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ON THE DIOPHANTINE EQUATION $(2^x-1)(p^y-1)=2z^2$

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Abstract. Let p be an odd prime. By using the elementary methods we prove that: (1) if $2 \nmid x$, $p \equiv \pm 3 \pmod 8$, the Diophantine equation $(2^x-1)(p^y-1)=2z^2$ has no positive integer solution except when $p=3$ or p is of the form $p=2a_0^2+1$, where $a_0>1$ is an odd positive integer. (2) if $2 \nmid x$, $2 \mid y$, $y \neq 2,4$, then the Diophantine equation $(2^x-1)(p^y-1)=2z^2$ has no positive integer solution.

Keywords: elementary method; Diophantine equation; positive integer solution

MSC 2020: 11B39, 11D61

1. Introduction to main results

In 2000, Szalay in [10] proved that the Diophantine equation $(2^n-1)(3^n-1)=x^2$ has no positive integer solution, and Walsh in [13] proved that the Diophantine equation $(2^n-1)(3^m-1)=x^2$ has no positive integer solution. The Diophantine equations $(a^n-1)(b^n-1)=x^2$ and $(a^m-1)(b^n-1)=x^2$ are studied in [3], [4], [5], [6], [7], [9], [11], [14] and important results are obtained. In this paper, we further generalize these results by studying the equation

$$(2^x-1)(p^y-1)=2z^2, \quad p>0. \tag{1.1}$$

We partly solve the situation with $2 \nmid x$ and the following results are proved:

Theorem 1.1. *Let p be an odd prime. When $2 \nmid x, p \equiv \pm 3 \pmod 8$, the Diophantine equation* (1.1) *has no positive integer solution except that when $p=3$ it*

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has only positive integer solutions $x=1$, $y=5$, $z=11$; $x=1$, $y=2$, $z=2$; $x=1$, $y=1$, $z=1$; *and when* $p=2a_0^2+1$ *it has only the positive integer solution* $x=1$, $y=1$, $z=a_0$ *with* $2\nmid a_0>1$.

Theorem 1.2. *Let* p *be an odd prime. When* $2\nmid x$, $2\mid y$, $y\neq 2,4$, *the Diophantine equation* (1.1) *has no positive integer solution.*

2. Lemmas

In order to prove main results in this paper, we first state the following lemmas.

Lemma 2.1 ([12])**.** *The Diophantine equation* $x^3-1=2y^2$ *has only the integer solution* $(x,y)=(1,0)$.

Lemma 2.2 ([1])**.** *The Diophantine equation*

$$x^p-1=2y^2,\quad p \text{ is a prime},\ p>3$$

has only the positive integer solution $p=5$, $x=3$, $y=11$.

Lemma 2.3 ([8])**.** *Let* D *be a non-square number, with the Pell equation*

$$x^2-Dy^2=1,\quad (x,y)=1,\quad D>0. \tag{2.1}$$

Let $x>0$, $y>0$ *be a solution of equation* (2.1), *and* $\varepsilon=x_0+y_0\sqrt{D}$ *be a fundamental solution of equation* (2.1). *If* $x\mid^* x_0$ (x_0 *is divisible by every prime factor of* x), *then* $x+y\sqrt{D}=\varepsilon$.

Lemma 2.4 ([2])**.** *The Diophantine equation* $x^4-2y^2=1$ *has only integer solutions* $x=\pm1$, $y=0$.

Lemma 2.5. *The Diophantine equation*

$$a^y-1=2z^2,\quad y\neq 2 \tag{2.2}$$

has only the positive integer solutions $a=3$, $y=5$, $z=11$ *and* $a=2a_0^2+1$, $y=1$, $z=a_0$.

Proof. If $4 \mid y$, from (2.2), we have $(a^{y/4})^4 - 2z^2 = 1$, by Lemma 2.4, equation (2.2) has only integer solutions $a^{y/4} = \pm 1$, $z = 0$, which contradicts $z > 0$, therefore $4 \nmid y$. If $2 \mid y$, let $y = 2y_1$, then $2 \nmid y_1$. If $y_1 > 1$, then there exists an odd prime p, such that $p \mid y_1$, let $y_1 = py_2$; then from (2.2), we have

$$(a^{2y_2})^p - 1 = 2z^2.$$

If $p = 3$, from Lemma 2.1, the above equation has only integer solution $(a^{2y_2}, z) = (1, 0)$, which contradicts $z > 0$. If $p > 3$, from Lemma 2.2, the above equation has only the positive integer solution $p = 5$, $a^{2y_2} = 3$, $z = 11$, which is obviously impossible. Therefore $y_1 = 1$, that is $y = 2$, which contradicts $y \neq 2$.

