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GENERALIZED CONNECTIVITY OF SOME TOTAL GRAPHS

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Abstract. We study the generalized k-connectivity $\kappa_k(G)$ as introduced by Hager in 1985, as well as the more recently introduced generalized k-edge-connectivity $\lambda_k(G)$. We determine the exact value of $\kappa_k(G)$ and $\lambda_k(G)$ for the line graphs and total graphs of trees, unicyclic graphs, and also for complete graphs for the case $k = 3$.

Keywords: generalized (edge-)connectivity; line graph; total graph; complete graph

MSC 2020: 05C05, 05C40, 05C70, 05C75

1. Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to book (see [3]) for graph theoretical notation and terminology not described here. For a graph G, we denote by $V(G)$, $E(G)$, $L(G)$ the set of vertices, the set of edges, and the line graph of G, respectively.

By the development of parallel and distributed computing, the design and analysis of various interconnection networks have been a main topic of research for the past decade, see [1]. Interconnection networks are often modelled by graphs (or digraphs). The vertices of the graph represent the nodes of the network, that is, processing elements, memory modules or switches, and the edges correspond to communication lines. We know that the connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$ of a graph G are the minimum number of vertices and edges that need to be removed to disconnect the remaining vertices from each other, respectively.

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These two concepts are important measures for the robustness of networks. An equivalent definition of connectivity (or edge-connectivity) was given: For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa_G(S)$ (or $\lambda_G(S)$) denote the maximum number of internally- (or edge-) disjoint paths from u to v in G. Then $\kappa(G) = \min\{\kappa_G(S) \colon S \subseteq V, |S| = 2\}$ (or $\lambda(G) = \min\{\lambda_G(S) \colon S \subseteq V, |S| = 2\}$). The generalized k-(edge-)connectivity was introduced in order to measure the capability of a graph G to connect any k vertices in G and not just any two.

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geqslant 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint Steiner trees connecting S in G. For an integer k with $2 \leqslant k \leqslant p$, the *generalized k-connectivity* (or *k-tree-connectivity*) is defined in [8] as $\kappa_k(G) = \min\{\kappa(S) \colon S \subseteq V(G), |S| = k\}$. Clearly, $\kappa_2(G) = \kappa(G)$. Table 1 shows how the generalization proceeds.

	Classical connectivity	Generalized connectivity
Vertex subset	$S = \{x, y\} \subseteq V(G)$ $(\lvert S\rvert = 2)$	$S \subseteq V(G)$ $(\lvert S\rvert \geqslant 2)$
Set of Steiner trees	$\begin{cases} \mathscr{P}_{x,y} = \{P_1, P_2, \ldots, P_l\} \\ \{x, y\} \subseteq V(P_i) \\ E(P_i) \cap E(P_j) = \emptyset \\ V(P_i) \cap V(P_j) = \{x, y\} \end{cases}$	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \ldots, T_l\} \\ S \subseteq V(T_i) \\ E(T_i) \cap E(T_j) = \emptyset \\ V(T_i) \cap V(T_j) = S \end{cases}$
Local parameter	$\kappa(x, y) = \max \lvert\mathscr{P}_{x,y}\rvert$	$\kappa(S) = \max \lvert\mathscr{T}_S\rvert$
Global parameter	$\kappa(G) = \min\limits_{x,y \in V(G)} \kappa(x, y)$	$\kappa_k(G) = \min\limits_{S \subseteq V(G), \lvert S\rvert = k} \kappa(S)$

Table 1. Classical connectivity and generalized connectivity

	Edge-connectivity	Generalized edge-connectivity
Vertex subset	$S = \{x, y\} \subseteq V(G)$ $(\lvert S\rvert = 2)$	$S \subseteq V(G)$ $(\lvert S\rvert \geqslant 2)$
Set of Steiner trees	$\begin{cases} \mathscr{P}_{x,y} = \{P_1, P_2, \ldots, P_l\} \\ \{x, y\} \subseteq V(P_i) \\ E(P_i) \cap E(P_j) = \emptyset \end{cases}$	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \ldots, T_l\}, \\ S \subseteq V(T_i), \\ E(T_i) \cap E(T_j) = \emptyset \end{cases}$
Local parameter	$\lambda(x, y) = \max \lvert\mathscr{P}_{x,y}\rvert$	$\lambda(S) = \max \lvert\mathscr{T}_S\rvert$
Global parameter	$\lambda(G) = \min\limits_{x,y \in V(G)} \lambda(x, y)$	$\lambda_k(G) = \min\limits_{S \subseteq V(G), \lvert S\rvert = k} \lambda(S)$

Table 2. Classical edge-connectivity and generalized edge-connectivity

As a natural counterpart of the generalized k-connectivity, Li, Mao and Sun in [18] introduced the concept of generalized k-edge-connectivity. For $S \subseteq V(G)$ and $|S| \geqslant 2$, two S-trees T and T' connecting S are said to be *edge disjoint* if $E(T) \cap E(T') = \emptyset$. And the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge disjoint S-trees connecting S in G. For an integer k with $2 \leqslant k \leqslant p$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) \colon S \subseteq V(G) \text{ and } |S| = k\}$, hence $\lambda_2(G) = \lambda(G)$. Table 2 shows how the generalization of the edge-version definition proceeds.

For results on the generalized connectivity, we refer to [2], [4], [7], [10], [11], [12], [13], [15], [16] and book [17].

The line graph $L(G)$ of a graph G has vertex set $E(G)$, and two vertices are adjacent in $L(G)$ if and only if the corresponding two edges in G have precisely one end vertex in common. The *total graph* $T(G)$ of G has vertex set $V(G) \cup E(G)$, and two vertices are adjacent in $T(G)$ if and only if the corresponding two elements of $V(G) \cup E(G)$ in G are

(i) adjacent vertices, or
(ii) a vertex and an incident edge, or
(iii) two edges that have precisely one end vertex in common.

In [5], Chartrand et al. showed that for any two integers n and k with $2 \leqslant k \leqslant n$, $\kappa_k(K_n) = n - \lceil \frac{1}{2}k \rceil$, and in [18], Li, Mao and Sun determined that $\lambda_k(K_n) = n - \lceil \frac{1}{2}k \rceil$. Hamada discussed the connectivity of total graphs in [9]. Motivated by these research, in this paper we investigate the generalized k-connectivity and k-edge-connectivity of line graphs and total graphs of trees, unicyclic graphs and complete graph. For the latter, we only consider the case $k = 3$. At the end of this paper, we give some bounds on these two parameters for general line graphs and total graphs.

For $S \subset V(G)$ we use $G[S]$ to denote the subgraph of G induced by S. In particular, if T is a tree in G, then $E(T) \subset V(L(G))$ and $L(G)[E(T)]$ is an induced subgraph of $L(G)$.

2. Preliminary results

The following observations are immediate.

Observation 1 ([17]). *If G is a connected graph, then $\kappa_k(G) \leqslant \lambda_k(G) \leqslant \delta(G)$.*

Observation 2 ([17]). *If H is a spanning subgraph of G, then $\kappa_k(H) \leqslant \kappa_k(G)$.*

Li, Li and Zhou gave the following sharp upper bound on $\kappa_3(G)$ in terms of the minimum degree $\delta(G)$ and connectivity $\kappa(G)$.

Proposition 2.1 ([14]).

(1) *Let G be a connected graph of order $n \geqslant 6$. Then $\kappa_3(G) \leqslant \kappa(G)$. Moreover, the upper bound is sharp.*

(2) *Let G be a connected graph of order n. If there are two adjacent vertices of degree $\delta(G)$, then $\kappa_3(G) \leqslant \delta(G) - 1$. Moreover, the upper bound is sharp.*

Similarly, the following sharp upper bound on $\lambda_k(G)$ has been obtained in [18].

Proposition 2.2 ([18]).

