligned}$$

This completes the proof of Claim 2.9. □

By Claims 2.4, 2.6, 2.8 and 2.9, we obtain that

$$\begin{aligned}\deg_G(U) &= \sum_{i=1}^{l}(\deg_G(a_i) + \deg_G(b_i)) \\ &\leqslant \sum_{i=1}^{l}(|\mathrm{lb}P_T(a_i)| - 1) + \sum_{i=1}^{l}(|\mathrm{lb}P_T(b_i)| - 1) + 2|\{y_i\colon\ 1 \leqslant i \leqslant l\}| \\ &\quad + \sum_{p\in L(T)-U}|\mathrm{lb}P_T(p) - 1| \\ &\leqslant \sum_{i=1}^{l}(|\mathrm{lb}P_T(a_i)|) + \sum_{i=1}^{l}(|\mathrm{lb}P_T(b_i)|) + \sum_{p\in L(T)-U}|\mathrm{lb}P_T(p)| \\ &= |G| - |\mathrm{R\_Stem}(T)|.\end{aligned}$$

On the other hand, we note that $|\mathrm{R\_Stem}(T)| \geqslant l + k + 1 \geqslant 2k + 4$. Hence,

$$\begin{aligned}\sum_{i=1}^{l}\deg_G(a_i) + \sum_{i=1}^{l}\deg_G(b_i) \leqslant |G| - 2k - 4 \Rightarrow \min\Big\{\sum_{i=1}^{l}\deg_G(a_i), \sum_{i=1}^{l}\deg_G(b_i)\Big\} \\ \leqslant \Big\lfloor\frac{|G| - 2k - 4}{2}\Big\rfloor.\end{aligned}$$

Combining the above inequality with Claim 2.7, we obtain

$$\sigma_l^4(G) \leqslant \min\Big\{\sum_{i=1}^{l} \deg_G(a_i), \sum_{i=1}^{l} \deg_G(b_i)\Big\} \leqslant \Big\lfloor \frac{|G|-2k-4}{2} \Big\rfloor.$$

Moreover, $l \geqslant k+3$ and we conclude that

$$\sigma_{k+3}^4(G) \leqslant \sigma_l^4(G) \leqslant \Big\lfloor \frac{|G|-2k-4}{2} \Big\rfloor.$$

This gives a contradiction of the assumption of Theorem 1.7.

The proof of Theorem 1.7 is completed. □

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