Applications of Mathematics

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Applications of Mathematics, Vol. 66 (2021), No. 4, 619–639

Persistent URL: http://dml.cz/dmlcz/148975

LOCAL WELL-POSEDNESS FOR A TWO-PHASE MODEL WITH MAGNETIC FIELD AND VACUUM

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Received September 3, 2019. Published online May 28, 2021.

Abstract. This paper proves the local well-posedness of strong solutions to a two-phase model with magnetic field and vacuum in a bounded domain $\Omega \subset \mathbb{R}^3$ without the standard compatibility conditions.

Keywords: two-phase flow; magnetic field; vacuum; local well-posedness

MSC 2020: 76T10, 35Q35, 35D35, 76N10

1. Introduction

In this paper, we consider the following two-phase model with magnetic field in a bounded domain $\Omega \subset \mathbb{R}^3$ (see [15]):

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0, \tag{1.1}$$

$$\partial_t((\varrho + n)u) + \operatorname{div}((\varrho + n)u \otimes u) + \nabla p(\varrho, n) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = \operatorname{rot} b \times b, \tag{1.2}$$

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.3}$$

$$\partial_t b + \operatorname{rot}(b \times u) = \Delta b, \tag{1.4}$$

$$\operatorname{div} b = 0 \tag{1.5}$$

with initial and boundary conditions

$$u = 0, \quad b \cdot \nu = 0, \quad \operatorname{rot} b \times \nu = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.6}$$

$$(\varrho, (\varrho + n)u, n, b)(\cdot, 0) = (\varrho_0, (\varrho_0 + n_0)u_0, n_0, b_0)(\cdot) \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.7}$$

The research has been supported by the Natural Science Foundation of China (Grant No. 11671193).

DOI: 10.21136/AM.2021.0222-19

Here ϱ, u, b and n denote the density of the fluid, the velocity of the fluid, the magnetic field and the density of the particles in the mixture, respectively. The parameters μ and λ are viscosity constants satisfying

$$\mu > 0, \quad 3\lambda + 2\mu \geqslant 0.$$

The pressure

$$p := a\varrho^\gamma + n$$

with constants $a > 0$ and $\gamma > 1$. The domain Ω is bounded in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and ν is the unit outward normal vector to $\partial\Omega$.

Recently, Wen and Zhu [15] considered the global small smooth solution of problem (1.1)–(1.7) with positive $\inf(\varrho_0 + n_0) > 0$.

When $b = 0$, system (1.1)–(1.3) can be derived, formally, by Carrillo and Goudon in [2] from the Vlasov-Fokker-Planck/compressible Navier-Stokes equations, and Vasseur et al. [14] obtained the global existence of weak solutions.

When $n = 0$, (1.1), (1.2), (1.4), and (1.5) are reduced to the isentropic compressible magnetohydrodynamic system. Zhu [16] proved the local well-posedness of strong solutions with the following natural compatibility condition:

$$\nabla(a\varrho_0^\gamma) - \mu\Delta u_0 - (\lambda+\mu)\nabla \operatorname{div} u_0 - \operatorname{rot} b_0 \times b_0 = \sqrt{\varrho_0}g \tag{1.8}$$

for some $g \in L^2(\Omega)$. Nowakowski and his coauthors [11], [12] obtained the local existence of strong solutions in bounded domain of $\mathbb{R}^3$ under some symmetric assumptions. Hong et al. [8] obtained global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum and assumptions (1.8) and some other constraints on the total initial energy.

If we take $n = 0$ and $b = 0$ in system (1.1)–(1.2), then it is reduced to the classical isentropic Navier-Stokes equations and there are many results on it. Below we just mention some of them related to our paper. Choe and his coauthors [3], [4] obtained the local well-posedness of strong solutions of the barotropic/istropic Navier-Stokes equations with vacuum in bounded or unbounded domains under a natural compatibility condition similar to (1.8). Recently, this compatibility condition was removed by Huang [9] and Gong et al. [7], independently. Enomoto and Shibata [5] obtained the local well-posedness of strong solution to the isentropic Navier-Stokes equations in the framework of maximal L_p-L_q regularity, see also [13] for recent progress on compressible two-component mixture flow.