(1) *Let G be a connected graph. If there are two adjacent vertices of degree $\delta(G)$, then $\lambda_k(G) \leqslant \delta(G) - 1$. Moreover, the upper bound is sharp.*

(2) *For any graph G of order $n \geqslant 6$, $\lambda_k(G) \leqslant \lambda(G)$. Moreover, the upper bound is sharp.*

Proposition 2.3 ([14]). *Let G be a connected graph. For every two integers k and r with $k \geqslant 0$ and $r \in \{0, 1, 2, 3\}$, if $\kappa(G) = 4k + r$, then $\kappa_3(G) \geqslant 3k + \lceil \frac{1}{2}r \rceil$. Moreover, the lower bound is sharp.*

Proposition 2.4 ([18]). *Let G be a connected graph with n vertices. For every two integers l and r with $k \geqslant 0$ and $r \in \{0, 1, 2, 3\}$, if $\lambda(G) = 4l + r$, then $\lambda_3(G) \geqslant 3l + \lceil \frac{1}{2}r \rceil$. Moreover, the lower bound is sharp.*

3. Generalized k-(edge-)connectivity of total graphs for trees and unicyclic graphs

In this section, we determine the exact value of the generalized k-(edge-)connectivity of the total graph for trees and unicycle graphs. First, we list two known results, which are due to Hamada, Nonaka, and Yoshimura in [9] and Nash-Williams in [19].

Theorem 3.1 ([9]). *Let G be a graph with $\kappa(G) \geqslant m$. Then $\kappa(T(G)) \geqslant 2m$ and $\lambda(T(G)) \geqslant 2m$.*

Theorem 3.2 ([19]). *Every $2k$-edge-connected graph contains a system of k edge-disjoint spanning trees.*

Theorem 3.3. *Let p, k be two integers with $p \geqslant 2$ and $3 \leqslant k \leqslant 2p - 1$. If T_p is a tree of order p, then*

$$\kappa_k(T(T_p)) = \begin{cases} 1 & \text{if } k = 2p - 1 \text{ and } \Delta(T_p) = 2, \\ 2 & \text{otherwise.} \end{cases}$$

P r o o f. We first consider the case when $k = 2p - 1$ and $\Delta(T_p) = 2$. Since T_p is a tree with $\kappa(T_p) = 1$, by Theorem 3.1, we get $\lambda(T(T_p)) \geqslant 2$. By Theorem 3.2 and $|T(T_p)| = 2p - 1$, $T(T_p)$ contains a spanning tree, and hence $\kappa_{2p-1}(T(T_p)) \geqslant 1$. It suffices to prove $\kappa_{2p-1}(T(T_p)) \leqslant 1$. Now we prove the following claim.

Claim 1. *$T(T_p)$ contains at most one edge-disjoint spanning tree for $\Delta(T_p) = 2$.*

P r o o f. Assume to the contrary that $T(T_p)$ contains at least two edge-disjoint spanning trees. This means $|E(T(T_p))| \geqslant 2(2p - 2)$. At the same time, consider $\Delta(T_p) = 2$. This yields that T_p is a path and thus $|E(T(T_p))| = p-1+p-2+2(p-1) = 4p - 5$, contradiction. □

Next, we consider the case when $k = 2p - 1$ and $\Delta(T_p) > 2$. Since the minimum degree of $T(T_p)$ is 2, by Observation 1, we get $\kappa_k(T(T_p)) = \kappa_{2p-1}(T(T_p)) \leqslant 2$. Since $\Delta(T_p) > 2$, there exists a vertex $v \in V(T_p)$ with $d_{T_p}(v) \geqslant 3$. Using $K_{d(v)}$ denote the clique in the line graph $L(T_p)$ arising from edges incident with v, noting $K_{d(v)}$ contains a triangle. Let CT_p be a spanning tree of $L(T_p)$. Then we directly get two edge-disjoint spanning trees T' and T'' based on T_p and CT_p by connecting all vertices of $L(T_p)$ with tree T_p and all vertices of T_p with tree CT_p. This implies that $\kappa_{2p-1}(T(T_p)) \geqslant 2$, and thus we get $\kappa_{2p-1}(T(T_p)) = 2$.

We now consider the general case when $3 \leqslant k \leqslant 2p - 2$. By Observation 1 and the fact that the minimum degree of $T(T_p)$ is 2, we directly get $\kappa_k(T(T_p)) \leqslant 2$. It suffices to prove that $\kappa_k(T(T_p)) \geqslant 2$ for $3 \leqslant k \leqslant 2p - 2$.

Suppose that $V(T_p) = \{u_1, u_2, \ldots, u_p\}$ and $V(L(T_p)) = \{e_{ij} \colon e_{ij} = u_iu_j \in E(T_p)\}$. Then $V(T(T_p)) = V(L(T_p)) \cup V(T_p)$. For convenience, the induced subgraph $T(T_p)[V(T_p)]$ is still denoted by T_p and $T(T_p)[V(L(T_p))]$ is denoted by $L(T_p)$. Let S be a k-subset of $V(T(T_p))$ and $S' = S \cap V(T_p)$, $S'' = S \cap V(L(T_p))$. Suppose $S' = \emptyset$, i.e., $S \subseteq V(L(T_p))$. Since $L(T_p)$ is connected, it follows that there exists one S-Steiner tree in $L(T_p)$, say T'. Then by connecting vertices of S with tree T_p we get another S-Steiner tree, say T''. Clearly, T' and T'' are two internally disjoint S-trees in $T(T_p)$. This means that $\kappa_k(T(T_p)) \geqslant 2$. By a similar consideration for the case $S'' = \emptyset$, we can also get two internally disjoint S-Steiner trees in $T(T_p)$, and then prove that $\kappa_k(T(T_p)) \geqslant 2$.

Next assume $S', S'' \neq \emptyset$. Since T_p is connected, so is $L(T_p)$. Now we denote one spanning tree of $L(T_p)$ as T. Thus, we can obtain two internally disjoint S-Steiner trees in $T(T_p)$ by connecting vertices of S' with spanning tree T and connecting vertices of S'' with tree T_p. In particular, if $\Delta(T_p) = 2$, then T_p is a path. Since $3 \leqslant k \leqslant 2p - 2$, it follows that $V(T(T_p)) \setminus S \neq \emptyset$, and hence we can still obtain two internally disjoint S-Steiner trees in $T(T_p)$. So we always get $\kappa_k(T(T_p)) \geqslant 2$, as desired. This completes the proof. □

Next we turn to determine the generalized k-edge-connectivity of $T(T_p)$.

Theorem 3.4. *Let p, k be two integers with $p \geqslant 2$ and $3 \leqslant k \leqslant 2p-1$. If T_p is a tree with order p, then*

$$\lambda_k(T(T_p)) = \begin{cases} 1 & \text{if } k = 2p-1 \text{ and } \Delta(T_p) = 2, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First, we consider the case when $k = 2p-1$ and $\Delta(T_p) = 2$. By Observation 1 and Theorem 3.3, we get $\lambda_{2p-1}(T(T_p)) \geqslant \kappa_{2p-1}(T(T_p)) = 1$. On the other hand, consider $|E(T(T_p))| = p-1+p-2+2(p-1) = 4p-5 < 2(2p-2)$. Then $T(T_p)$ does not contain two edge-disjoint spanning trees, which implies that $\lambda_{2p-1}(T(T_p)) \leqslant 1$. Thus, we get $\lambda_{2p-1}(T(T_p)) = 1$.

Next, we consider the case when $k = 2p-1$ and $\Delta(T_p) > 2$ and the general case when $3 \leqslant k \leqslant 2p-2$. On the one hand, since the minimum degree of $T(T_p)$ is 2, by Observation 1, we have $\lambda_k(T(T_p)) \leqslant 2$. On the other hand, by Observation 1 and Theorem 3.3, we have $\lambda_k(T(T_p)) \geqslant \kappa_k(T(T_p)) = 2$. Thus, we get $\lambda_k(T(T_p)) = 2$. This completes the proof. □

By Theorems 3.3 and 3.4, we directly get the generalized 3-connectivity and generalized 3-edge-connectivity of the total graph of tree T_p.