If $2 \nmid y$, $y > 1$, then there exists an odd prime p, such that $p \mid y$, let $y = py_1$; then from (2.2), we have

$$(a^{y_1})^p - 1 = 2z^2.$$

If $p = 3$, from Lemma 2.1, the above equation has only the integer solution $(a^{y_1}, z) = (1, 0)$, which contradicts $z > 0$. If $p > 3$, from Lemma 2.2, the above equation has only the positive integer solution $p = 5$, $a^{y_1} = 3$, $z = 11$, and so $a = 3$, $y_1 = 1$, $y = py_1 = 5$, and equation (2.2) has only the positive integer solution $a = 3$, $y = 5$, $z = 11$.

If $y = 1$, then $a = 2a_0^2 + 1, z = a_0$; here equation (2.2) has only positive integer solution $(a, y, z) = (2a_0^2 + 1, 1, a_0)$. □

Lemma 2.6 ([2]). *The Diophantine equation* $x^4 - 2y^2 = -1$ *has only the integer solutions* $x = \pm 1$, $y = \pm 1$.

Lemma 2.7 ([2]). *Let* p *be an odd prime. If the Diophantine equation* $x^p + 1 = 2y^2$ *has a positive integer solution, then* $2p \mid y$ *is true except that it has the positive integer solution* $(x, y) = (1, 1)$ *and when* $p = 3$ *it has only positive integer solutions* $(x, y) = (1, 1), (23, 78)$.

3. Proof of the theorems

Proof of the Theorem 1.1. We prove our result in two situations.

(I) *The case of* $2 \nmid z$. Since $2 \nmid z$, therefore $2 \nmid y$. In fact, if $2 \mid y$, from (1.1), we have $2z^2 \equiv 0 \pmod 8$, then $2 \mid z$, which contradicts $2 \nmid z$, therefore $2 \nmid y$. If $p \equiv -3 \pmod 8$, from equation (1.1), we have $2z^2 \equiv (2^x - 1)(p - 1) \equiv 4(2^x - 1) \pmod 8$, and so $z^2 \equiv 2(2^x - 1) \pmod 4$, which contradicts $2 \nmid z$. If $p \equiv 3 \pmod 8$, from equation (1.1), we have $2z^2 \equiv (2^x - 1)(p - 1) \equiv 2(2^x - 1) \pmod 8$, and so

$z^2 \equiv 2^x - 1 \pmod 4$. If $x \geqslant 2$, then $z^2 \equiv -1 \pmod 4$, we have a contradiction; therefore $x = 1$. Here from equation (1.1), we have

$$p^y - 1 = 2z^2, \quad p \equiv 3 \pmod 8.$$

From Lemma 2.5, the above equation has only the positive integer solutions $p = 3$, $y = 5$, $z = 11$ and $p = 2a_0^2 + 1$, $y = 1$, $z = a_0$, $2 \nmid a_0$; that is equation (1.1) has only the positive integer solutions $p = 3$, $x = 1$, $y = 5$, $z = 11$ and $p = 2a_0^2 + 1$, $x = 1$, $y = 1$, $z = a_0$, $2 \nmid a_0$.

(II) *The case of* $2 \mid z$. Since $2 \mid z$, therefore $2 \mid y$. In fact, if $2 \nmid y$, from (1.1), we have $2z^2 \equiv (2^x - 1)(\pm 3 - 1) \pmod 8$, and so $z^2 \equiv 2^x - 1$, $2(2^x - 1) \equiv \pm 1, 2 \pmod 4$, which contradicts $2 \mid z$; therefore $2 \mid y$. Assume $(2^x - 1, p^y - 1) = d$, then from equation (1.1), we have

$$\begin{cases} 2^x - 1 = ds^2, \\ p^y - 1 = 2dt^2, \\ z = dst, \end{cases} \tag{3.1}$$

where $(s,t) = 1$, $2 \nmid s$, $2 \nmid d$.

From the assumption of $2 \nmid x$, again from the first equation in (3.1), we have $2(2^{(x-1)/2})^2 - 1 = ds^2$, and it is converted into

$$(2(2^{(x-1)/2})^2 + ds^2)^2 - 2d(2^{(x+1)/2}s)^2 = 1. \tag{3.2}$$

Let the Pell equation be

$$X^2 - 2dY^2 = 1, \quad (X,Y) = 1.$$

We assume that $\varepsilon_{2d} = T + U\sqrt{2d}$ is a fundamental solution of the above equation, then all positive integer solutions of the above equation are $X_k + Y_k\sqrt{2d} = (T + U\sqrt{2d})^k$, $k \geqslant 0$. Therefore from equation (3.2) and second equation in (3.1), we obtain the simultaneous equations

$$\begin{cases} 2(2^{(x-1)/2})^2 + ds^2 + (2^{(x+1)/2}s)\sqrt{2d} = X_r + Y_r\sqrt{2d}, \\ p^{y/2} + t\sqrt{2d} = X_l + Y_l\sqrt{2d}. \end{cases} \tag{3.3}$$