Motivated by the results on isentropic Navier-Stokes equations [7], [9], in this paper, we establish the local well-posedness of strong solutions to system (1.1)–(1.5) with initial vacuum in a bounded domain $\Omega \subset \mathbb{R}^3$ without the compatibility condition

$$\nabla(a\varrho_0^\gamma + n_0) - \mu\Delta u_0 - (\lambda+\mu)\nabla \operatorname{div} u_0 - \operatorname{rot} b_0 \times b_0 = \sqrt{\varrho_0}g \quad \text{for some } g \in L^2(\Omega).$$

We will prove the following theorem.

Theorem 1.1. *Let $0 \leqslant \varrho_0$, $n_0 \in W^{1,q}$ $(3 < q < 6)$, $u_0 \in H_0^1$, $b_0 \in H^1$ with $\operatorname{div} b_0 = 0$ in Ω and $b_0 \cdot \nu = 0$, $\operatorname{rot} b_0 \times \nu = 0$ on $\partial\Omega$. Then problem (1.1)–(1.7) has a unique local strong solution (ϱ, n, u, b) satisfying*

$$\begin{aligned}(1.9)\quad & 0 \leqslant \varrho,\ n \in L^\infty(0,T;W^{1,q}),\ \partial_t\varrho, \partial_t n \in L^\infty(0,T;L^2),\\ & u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2),\ \sqrt{\varrho+n}\partial_t u \in L^2(0,T;L^2),\\ & \sqrt{t}\sqrt{\varrho+n}\partial_t u \in L^\infty(0,T;L^2),\ \sqrt{t}\partial_t u \in L^2(0,T;H_0^1),\\ & b \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2),\ \partial_t b \in L^2(0,T;L^2),\ \sqrt{t}\partial_t b \in L^\infty(0,T;L^2)\end{aligned}$$

for some $0 < T \leqslant \infty$.

Remark 1.1. Theorem 1.1 still holds for more general pressure like $p(\varrho, n) = a\varrho^\gamma + bn^\beta$, $a > 0$, $b \geqslant 0$, $\gamma > 1$ and $\beta > 1$, and the whole proof can be fulfilled by modifying slightly the arguments below. Here we choose the form $p(\varrho, n) = a\varrho^\gamma + n$, $a > 0$, $\gamma > 1$ just because it was given in [15] for system (1.1)–(1.5), which can be derived from a fluid-particle model (see [15] for more details).

We will prove Theorem 1.1 in the following way: For $\delta > 0$ we choose $0 < \delta \leqslant \varrho_0^\delta \in H^2$, $0 < \delta \leqslant n_0^\delta \in H^2$, $u_0^\delta \in H_0^1 \cap H^2$ and $b_0^\delta \in H^2$ satisfying

$$(1.10)\qquad (\varrho_0^\delta, n_0^\delta) \to (\varrho_0, n_0) \text{ in } W^{1,q} \quad \text{and} \quad (u_0^\delta, b_0^\delta) \to (u_0, b_0) \text{ in } H^1 \text{ as } \delta \to 0.$$

Then we can verify that the problem has a unique local strong solution $(\varrho^\delta, n^\delta, u^\delta, b^\delta)$ in $[0, T_\delta)$. In fact, we shall prove the local well-posedness of strong solutions to problem (1.1)–(1.7) with positive and bounded ϱ_0 and n_0 for completeness in Appendix A.

Now we define

$$\begin{aligned}(1.11)\qquad M^\delta(t) := 1 + \sup_{0\leqslant s\leqslant t} \Big\{ & \|(\varrho^\delta, n^\delta)(\cdot, s)\|_{W^{1,q}} + \|u^\delta(\cdot,s)\|_{H^1} + \|b^\delta(\cdot,s)\|_{H^1}\\ & + \sqrt{s}\|\sqrt{\varrho^\delta + n^\delta}u_t^\delta(\cdot,s)\|_{L^2} + \sqrt{s}\|b_t^\delta(\cdot,s)\|_{L^2}\Big\}\\ & + \|u^\delta\|_{L^2(0,t;H^2)} + \|\sqrt{\varrho^\delta + n^\delta}u_t^\delta\|_{L^2(0,t;L^2)}\\ & + \|\sqrt{s}\nabla u_t^\delta\|_{L^2(0,t;L^2)} + \|b^\delta\|_{L^2(0,t;H^2)} + \|b_t^\delta\|_{L^2(0,t;L^2)}.\end{aligned}$$

We can prove the following theorem.