Corollary 3.5. *Let p be an integer with $p \geqslant 2$. If $T(T_p)$ is a total graph of tree T_p with order p, then*

$$\kappa_3(T(T_p)) = \lambda_3(T(T_p)) = \begin{cases} 1 & \text{if } p = 2, \\ 2 & \text{if } p \geqslant 3. \end{cases}$$

Following this, we determine the exact value of generalized connectivity of the total graph of unicyclic graphs.

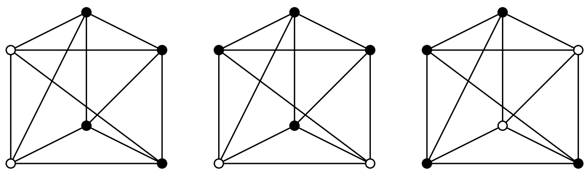
Theorem 3.6. *Let p, k, l be integer numbers with $p \geqslant 3$, $3 \leqslant l \leqslant p$ and $3 \leqslant k \leqslant 2p$. If G_p is an unicyclic graph with order p and unique cycle C_l, then*

$$\kappa_k(T(G_p)) = \begin{cases} 3 & \text{if } p = l \text{ and } k = 3 \text{ or } p = l = 3 \text{ and } k = 4, \\ 2 & \text{otherwise.} \end{cases}$$

P r o o f. We complete the proof by distinguishing two cases according to $p = l$ and $p > l$.

Case 1: $p = l$. $p = l$ means $G_p = C_l$. We first consider the case when $G_p = C_l$ and $k = 3$. Since $T(G_p)$ is 4-regular graph, by Proposition 2.1 (2), we get $\kappa_3(T(G_p)) \leqslant 3$. On the other hand, since $T(G_p)$ is 4-connected, by Proposition 2.3, we get $\kappa_3(T(G_p)) \geqslant 3$. Thus, we get $\kappa_3(T(G_p)) = \kappa_3(T(C_p)) = 3$.

Next we consider the case when $G_p = C_3$ and $k = 4$. Similarly, by Observation 1 and Proposition 2.2 (1), combining the fact that $T(G_p)$ is 4-regular graph, we get $\kappa_4(T(G_p)) \leqslant \lambda_4(T(G_p)) \leqslant 3$. On the other hand, let S be a 4-subset of $V(T(G_p))$. By the symmetry of $T(G_p)$, there are only three different choices of S, see Figure 1 (a)–1 (c). By simple checking, we always find 3 internally disjoint S-trees in $T(G_p)$ for every S. Thus, we have $\kappa_4(T(G_p)) \geqslant 3$. Therefore, $\kappa_4(T(G_p)) = \kappa_4(T(C_3)) = 3$.



(a) (b) (c)

Figure 1. Three different choices of 4-subset S in $V(T(C_3))$ (black dot)

Now we consider the case when $G_p = C_3$ and $k = 5, 6$. Since the choice of the k-subset S in $V(T(C_3))$ is unique, by simple checking, there exist at most two internally disjoint S-trees in $T(C_3)$. Thus, we get $\kappa_5(T(C_3)) = \kappa_6(T(C_3)) = 2$.

As for the general case when $G_p = C_l$ with $p \geqslant 4$ and $4 \leqslant k \leqslant 2p$, note that the induced subgraphs $T(G_p)[V(G_p)]$ and $T(G_p)[V(L(G_p))]$ in $T(G_p)$ are both p-cycles. We use C_p' and C_p'' to denote graphs $T(G_p)[V(G_p)]$ and $T(G_p)[V(L(G_p))]$, respectively. We first choose a k-subset S_0 in $V(T(G_p))$ such that $|S_0 \cap V(C_p')|$ is as large as possible and the induced subgraph $T(G_p)[S_0 \cap V(C_p')]$ are a path. By simple checking, $T(G_p)$ has at most two internally disjoint S_0-trees. This means $\kappa_k(T(G_p)) \leqslant 2$. On the other hand, for any k-subset $S \subset V(T(G_p))$ we always obtain two internally disjoint S-trees in $T(T_p)$ by connecting vertices of $S \setminus V(C_p'')$ with one spanning tree of C_p' and connecting vertices of $S \setminus V(C_p')$ with one spanning tree of C_p''. This implies that $\kappa_k(T(G_p)) \geqslant 2$. Thus, $\kappa_k(T(G_p)) = 2$.

Case 2: $p > l$. In this case, since the minimum degree of $T(G_p)$ is 2, by Observation 1, we have $\kappa_k(T(G_p)) \leqslant 2$ for $3 \leqslant k \leqslant 2p$. It suffices to prove that $\kappa_k(T(G_p)) \geqslant 2$ for $3 \leqslant k \leqslant 2p$. Similarly, we denote $T(G_p)[V(G_p)]$ as G_p and $T(G_p)[V(L(G_p))]$ as $L(G_p)$, and suppose S be a k-subset of $V(T(G_p))$ with $S' = S \cap V(G_p)$, $S'' = S \cap V(L(G_p))$.

If $S' = \emptyset$, i.e., $S \subseteq V(L(G_p))$, since $L(G_p)$ is connected, there exists one S-Steiner tree in $L(G_p)$, say T'. Then by connecting vertices of S with a spanning tree of G_p, we again get a S-Steiner tree, say T''. Clearly, T' and T'' are two internally disjoint S-trees in $T(G_p)$. This means $\kappa_k(T(T_p)) \geqslant 2$. By similar consideration, if $S'' = \emptyset$, we also get two internally disjoint S-Steiner trees in $T(G_p)$. Thus, we get $\kappa_k(T(G_p)) \geqslant 2$.

As for the case $S', S'' \neq \emptyset$, note that G_p and $L(G_p)$ are both connected, we denote by T_1 and T_2 two spanning trees of G_p and $L(G_p)$, respectively. Then we obtain two internally disjoint S-trees in $T(G_p)$ by connecting vertices of S' with T_2 and connecting vertices of S'' with T_1. This proves that $\kappa_k(T(G_p)) \geqslant 2$, as desired. This completes the proof. □

Now we determine the generalized k-edge-connectivity of $T(G_p)$ similarly.

Theorem 3.7. *Let p, k, l be integer numbers with $p \geqslant 3$, $3 \leqslant l \leqslant p$ and $3 \leqslant k \leqslant 2p$. If G_p is an unicyclic graph with order p and unique cycle C_l, then*

$$\lambda_k(T(G_p)) = \begin{cases} 3 & \text{if } p = l \text{ and } 3 \leqslant k \leqslant 4, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. First, we consider the general case when $l < p$ and $3 \leqslant k \leqslant 2p$. Consider the minimum degree of $T(G_p)$ is 2, by Observation 1, we have $\lambda_k(T(G_p)) \leqslant 2$. On the other hand, by Observation 1 and Theorem 3.6, we have $\lambda_k(T(G_p)) \geqslant \kappa_k(T(G_p)) = 2$. Thus, we have $\lambda_k(T(G_p)) = 2$ for $l < p$ with $3 \leqslant k \leqslant 2p$. In the following we discuss the case for $l = p$ in details. Clearly, $G_p = C_l = C_p$.

For convenience of narration, suppose

$$C_p = v_1 v_2 \dots v_p v_1 \quad \text{and} \quad L(C_p) = e_{12} e_{23} \dots e_{i(i+1)} \dots e_{(p-1)p} e_{p1} e_{12}$$

for $e_{i(i+1)} \in V(L(C_p))$. Similarly, we use C'_p and C''_p to denote graphs $T(G_p)[V(G_p)]$ and $T(G_p)[V(L(G_p))]$, respectively, value of $|S \cap V(C'_p)|$.