From the second equality in (3.3), if $l = 2h$, then $p^{y/2} = X_{2h} = 2X_h^2 - 1$, and so

$$p^{y/2} = 2X_h^2 - 1. \tag{3.4}$$

Hence, from (3.4), we have

$$1 = \Big(\frac{1}{p}\Big) = \Big(\frac{2X_h^2 - p^{y/2}}{p}\Big) = \Big(\frac{2X_h^2}{p}\Big) = -1,$$

which is impossible. Therefore $2 \nmid l$. Let $l = 2h + 1$; then $p^{y/2} = X_{2h+1}$. Since $X_1 \mid X_{2h+1}$, hence $T \mid p^{y/2}$. Since p is an odd prime, if $T = 1$, then $t = 0$, and so $z = 0$, it is not positive integer solution. Hence $T > 1$, and so $T = p^{T_0}$, $T_0 \geqslant 1$. From Lemma (2.3), the second equality in (3.3) has only the positive integer solution $(p^{y/2}, t) = (p^{T_0}, U)$, and so the second equation in (3.1) has only the positive integer solution $(y, t) = (2T_0, U)$.

From the first equality in (3.3), it follows that X_r is an odd number. Now, if $r = 2h$, then $X_r = 2X_h^2 - 1$ and $2^{(x+1)/2}s = Y_r = 2X_hY_h$, since $2 \nmid X_h$, $(X_h, Y_h) = 1$, hence $X_h = b_1$, $Y_h = 2^{(x-1)/2}b_2$, $(b_1, b_2) = 1, 2 \nmid b_1$. Therefore we have

$$X_r = 2b_1^2 - 1. \tag{3.5}$$

From the first equation in (3.1), we have

$$ds^2 = 2^x - 1. \tag{3.6}$$

From the first equality in (3.3), we have $X_r = 2(2^{(x-1)/2})^2 + ds^2$, and it is converted into

$$X_r = 2^x + ds^2. \tag{3.7}$$

Plugging equation (3.6) into (3.7), we obtain $X_r = 2^x + 2^x - 1 = 2 \cdot 2^x - 1$. From (3.5),we easily know $2b_1^2 - 1 = 2 \cdot 2^x - 1$, that is $b_1^2 = 2^x$, which is impossible. Hence $2 \nmid r$.

Since $2 \nmid d$, if $2 \nmid Y$, then $X^2 = 2dY^2 + 1 \equiv 2d + 1 \equiv 3 \pmod 4$; this is impossible. Hence $2 \mid Y$. Therefore $U = 2^{U_0}U_1$, $2 \nmid U_1$, $U_0 \geqslant 1$, and so $Y_1 = U = 2^{U_0}U_1$, $Y_2 = 2X_1Y_1 = p^{T_0} \cdot 2^{U_0+1}U_1$. Let $r = 2h + 1$. Since $Y_1 \mid Y_r$, hence we have

$$\begin{aligned} Y_{2h+1} &= 2X_1Y_{2h} - Y_{2h-1} = 2X_1Y_1\frac{Y_{2h}}{Y_1} - Y_{2h-1} = p^{T_0} \cdot 2^{U_0+1}U_1\frac{Y_{2h}}{Y_1} - Y_{2h-1} \\ &\equiv -Y_{2h-1} \pmod{2^{U_0+1}}. \end{aligned} \tag{3.8}$$

and

$$Y_{2h+1} \equiv Y_{2h-1} \equiv Y_{2h-3} \equiv \ldots \equiv Y_3 \equiv Y_1 \equiv 2^{U_0}U_1 \equiv 2^{U_0} \pmod{2^{U_0+1}}. \tag{3.9}$$

Hence, $2^{(x+1)/2}s = Y_{2h+1} \equiv 2^{U_0} \pmod{2^{U_0+1}}$, when $(x+1)/2 > U_0$, $0 \equiv 2^{U_0} \pmod{2^{U_0+1}}$, this is impossible. Therefore, $(x+1)/2 \leqslant U_0$, and so $x \leqslant 2U_0 - 1$. Here, $X_{2h+1} = 2^x + ds^2 = 2^{x+1} - 1$. When $h = 0$, $2^{x+1} - 1 = X_{2h+1} = X_1 = p^{T_0}$, since $2 \mid (x+1)$, hence $(2^{(x+1)/2}+1)(2^{(x+1)/2}-1) = p^{T_0}$. Since $(2^{(x+1)/2}+1, 2^{(x+1)/2}-1) = 1$, p is an odd prime, hence we have:

$$\begin{cases} 2^{(x+1)/2}+1 = p^{T_0}, \\ 2^{(x+1)/2}-1 = 1. \end{cases} \tag{3.10}$$

From the second equation in (3.10), we have $2^{(x+1)/2} = 2$, and so $x = 1$. From the first equation in (3.10), we have $p^{T_0} = 3$, and so $p = 3$, $T_0 = 1$. Here, $y = 2T_0 = 2$, and from (3.1), we have $s = d = 1$, $t = 2$, $z = dst = 2$. Therefore we have the positive integer solution $p = 3$, $x = 1$, $y = 2$, $z = 2$.