Theorem 1.2. *For any $t \in [0, T_\delta)$ we have that*

$$(1.12)\qquad M^\delta(t) \leqslant C_0(M_0^\delta)\exp(t^{(6-q)/(4q)}C(M^\delta(t)))$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$, where M_0^δ is defined through (1.11) by replacing $(\varrho^\delta, n^\delta, u^\delta, b^\delta)$ with $(\varrho_0^\delta, n_0^\delta, u_0^\delta, b_0^\delta)$ and taking $t = 0$.

It follows from (1.12) that (see argument on page 70 of [10]):

$$M^\delta(t) \leqslant C. \tag{1.13}$$

Once the sufficient a priori estimates (1.13) are established, we can take the same arguments as those in [7] on the isentropic Navier-Stokes equations with vacuum to obtain the existence part of Theorem 1.1 through taking $\delta \to 0$ and the standard compactness principle. (The structures of system (1.1)–(1.4) will be used in the arguments.) Moreover, the proof of the uniqueness part with regularity (1.9) is also easy and we omit it here and suggest the interested readers refer to [7] for more details.

Thus, to complete the proof of Theorem 1.1, we only need to prove Theorem 1.2. We shall present it in the next section.

2. Proof of Theorem 1.2

Below, we shall drop the superscript "δ" of $\varrho^\delta, n^\delta, u^\delta, b^\delta, M^\delta(t)$ and M_0^δ for the sake of simplicity. We also point out that the nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$ may change from line to line.

First, it is easy to see that

$$0 < \varrho, \quad 0 < n, \quad \int \varrho \,\mathrm{d}x = \int \varrho_0 \,\mathrm{d}x, \quad \int n \,\mathrm{d}x = \int n_0 \,\mathrm{d}x, \tag{2.1}$$

where the symbol $\int$ stands for the integration over Ω.

Testing (1.3) by $b_t - \Delta b$, we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int |\mathrm{rot}\, b|^2 \,\mathrm{d}x + \int (|b_t|^2 + |\Delta b|^2) \,\mathrm{d}x \\ = \int |\mathrm{rot}(b \times u)|^2 \,\mathrm{d}x \leqslant C(\|u\|_{L^6} \|\nabla b\|_{L^3} + \|b\|_{L^6} \|\nabla u\|_{L^3})^2 \\ \leqslant C(M)(\|\nabla u\|_{L^3}^2 + \|\nabla b\|_{L^3}^2) \leqslant C(M)(\|u\|_{H^2}^2 + \|b\|_{H^2}^2), \end{aligned}$$

which gives

$$\int |\mathrm{rot}\, b|^2 \,\mathrm{d}x + \int_0^t \int (|b_t|^2 + |\Delta b|^2) \,\mathrm{d}x \,\mathrm{d}s \leqslant C_0(M_0) + C(M)\sqrt{t}. \tag{2.2}$$

We will use the Poincaré inequality (see [1], Lemma 8):

$$\|b\|_{L^2} \leqslant C \|\mathrm{rot}\, b\|_{L^2}, \tag{2.3}$$

and estimates (see [1], Lemma 9):

$$\|b\|_{H^1} \leqslant C \|\mathrm{rot}\, b\|_{L^2}, \tag{2.4}$$

$$\|b\|_{H^2} \leqslant C \|\mathrm{rot}^2 b\|_{L^2} = C \|\Delta b\|_{L^2}. \tag{2.5}$$

Recall that we have the constraint $\operatorname{div} b = 0$. Equation (1.2) can be written as

$$-\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = f := -(\varrho+n)\partial_t u - (\varrho+n)u\cdot\nabla u - \nabla p + \operatorname{rot} b\times b. \tag{2.6}$$