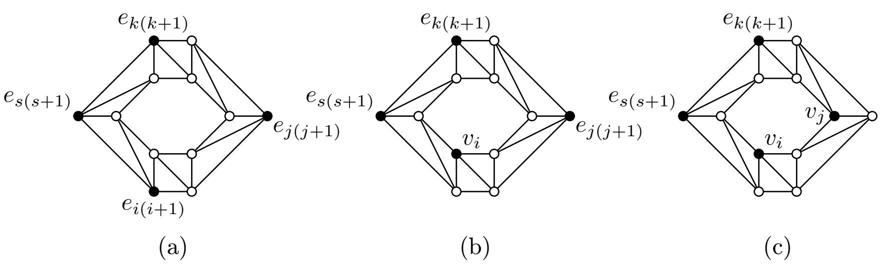


Figure 2. 4-subset S of $V(T(C_p))$ (black dot)

Now we consider the case when $G_p = C_l$ and $3 \leqslant k \leqslant 4$. Since $T(C_p)$ is 4-regular graph, by Proposition 2.2, we directly get $\lambda_k(T(C_p)) \leqslant 3$. It suffices to prove that $\lambda_k(T(C_p)) \leqslant 3$ for $3 \leqslant k \leqslant 4$. If $k = 3$, by Observation 1 and Theorem 3.6, we get $\lambda_3(T(C_p)) \geqslant \kappa_3(T(C_p)) = 3$. If $k = 4$ and $p = 3$, similarly, by Observation 1 and Theorem 3.6, we get $\lambda_4(T(C_3)) \geqslant \kappa_4(T(C_3)) = 3$. In the following we mainly consider the case when $k = 4$ and $p \geqslant 4$. Let S be a 4-subset of $V(T(C_p))$, we show that there exist 3 edge disjoint S-trees in $T(C_p)$ to prove $\lambda_4(T(C_p)) \geqslant 3$. We distinguish three cases by the

(1) $|S \cap V(C'_p)| = 0$ means $S \subseteq V(C''_p)$. Without loss generality, suppose $S = \{e_{i(i+1)}, e_{j(j+1)}, e_{k(k+1)}, e_{s(s+1)}\}$ with $i < j < k < s$, see Figure 2 (a). We form 3 edge disjoint S-trees T_1, T_2, T_3 in $T(G_p)$ as follows and get $\lambda_4(T(C_p)) \geqslant 3$:

$$\begin{aligned}
T_1 &= e_{i(i+1)} \ldots e_{j(j+1)} \ldots e_{k(k+1)} \ldots e_{s(s+1)},\\
T_2 &= e_{j(j+1)} v_j \ldots v_{i+1} e_{i(i+1)} \ldots e_{s(s+1)} v_s \ldots v_{k+1} e_{k(k+1)},\\
T_3 &= e_{j(j+1)} v_{j+1} \ldots v_k e_{k(k+1)} \cup e_{s(s+1)} v_{s+1} \ldots v_i e_{i(i+1)}\\
&\quad \cup v_i v_{i+1} e_{(i+1)(i+2)} v_{i+2} \ldots v_j v_{j+1}.
\end{aligned}$$

(2) $|S \cap V(G_p)| = 1$. Similarly, suppose $S = \{v_i, w_{j(j+1)}, w_{k(k+1)}, w_{s(s+1)}\}$, see Figure 2 (b). Now we form 3 edge disjoint S-trees T_1, T_2, T_3 in $T(G_p)$ as follows and get $\lambda_4(T(C_p)) \geqslant 3$:

$$\begin{aligned}
T_1 &= v_i e_{i(i+1)} \ldots e_{j(j+1)} \ldots e_{s(s+1)},\\
T_2 &= e_{j(j+1)} v_j \ldots v_i e_{(i-1)i} \ldots e_{s(s+1)} v_s \ldots v_{k+1} e_{k(k+1)},\\
T_3 &= e_{j(j+1)} v_{j+1} \ldots v_k e_{k(k+1)} \cup v_i \ldots v_{s+1} e_{s(s+1)}\\
&\quad \cup v_{s+1} v_s e_{(s-1)s} v_{s-1} \ldots v_{k+1} v_k.
\end{aligned}$$

(3) $|S \cap V(G_p)| = 2$. Suppose $S = \{v_i, v_j, w_{k(k+1)}, w_{s(s+1)}\}$, see Figure 2 (c). Now we form 3 edge disjoint S-trees T_1, T_2, T_3 in $T(G_p)$ as follows and get $\lambda_4(T(C_p)) \geqslant 3$:

$$\begin{aligned}
T_1 &= v_i \ldots v_j e_{j(j+1)} \ldots e_{k(k+1)} \ldots e_{s(s+1)},\\
T_2 &= v_i \ldots v_{s+1} v_s \ldots v_{k+1} v_k \ldots v_j \cup v_{k+1} e_{k(k+1)} \cup v_s e_{s(s+1)},\\
T_3 &= e_{k(k+1)} v_k e_{(k-1)k} v_{k-1} \ldots v_{j+1} e_{j(j+1)} \ldots e_{s(s+1)}\\
&\quad \cup v_i e_{i(i+1)} \cup v_j e_{j(j+1)}.
\end{aligned}$$

Finally, we consider the case when $G_p = C_l$ with $5 \leqslant k \leqslant 2p$. On the one hand, by Observation 1 and Theorem 3.6, we have $\lambda_k(T(C_p)) \geqslant \kappa_k(T(C_p)) = 2$. On the other hand, choose a k-subset S_0 of $V(T(C_p))$ such that $v_2, v_3, v_i, e_{12}, e_{23} \in S_0$ and

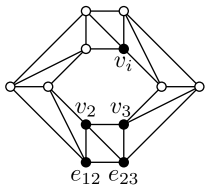


Figure 3. A k-subset S_0 of $V(T(C_p))$ for $5 \leqslant k \leqslant 2p$ (black dot)

$i \neq 2, 3$, see Figure 3. By simple checking, there exist at most 2 edge disjoint S_0-trees in $T(C_p)$, which implies $\lambda_k(T(C_p)) \leqslant 2$. Thus, we get $\lambda_k(T(C_p)) = 2$ for $5 \leqslant k \leqslant 2p$. This completes the proof. □

By Theorems 3.6 and 3.7, we get the generalized 3-connectivity and generalized 3-edge-connectivity of the total graph of unicyclic graph G_p.

Corollary 3.8. *Let p, l be two integers with $p \geqslant 3$, $3 \leqslant l \leqslant p$. If G_p is an unicyclic graph with order p and unique cycle C_l, then*

$$\kappa_3(T(G_p)) = \lambda_3(T(G_p)) = \begin{cases} 3 & \text{if } p = l, \\ 2 & \text{otherwise.} \end{cases}$$

4. Generalized 3-(edge-)connectivity of line graph and total graph for complete graph

For $2 \leqslant k \leqslant p$, it is known that $\kappa_k(K_p) = \lambda_k(K_p) = p - \lceil \frac{1}{2}k \rceil$. Motivated by this, we consider to determine the generalized k-(edge-)connectivity of a line graph and a total graph of complete graph K_p. In [9] and [6], Hamada and Chartrand, respectively, discussed this problem for $k = 2$. Here we only consider the case when $k = 3$.

Lemma 4.1. *Let $L(K_p)$ be a line graph of complete graph K_p with $V(K_p) = \{u_i | 1 \leqslant i \leqslant p\}$ and $V(L(K_p)) = \{e_{ij} | e_{ij} = u_i u_j \in E(K_p)\}$. If $S_0 = \{e_{ab}, e_{bc}, e_{ac}\}$ is a 3-subset of $V(L(K_p))$, then the generalized local connectivity $\kappa(S_0) = \lfloor \frac{3}{2}(p-2) \rfloor$.*

Proof. We proceed by induction on p. Clearly, the conclusion holds for $p = 3$, since $L(K_3)$ contains one S_0-tree and $\lfloor \frac{3}{2}(p-2) \rfloor = 1$. Of course, the conclusion also holds for $p = 4$. In fact, suppose $V(K_4) = \{u_a, u_b, u_c, u_d\}$. Then $e_{ac}e_{ab}e_{bc}$, $e_{ac}e_{bc}e_{bd}e_{ab}$ and $e_{ac}e_{cd}e_{ad}e_{ab} \cup e_{cd}e_{bc}$ are internally disjoint S_0-trees in $L(K_4)$, and nothing else. Thus, we get $\lfloor \frac{3}{2}(p-2) \rfloor = 3$.