When $h > 0$, since $x+1 \leqslant 2U_0$, $2^{x+1}-1 = X_{2h+1}$, hence

$$2^{x+1}-1 = \sum_{t=0}^{h} C_{2h+1}^{2t}(p^{T_0})^{2h+1-2t}2^{2tU_0}U_1^{2t}(2d)^t \equiv (p^{2h+1})^{T_0} \pmod{2^{2U_0+1}}. \tag{3.11}$$

Therefore, $-1 \equiv (p^{2h+1})^{T_0} \pmod{2^{x+1}}$, when $x > 1$, $-1 \equiv (p^{2h+1})^{T_0} \equiv (\pm 3)^{T_0} \equiv \pm 3, 1 \pmod 8$, this is impossible. Hence $x = 1$. Since $p \geqslant 3$, we have:

$$3 = 2^{x+1}-1 = X_{2h+1} = \sum_{t=0}^{h} C_{2h+1}^{2t}(p^{T_0})^{2h+1-2t}2^{2tU_0}U_1^{2t}(2d)^t > 3^{T_0}+C_3^2 \geqslant 6, \tag{3.12}$$

which is impossible. □

Proof of the Theorem 1.2. Since $2 \nmid x$, $2 \mid y$, let $(2^x-1, p^y-1) = d$, from equation (1.1), we know that simultaneous equation (3.1) is true. Hence the proof methods of Theorem 1.2 with the exception of equations (3.4) and (3.11) are the same as those of Theorem 1.1 following equation (3.1). The proofs for equations (3.4) and (3.11) are given below only when $p \equiv \pm 1 \pmod 8$.

Since $2 \mid y$, hence let $y = 2^w y_1$, $w \geqslant 1$, $2 \nmid y_1 > 0$. If $y_1 > 1$, then there exists an odd prime q, such that $q \mid y_1$, and when $q = 3$, equation (3.4) is converted into $(p^{y/6})^3 + 1 = 2X_h^2$, and from Lemma 2.7, it has only the positive integer solutions $(p^{y/6}, X_h) = (1,1), (23,78)$, and since p is an odd prime and $2 \nmid X_h$, this is impossible. When $q > 3$, equation (3.4) is converted into $(p^{y/2q})^q + 1 = 2X_h^2$. Since $(p^{y/2q}, X_h) \neq (1,1)$, from Lemma 2.7, we know $2q \mid X_h$, which contradicts $2 \nmid X_h$, hence $y_1 = 1$. Therefore $y = 2^w$. If $w \geqslant 3$, then equation (3.4) is converted into $(p^{2^{w-3}})^4 - 2X_h^2 = -1$, and from Lemma 2.6, it has only the positive integer solution $p^{2^{w-3}} = 1$, $X_h = 1$, which contradicts that p is an odd prime. Hence $w \leqslant 2$. Then $y = 2^w = 2, 4$, which contradicts $y \neq 2, 4$.

From the proof of Theorem 1.1, when $p \equiv 1 \pmod 8$, equation (3.11) is obviously impossible. When $p \equiv -1 \pmod 8$, that is $p = 2^u v - 1$, $u \geqslant 3$, $2 \nmid v > 0$, we have $p^2 = (2^u v - 1)^2 \equiv 1 \pmod{2^{u+1}}$, and from equation (3.11), we know $-1 \equiv (p^{2h+1})^{T_0} \pmod{2^{x+1}}$ is true, and when $x \geqslant u$, $-1 \equiv (p^{2h+1})^{T_0} \equiv p^{T_0} \equiv 1$, $p \equiv 1$, $2^u - 1 \pmod{2^{u+1}}$ are contradictory. Hence $x \leqslant u - 1$. From equation (3.11), we have

$$(3.13)\quad 2^u - 1 \geqslant 2^{x+1} - 1 = \sum_{t=0}^{h} C_{2h+1}^{2t} (p^{T_0})^{2h+1-2t} 2^{2tU_0} U_1^{2t} (2d)^t > (p^{2h+1})^{T_0} > p^2$$
$$= 2^{2u} v^2 - 2^{u+1} v + 1 > 2^{u+1} + 1,$$

which is contradictory. This completes the proof. □

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