Now, we assume that the conclusion holds for $p = k$ ($\geqslant 5$). In the following we show the conclusion holds for $p = k+1$. Here we distinguish two cases by the parity of k. For convenience of narration, choose a vertex $u_{k+1} \in V(K_{k+1})$ such that $k+1 \neq a, b, c$. Then $K_{k+1} = K_k + u_{k+1}$ and $S_0 = \{e_{ab}, e_{bc}, e_{ac}\} \subset V(L(K_k))$.

Case 1: k is even. By the induction hypothesis, there exist at most $\lfloor \frac{3}{2}(k-2) \rfloor$ internally disjoint S_0-trees in $L(K_k)$. Then add a new S_0-tree $e_{bc}e_{c(k+1)}e_{a(k+1)}e_{ab} \cup e_{a(k+1)}e_{ac}$ together, thus we get $\lfloor \frac{3}{2}(k-2) \rfloor + 1 = \lfloor \frac{3}{2}(k-1) \rfloor = \lfloor \frac{3}{2}((k+1)-2) \rfloor$. The conclusion holds for $p = k+1$. In addition, by the procedure of this construction, vertex $e_{b(k+1)}$ remains.

Case 2: k is odd. Since $k-1$ is even, by Case 1, there exist $\lfloor \frac{3}{2}(k-2) \rfloor$ internally disjoint S_0-trees in $L(K_k)$ and vertex e_{bk} remains in forming these internally disjoint S_0-trees. Now we add two new S_0-trees such as $e_{bc}e_{c(k+1)}e_{b(k+1)}e_{ab} \cup e_{c(k+1)}e_{ac}$ and $e_{ab}e_{a(k+1)}e_{k(k+1)}e_{bk}e_{bc} \cup e_{a(k+1)}e_{ac}$ together, and thus get $\lfloor \frac{3}{2}(k-2) \rfloor + 2 = \lfloor \frac{3}{2}(k-1) \rfloor$. The conclusion holds for $p = k+1$.

Now we analyze the maximality to show $\kappa(S_0) = \lfloor \frac{3}{2}(p-2) \rfloor$. Denote $\{u_a, u_b, u_c\} = V_{S_0}$, then $|E(V_{S_0}, \overline{V_{S_0}})| = 3(p-3)$. By the above construction, we know that each S_0-tree needs to consume at least two edges in $E(V_{S_0}, \overline{V_{S_0}})$ except for S_0-trees $e_{ac}e_{ab}e_{bc}$ and $e_{bc}e_{cx}e_{ax}e_{ab} \cup e_{ax}e_{ac}$. By this we know that $L(K_p)$ contains at most $\frac{1}{2}(3(p-3)-1) + 1 + 1 = \frac{3}{2}(p-2)$ internally disjoint S_0-trees, and thus $\kappa(S_0) = \lfloor \frac{3}{2}(p-2) \rfloor$. This completes the proof. $\square$

First, we determine the generalized 3-(edge-)connectivity of a line graph of a complete graph.

Theorem 4.1. *Let $L(K_p)$ be the line graph of K_p with order p ($\geqslant 3$). Then $\kappa_3(L(K_p)) = \lfloor \frac{3}{2}(p-2) \rfloor$.*

Proof. Suppose $V(K_p) = \{u_1, u_2, \ldots, u_p\}$ and $V(L(K_p)) = \{e_{ij} \colon e_{ij} = u_iu_j, 1 \leqslant i \neq j \leqslant p\}$. The case for $p = 3$ is trivial, here consider $p \geqslant 4$.

We first consider the case when $p = 4$. Since $L(K_4)$ is 4-regular, by Proposition 2.1 (2), we get $\kappa_3(L(K_4)) \leqslant 3$. On the other hand, let $S = \{x, y, z\}$ be a 3-subset of $V(L(K_4))$. Then the induced subgraphs $L(K_4)[S]$ are either K_3 or P_3. If $L(K_4)[S] = K_3$, let $x = e_{12}$, $y = e_{13}$, $z = e_{23}$, then xzy, $xye_{34}z$ and $xe_{14}e_{24}y \cup ze_{14}$ are 3 internally disjoint S-trees in $L(K_4)$. If $L(K_4)[S] = P_3$, let $x = e_{12}$, $y = e_{23}$, $z = e_{34}$, then xyz, $xe_{24}z \cup e_{24}y$ and $xe_{13}z \cup ye_{13}$ are 3 internally disjoint S-trees in $L(K_4)$. Thus, we get $\kappa_3(L(K_4)) \geqslant 3$. Combine $\lfloor \frac{3}{2}(p-2) \rfloor = 3$ for $p = 4$, the conclusion holds for $p = 4$. In the following we investigate the cases for $p \geqslant 5$.

Let $S = \{x, y, z\} \subseteq V(L(K_p))$ and assume $x = e_{ab}$, $y = e_{cd}$, $z = e_{ef}$ and $V_S = \{u_a, u_b, u_c, u_d, u_e, u_f\}$ for $1 \leqslant a < b < c < d < e < f \leqslant p$. Then the induced subgraph $K_p[V_S]$ is just one of $3K_2, K_{1,3}, K_2 \cup P_3, P_4$ and K_3. If $K_p[V_S] = 3K_2$, i.e., $K_p[V_S] = u_a u_b \cup u_c u_d \cup u_e u_f$, we first construct 6 internally disjoint S-trees as $ye_{bc}xe_{be}z$, $ye_{ad}xe_{af}z$, $xe_{ac}ye_{ce}z$, $xe_{db}ye_{df}z$, $xe_{ae}ze_{ed}y$ and $xe_{bf}ze_{fc}y$. In addition to these S-trees, for each $i \in \{1, 2, \ldots, p\} \setminus \{a, b, c, d, e, f\}$ we can construct 2 internally disjoint S-trees such as $T_{i1} = xe_{ia}e_{ic}y \cup e_{ic}e_{ie}z$ and $T_{i2} = xe_{ib}e_{id}y \cup e_{id}e_{if}z$. Then we get in total $2(p-6)+6 = 2p-6$ internally disjoint S-trees in $L(K_p)$. Note that $2p-6 \geqslant \lfloor \frac{3}{2}(p-2) \rfloor$ for $p \geqslant 5$, thus we get $\kappa_3(L(K_p)) \geqslant \lfloor \frac{3}{2}(p-2) \rfloor$ while $K_p[V_S] = 3K_2$.

By similar consideration, if $K_p[V_S] = K_{1,3}, K_2 \cup P_3$ and P_4, we get $2(p-4)+2$, $2(p-5)+5$ and $2(p-4)+3$ internally disjoint S-trees in $L(K_p)$, respectively. In particular, as to the case for $K_p[V_S] = K_3$, by Lemma 4.1, we can get $\lfloor \frac{3}{2}(p-2) \rfloor$ internally disjoint S-trees in $L(K_p)$. Therefore, we get $\kappa_3(L(K_p)) \geqslant \lfloor \frac{3}{2}(p-2) \rfloor$. On the other hand, choose a 3-subset $S_0 = \{e_{ab}, e_{bc}, e_{ac}\} \subseteq V(L(K_p))$. By Lemma 4.1, we get $\kappa_3(L(K_p)) \leqslant \kappa(S_0) = \lfloor \frac{3}{2}(p-2) \rfloor$. This completes the proof. □

Theorem 4.2. *Let $L(K_p)$ be the line graph of K_p with order p ($\geqslant 3$). Then $\lambda_3(L(K_p)) = 2p-5$.*

Proof. Since $L(K_p)$ is $(2p-4)$-regular, by Proposition 2.2 (1), we get

$$\lambda_3(L(K_p)) \leqslant 2p-5.$$

It suffices to prove $\lambda_3(L(K_p)) \geqslant 2p-5$.

Let $V(K_p) = \{u_1, u_2, \ldots, u_p\}$, $V(L(K_p)) = \{e_{ij}\colon e_{ij} = u_i u_j, 1 \leqslant i \neq j \leqslant p\}$. Suppose $S = \{x, y, z\}$ be a 3 subset of $V(L(K_p))$. Without loss generality, assume $x = e_{ab}$, $y = e_{cd}$, $z = e_{ef}$ for $1 \leqslant a, b, c, d, e, f \leqslant p$ and $V_S = \{u_a, u_b, u_c, u_d, u_e, u_f\}$. Then the induced subgraph $K_p[V_S]$ is one of $3K_2, K_{1,3}, K_2 \cup P_3, P_4$ and K_3.

If $K_p[V_S] = 3K_2$, suppose $K_p[V_S] = u_a u_b \cup u_c u_d \cup u_e u_f$. We first get 7 internally disjoint S-trees as $xe_{ac}e_{ae}z \cup e_{ac}y$, $xe_{bc}y \cup e_{bc}e_{be}z$, $xe_{ad}ye_{ce}z$, $xe_{bd}e_{fd}z \cup e_{bd}y$, $xe_{bf}e_{cf}z \cup e_{cf}y$, $xe_{af}z \cup ye_{de}z$ and $xe_{be}e_{bf}e_{df}y \cup e_{bf}z$. Then for every $i \in [p] \setminus \{a, b, c, d, e, f\}$ we get two edge disjoint S-trees such as $xe_{ia}e_{ic}e_{ie}z \cup e_{ic}y$ and $xe_{ib}e_{id}e_{if}z \cup e_{id}y$. Total up altogether, we get $2(p-6)+7 = 2p-5$ edge disjoint S-trees in $L(K_p)$. Thus, we have $\lambda_3(L(K_p)) \geqslant 2p-5$ while $K_p[V_S] = 3K_2$.

Similarly, if $K_p[V_S] = K_2 \cup P_3$, suppose $K_p[V_S] = u_a u_b \cup u_c u_d u_e$. This means $d = f$ in S. We first get 5 edge disjoint S-trees as: $xe_{ad}y \cup e_{ad}z$, $xe_{be}yz$, $xe_{cb}e_{ce}z \cup e_{ce}y$, $xe_{db}z \cup e_{db}y$ and $xe_{ae}e_{ac}z \cup e_{ae}y$. Then for every $i \in [n] \setminus \{a, b, c, d, e\}$ we get two edge disjoint S-trees as $xe_{ia}e_{id}z \cup e_{id}y$ and $xe_{ib}e_{ie}e_{ic}z \cup e_{ie}y$. Adding up all, we get $2(p-5)+5 = 2p-5$ edge disjoint S-trees in $L(K_p)$. Thus, we have $\lambda_3(L(K_p)) \geqslant 2p-5$ while $K_p[V_S] = K_2 \cup P_3$.

If $K_p[V_S] = P_4$, suppose $K_p[V_S] = u_a u_b u_c u_d$, which means $e = b$, $c = f$ in S. We first get 3 edge disjoint S-trees as: xyz, $xe_{bc}z \cup e_{bc}y$ and $xe_{ad}z \cup e_{ad}y$. Then for every $i \in [p] \setminus \{a, b, c, d\}$ we can get two edge disjoint S-trees as $xe_{ia}e_{id}z \cup e_{ia}y$ and $xe_{ib}e_{ic}z \cup e_{ic}y$. Adding up all, we get $2(p-4)+3 = 2p-5$ edge disjoint S-trees in $L(K_p)$. Thus, we have $\lambda_3(L(K_p)) \geqslant 2p-5$ while $K_p[V_S] = P_4$.

If $K_p[V_S] = K_{1,3}$, suppose $K_p[V_S] = u_au_b \cup u_au_c \cup u_au_e$, which means $a = d = f$ in S. We first get 3 edge disjoint S-trees as: xyz, $xe_{bc}e_{be}z \cup e_{bc}y$ and $xe_{be}e_{ec}z \cup e_{ec}y$. Then for every $i \in [p] \setminus \{a, b, c, e\}$ we can get two edge disjoint S-trees as $xe_{ia}z \cup e_{ia}y$ and $xe_{ib}e_{ic}e_{ie}z \cup e_{ic}y$. Add up all, we get $2(p-4)+3 = 2p-5$ edge disjoint S-trees in $L(K_p)$. Thus, we have $\lambda_3(L(K_p)) \geqslant 2p-5$ while $K_p[V_S] = K_{1,3}$.

If $K_p[V_S] = K_3$, let $K_p[V_S] = u_au_bu_cu_a$, that is to say $e = a$, $d = b$, $f = c$ in S. Then for every $i \in [p] \setminus \{a, b, c\}$ we can get two edge disjoint S-trees as $xe_{ia}e_{ic}y \cup e_{ia}z$ and $xe_{ib}e_{ic}z \cup e_{ib}y$. Add S-tree xyz with them, we get $2(p-3)+1 = 2p-5$ edge disjoint S-trees in $L(K_p)$. Thus, we have $\lambda_3(L(K_p)) \geqslant 2p-5$ while $K_p[V_S] = K_3$.

By the above argument, we have $\lambda_3(L(K_p)) \geqslant 2p-5$. This completes the proof. □

Next, we determine the generalized 3-(edge-)connectivity of total graph for complete graph.

Theorem 4.3. *Let $T(K_p)$ be the total graph of K_p with order p $(\geqslant 2)$. Then*

$$\kappa_3(T(K_p)) = \begin{cases} 3 & \text{if } p = 3,\\ \left\lfloor \dfrac{3(p-2)}{2} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Proof. By Corollaries 3.5 and 3.8, the conclusion holds for cases $p = 2, 3$. Here we consider $p \geqslant 4$. Suppose $V(K_p) = \{u_1, u_2, \ldots, u_p\}$ and $V(L(K_p)) = \{e_{ij}\colon e_{ij} = u_iu_j \in E(K_p)\}$. Then $V(T(K_p)) = V(L(K_p)) \cup V(K_p)$. For convenience of narration, we denote the induced subgraphs $T(K_p)[V(L(K_p))]$ and $T(K_p)[V(K_p)]$ as $L(K_p)$ and K_p, respectively. We first choose a 3-subset $S_0 = \{e_{ij}, e_{jk}, e_{ik}\}$. By Lemma 4.1, $L(K_p)$ contains at most $\lfloor \frac{3}{2}(p-2) \rfloor$ internally disjoint S_0-trees. Adding S_0-tree $e_{ij}u_iu_je_{jk} \cup e_{ik}u_i$ together, we get at most $\lfloor \frac{3}{2}(p-2) \rfloor + 1$ internally disjoint S_0-trees in $T(K_p)$. By the definition of generalized connectivity, we have $\kappa_3(T(K_p)) \leqslant \lfloor \frac{3}{2}(p-2) \rfloor + 1$. It suffices to prove that $\kappa_3(T(K_p)) \geqslant \lfloor \frac{3}{2}(n-2) \rfloor + 1$.

Let $S = \{x, y, z\}$ be a 3-subset of $V(T(K_p))$. Now we prove that there exist at least $\lfloor \frac{3}{2}(p-2) \rfloor + 1$ internally disjoint S-trees in $T(K_p)$. Here we need to distinguish four cases.

Case 1: $|S \cap V(K_p)| = 3$. This means $x, y, z \in V(K_p)$, so assume $x = u_a$, $y = u_b$, $z = u_c$, where $1 \leqslant a, b, c \leqslant p$. We first get some internally disjoint S-trees in $T(K_p)$ such as path zxy and trees $T_i = u_iz \cup u_ix \cup u_iy$ for $i \in \{1, 2, \ldots, p\} \setminus \{a, b, c\}$. Then we obtain internally disjoint S-trees such as paths $xe_{ab}yz$, $xe_{ac}ze_{bc}y$ and trees

$T_j = xe_{ja}e_{jb}y \cup e_{jb}e_{jc}z$ for $j \in \{1, 2, \dots, p\} \setminus \{a, b, c\}$. Total up all, we will get $2p-3$ internally disjoint S-trees in $T(K_p)$. Note that $2p - 3 > \lfloor \frac{3}{2}(p-2) \rfloor + 1$, as desired.

Case 2: $|S \cap V(K_p)| = 2$. Assume $x, y \in V(K_p)$ and $z \in V(L(K_p))$. Without loss of generality, let $x = u_a$, $y = u_b$, $z = e_{cd}$ with $1 \leqslant a, b, c, d \leqslant n$. If $|\{u_a, u_b\} \cap \{u_c, u_d\}| = 0$, this means edges $u_a u_b$ and $u_c u_d$ are nonadjacent in K_p, then for every $i \in \{1, 2, \dots, p\} \setminus \{a, b\}$, trees $T_i = xu_i y \cup u_i e_{ic} z$ are $p-2$ internally disjoint S-trees for every $i \in \{1, 2, \dots, p\} \setminus \{a, b, d\}$, trees $T'_i = xe_{ai}e_{bi}y \cup e_{bi}e_{id}z$ are $p-3$ internally disjoint S-trees. Putting all T_i, T'_i with trees $yxe_{ad}z$ and $xe_{ab}ye_{bd}z$ together, we get $2p-3$ internally disjoint S-trees in $T(K_p)$. If $|\{u_a, u_b\} \cap \{u_c, u_d\}| = 1$, the edges $u_a u_b$ and $u_c u_d$ are adjacent in K_p. By similar discussion as above, we also get $2p-3$ internally disjoint S-trees in $T(K_p)$. If $|\{u_a, u_b\} \cap \{u_c, u_d\}| = 2$, this means $u_a u_b = u_c u_d$, then for every two integers $i, j \in \{1, 2, \dots, p\} \setminus \{a, b\}$ we can get three internally disjoint S-trees such as $xu_i e_{ai} z \cup u_i y$, $xu_j e_{bj} z \cup u_j y$ and $xe_{aj} z e_{bi} y$, and thus we get at least $\lfloor \frac{3}{2}(p-2) \rfloor$ internally disjoint S-trees. Putting these trees with xyz together, we obtain at least $\lfloor \frac{3}{2}(p-2) \rfloor + 1$ internally disjoint S-trees in $T(K_p)$. Note that $2p - 3 > \lfloor \frac{3}{2}(p-2) \rfloor + 1$, as desired.

Case 3: $|S \cap V(K_p)| = 1$. Assume $x \in V(K_p)$, $y, z \in V(L(K_p))$, let $x = u_a$, $y = e_{bc}$, $z = e_{df}$ with $1 \leqslant a, b, c, d, f \leqslant p$. If $|\{u_a\} \cap \{u_b, u_c\} \cap \{u_d, u_f\}| = 0$, then for every $i \in \{1, 2, \dots, p\} \setminus \{a, d\}$, trees $T_i = xu_i e_{ic} y \cup u_i e_{id} z$ are $p-2$ internally disjoint S-trees, for every $i \in \{1, 2, \dots, p\} \setminus \{a, b\}$, trees $T'_i = xe_{ai} \cup ze_{if}e_{ai}e_{ib}y$ are $p-2$ internally disjoint S-trees. Putting all T_i, T'_i with tree $ye_{ab}xu_d z$ together, we can get $2n-3$ internally disjoint S-trees in $T(K_p)$. If $|\{u_b, u_c\} \cap \{u_d, u_f\}| = 1$ and $a \notin \{b, c, d, f\}$, suppose $c = d$, then for every $i \in \{1, 2, \dots, p\} \setminus \{a, f\}$, trees $T_i = xu_i e_{ic} y \cup e_{ic} z$ are $p-2$ internally disjoint S-trees, for every $i \in \{1, 2, \dots, p\} \setminus \{a, b, f\}$, trees $T'_i = xe_{ai} \cup ze_{if}e_{ai}e_{ib}y$ are $p-3$ internally disjoint S-trees. Putting all T_i, T'_i with trees $xu_f e_{bf} y \cup e_{bf} z$ and $xe_{ab}e_{af} z \cup e_{ab} y$ together, we will get $2p-3$ internally disjoint S-trees in $T(K_p)$. If $|\{u_b, u_c\} \cap \{u_d, u_f\}| = 1$ and $a \in \{b, c, d, f\} \setminus \{u_b, u_c\} \cap \{u_d, u_f\}$, suppose $c = d$ and $a = b$, then the trees $T_i = xu_i e_{ic} y \cup e_{ic} z$ for every $i \in \{1, 2, \dots, p\} \setminus \{f\}$ and trees $T'_i = xe_{ai} \cup ze_{if}e_{ai}e_{ib}y$ for every $i \in \{1, 2, \dots, p\} \setminus \{a, c\}$ are $2p-3$ internally disjoint S-trees in $T(K_p)$. If $|\{u_a\} \cap \{u_b, u_c\} \cap \{u_d, u_f\}| = 1$, suppose $a = d = c$. Then trees $T_i = xu_i e_{ib} y \cup u_i e_{if} z$ for every $i \in \{1, 2, \dots, p\} \setminus \{a, f\}$ and $T'_i = xe_{ai} y \cup e_{ai} z$ for every $i \in \{1, 2, \dots, p\} \setminus \{c, f\}$ are $2p-4$ internally disjoint S-trees. In addition to these, adding tree $yu_b xz$ together, we will get $2p-3$ internally disjoint S-trees in $T(K_p)$. Note that $2p-3 > \lfloor \frac{3}{2}(p-2) \rfloor + 1$, as desired.

Case 4: $|S \cap V(K_p)| = 0$. This means $S \subseteq V(L(K_p))$ in this case. By Lemma 4.1 there exist at most $\lfloor \frac{3}{2}(p-2) \rfloor$ internally disjoint S-trees in $L(K_p)$. Putting these S-trees with $e_{ab}u_a u_c e_{cd} \cup u_c u_g e_{gf}$ together, we get $\lfloor \frac{3}{2}(n-2) \rfloor + 1$ internally disjoint S-trees in $T(K_p)$, as desired. This completes the proof. $\square$

Theorem 4.4. *Let $T(K_p)$ be the total graph of K_p with order p ($\geqslant 2$). Then $\lambda_3(T(K_p)) = 2p - 3$.*

P r o o f. Since $T(K_p)$ is $(2p-2)$-regular, by Proposition 2.2 (1), we get

$$\lambda_3(T(K_p)) \leqslant 2p - 3.$$

It suffices to prove that $\lambda_3(T(K_p)) \geqslant 2p - 3$. Let $S = \{x, y, z\}$ be a 3-subset of $V(T(K_p))$. We prove that there exist at least $2p - 3$ internally disjoint S-trees in $T(K_p)$.

Recall of the proof of Theorem 4.3. For all cases except $|S \cap V(K_p)| = 0$ and $|\{u_a, u_b\} \cap \{u_c, u_d\}| = 2$ with $x = u_a$, $y = u_b$ and $z = e_{cd}$, we have proved that there exist at least $2p-3$ internally disjoint S-trees in $T(K_p)$, which are also edge disjoint S-trees in $T(K_p)$, as desired. So here we mainly consider these two exceptional cases.

For the exceptional when case $|S \cap V(K_p)| = 0$, assume $S = \{e_{ab}, e_{cd}, e_{ef}\} \subset V(L(K_p))$. By Theorem 4.2, there exist at least $2p-5$ edge disjoint S-trees in $L(K_p)$. Putting these S-trees with $e_{ab}u_au_ce_{cd} \cup u_cu_ee_{ef}$ and $e_{ab}u_bu_de_{cd} \cup u_du_fe_{ef}$ together, we obtain $2p - 3$ edge disjoint S-trees in $T(K_p)$, as desired.

For the exceptional when case $|\{u_a, u_b\} \cap \{u_c, u_d\}| = 2$ with $x = u_a$, $y = u_b$ and $z = e_{cd}$, $S = \{u_a, u_b, e_{ab}\}$. Then for every $i \in \{1, 2, \ldots, p\} \setminus \{a, b\}$, trees $T_i^1 = xu_ie_{bi}z \cup e_{bi}y$ and $T_i^2 = xe_{ai}u_iy \cup e_{ai}z$ are $2p-4$ edge disjoint S-trees. Putting all T_i^1 and T_i^2 with tree xzy together, we get $2p-3$ edge disjoint S-trees in $T(K_p)$, as desired.

By the above argument, there exist at least $2p - 3$ edge disjoint S-trees in $T(K_p)$ and thus we get $\lambda_3(T(K_p)) \geqslant 2p - 3$. This completes the proof. □

5. Bound for generalized 3-connectivity of line graphs

In fact, it is not easy to determine the generalized k-connectivity for general graph G even if $k = 3$. So in this section, we discuss the bounds of the generalized 3-(edge-)connectivity for line graph $L(G)$ of graph G.

First, we denote $K_p^- = K_p \setminus \{e\}$, where $e \in E(K_p)$. Clearly, by Proposition 2.3, we have $\lambda_3(K_p^-) = p - 2 = \lambda(K_p^-)$. Now we determine the value of $\lambda_3(L(K_p^-))$.

Theorem 5.1. *Let $L(K_p^-)$ be a line graph of K_p^- with $p \geqslant 4$. Then $\lambda_3(L(K_p^-)) = 2p - 6$.*

P r o o f. Since the minimum degree $\delta(L(K_p^-))$ is $2p - 5$ and there exist two adjacent vertices in $L(K_p^-)$ with degree $2p-5$, by Proposition 2.2 (1), we get $\lambda_3(L(K_p^-)) \leqslant 2p - 6$. Now we prove $\lambda(S) \geqslant 2p - 6$ for any 3-subset $S \subseteq V(L(K_p^-))$. And thus $\lambda_3(L(K_p^-)) \geqslant 2p - 6$. This completes the proof.

Suppose $V(K_p^-) = \{u_1, u_2, \ldots, u_p\}$, $V(L(K_p^-)) = \{e_{ij}\colon\ e_{ij} = u_iu_j \in E(K_p)\} - \{e_{12}\}$ and $S = \{x, y, z\} \subseteq V(L(K_p^-))$. Note that the graph K_n^- can be seen as $2K_1 + K_{p-2}$ with $2K_1 = u_1 \cup u_2$. For convenience of narration, we color the edges of K_p^- incident to u_1 red, incident to u_2 blue and the others green, and thus every vertex of $V(L(K_p^-))$ that corresponds to an edge in K_p^- meets the corresponding color automatically. Thus, we use S_g, S_b, S_r, S_{bg}, S_{rg}, S_{br} to denote S which consists of green, blue, red, blue and green, red and green, blue and red vertices, respectively.

It is clear that $S \subseteq V(L(K_{p-1}))$ for $S \in \{S_g, S_b, S_r, S_{bg}, S_{rg}\}$. By Theorem 4.2, it follows that there are at least $2(p-1) - 5 = 2p - 7$ edge disjoint S-trees in $L(K_{p-1}) \subset L(K_p^-)$. In addition to these S-trees, by using red or blue vertices in $L(K_p^-)$ we can also get one S-tree. Putting all together, we obtain at least $2p - 6$ edge disjoint S-trees in $L(K_p^-)$, i.e., $\lambda(S) \geqslant 2p - 6$.

As for the case for $S = S_{br}$, suppose $S = \{e_{2i}, e_{2j}, e_{1k}\}$. Since $e_{2i}, e_{2j} \in L(K_{p-1})$ and $\lambda(L(K_{p-1})) = 2p-6$, then there exist at least $2p-6$ edge disjoint $e_{2i} - e_{2j}$ paths in $L(K_{p-1})$. Based on these $2p - 6$ edge disjoint $e_{2i} - e_{2j}$ paths, by using neighbor vertices of e_{1k} to connect e_{1k} with each $e_{2i} - e_{2j}$ path we get $2p - 6$ edge disjoint S-trees in $L(K_p^-)$. This implies $\lambda(S) \geqslant 2p - 6$. Thus we get $\lambda_3(L(K_p^-)) = 2p - 6$. This completes the proof. □

Now we discuss the bounds of the generalized 3-(edge-)connectivity for line graph $L(G)$.

Lemma 5.1 ([6]). *Let G be a graph with $\lambda(G) \neq 2$. Then $\lambda(L(G)) = 2\lambda(G) - 2$ if and only if there exist two adjacent vertices in G with degree $\lambda(G)$.*

Lemma 5.2 ([6]). *Let G be a graph for which $\lambda(G) \neq 1, 2$. Then $\lambda(L(G)) = 2\lambda(G) - 1$ if and only if there exist two adjacent vertices in G with one degree $\lambda(G)$ and the other degree $\lambda(G) + 1$.*

Theorem 5.2. *Let $L(G)$ be a line graph of G. Then $\lambda_3(G) \leqslant \kappa_3(L(G))$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and then $V(L(G)) = \{e_{ij}\colon\ e_{ij} = u_iu_j \in E(G)\}$ for $u_i, u_j \in V(G)$. Assume $\lambda_3(G) = m$, now prove $\kappa_3(L(G)) \geqslant m$. Suppose $S = \{e_{pq}, e_{rs}, e_{tk}\}$ be a 3-element vertex set of $L(G)$ and also a 3-element edge set of G. Then the edge induced subgraph $G[S]$ may be one of $3K_2, K_{1,3}, K_2 \cup P_3, P_4$ and K_3. If $G[S] = 3K_2$, suppose $u_pu_q \cup u_ru_s \cup u_tu_k$. Let $S' = \{u_p, u_r, u_t\}$. Since $\lambda_3(G) = m$, there exist at least m edge disjoint S'-trees in G and each S'-tree corresponds to the unique S-tree in $L(G)$. Thus, there exist at least m internally disjoint S-trees in $L(G)$. If $G[S] = K_{1,3}, K_2 \cup P_3, P_4, K_3$, by similar consideration as above, we can prove that there exist at least m internally disjoint S-trees in $L(G)$. So we get $\kappa(S) \geqslant m$ for any 3-element vertex set S of $L(G)$. Thus, $\kappa_3(L(G)) \geqslant m$. □

Theorem 5.3. *Let G be a connected graph with $\lambda_3(G) = \lambda(G) \neq 2$ and there exist two adjacent vertices in G with degree $\lambda(G)$. Then*

$$\lambda(G) \leqslant \kappa_3(L(G)) \leqslant \lambda_3(L(G)) \leqslant 2\lambda(G) - 2.$$

Proof. By Lemma 5.1, we get $\lambda(L(G)) = 2\lambda(G) - 2$. On the one hand, by Observation 1 and Proposition 2.2 (2), we get $\kappa_3(L(G)) \leqslant \lambda_3(L(G)) \leqslant \lambda(L(G)) = 2\lambda(G) - 2$. On the other hand, by Theorem 5.2 and combining $\lambda_3(G) = \lambda(G)$, we get $\lambda(G) = \lambda_3(G) \leqslant \kappa_3(L(G))$. Thus, we have $\lambda(G) \leqslant \kappa_3(L(G)) \leqslant \lambda_3(L(G)) \leqslant 2\lambda(G) - 2$. □

Remark 1. In Theorem 5.3, the upper bound is sharp for graph K_p^- with $p > 3$, see Theorem 5.1. The condition "two adjacent vertices in G with degree $\lambda(G)$" is necessary since the conclusion is not right for a tree.

By Theorem 5.3 and Lemma 5.2, we immediately get:

Theorem 5.4. *Let G be a connected graph with $\lambda_3(G) = \lambda(G) \neq 1, 2$ and there exist two adjacent vertices in G with one degree $\lambda(G)$ and the other degree $\lambda(G)+1$. Then $\lambda(G) \leqslant \kappa_3(L(G)) \leqslant \lambda_3(L(G)) \leqslant 2\lambda(G) - 1$.*

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