Then we have

$$\begin{aligned}
\|u\|_{W^{2,q}} &\leqslant C\|f\|_{L^q} \leqslant C\|(\varrho+n)\partial_t u\|_{L^q} + C\|(\varrho+n)u\cdot\nabla u\|_{L^q} \\
&\quad + C\|\nabla p\|_{L^q} + C\|\operatorname{rot} b\times b\|_{L^q} \\
&\leqslant C\|(\varrho+n)\|_{L^\infty}^{(5q-6)/(4q)}\|\sqrt{\varrho+n}u_t\|_{L^2}^{(6-q)/(2q)}\|u_t\|_{L^6}^{(3q-6)/(2q)} \\
&\quad + C(M)\|u\|_{L^\infty}\|\nabla u\|_{L^q} + C(M) + C\|\operatorname{rot} b\|_{L^q}\|b\|_{L^\infty} \\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^{(6-q)/(2q)}\|\nabla u_t\|_{L^2}^{(3q-6)/(2q)} + C(M)\|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{3/2} \\
&\quad + C(M) + C\|\nabla b\|_{L^2}^{1/2}\|b\|_{H^2}^{3/2} \\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^{(6-q)/(2q)}\|\nabla u_t\|_{L^2}^{(3q-6)/(2q)} + C(M)\|u\|_{H^2}^{3/2} \\
&\quad + C(M) + C(M)\|b\|_{H^2}^{3/2},
\end{aligned} \tag{2.7}$$

which gives

$$\begin{aligned}
&\int_0^t \|u\|_{W^{2,q}}\,\mathrm{d}s \\
&\quad\leqslant C(M(t))\int_0^t \|\sqrt{\varrho+n}u_t\|_{L^2}^{(6-q)/(2q)}\|\nabla u_t\|_{L^2}^{(3q-6)/(2q)}\,\mathrm{d}s \\
&\qquad + C(M(t))\int_0^t \|u\|_{H^2}^{3/2}\,\mathrm{d}s + C(M(t))t + C(M(t))\int_0^t \|b\|_{H^2}^{3/2}\,\mathrm{d}s \\
&\quad\leqslant C(M(t))\int_0^t s^{-(3q-6)/(4q)}(\sqrt{s}\|\nabla u_t\|_{L^2})^{(3q-6)/(2q)}\|\sqrt{\varrho+n}u_t\|_{L^2}^{(6-q)/(2q)}\,\mathrm{d}s \\
&\qquad + C(M(t))\Big(\int_0^t \mathrm{d}s\Big)^{1/4}\Big(\int_0^t \|u\|_{H^2}^2\,\mathrm{d}s\Big)^{3/4} + C(M)t \\
&\qquad + C(M(t))\Big(\int_0^t \mathrm{d}s\Big)^{1/4}\Big(\int_0^t \|b\|_{H^2}^2\,\mathrm{d}s\Big)^{3/4} \\
&\quad\leqslant C(M(t))\Big(\int_0^t s^{-(3q-6)/(2q)}\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t s\|\nabla u_t\|_{L^2}^2\,\mathrm{d}s\Big)^{(3q-6)/(4q)} \\
&\qquad \times\Big(\int_0^t \|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}s\Big)^{(6-q)/(4q)} + C(M(t))t^{1/4} \\
&\quad\leqslant C(M(t))t^{(6-q)/(4q)} + C(M(t))t^{1/4} \leqslant C(M)t^{(6-q)/(4q)} \quad (3<q<6)
\end{aligned} \tag{2.8}$$

for all $0 < t \leqslant 1$.

Using the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^\infty} \leqslant C\|\nabla u\|_{L^2}^{(2q-6)/(5q-6)}\|u\|_{W^{2,q}}^{3q/(5q-6)} \tag{2.9}$$

and estimates (2.7), we observe that

$$
\begin{aligned}
(2.10)\qquad \int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}s &\leqslant C(M)\int_0^t \|u\|_{W^{2,q}}^{3q/(5q-6)}\,\mathrm{d}s\\
&\leqslant C\Big(\int_0^t \mathrm{d}s\Big)^{(2q-6)/(5q-6)}\Big(\int_0^t \|u\|_{W^{2,q}}\,\mathrm{d}s\Big)^{3q/(5q-6)}\\
&\leqslant C(M(t))t^{(2q-6)/(5q-6)}\cdot t^{(6-q)/(4q)\cdot 3q/(5q-6)}\\
&= C(M(t))t^{1/4}\leqslant C(M(t))t^{(6-q)/(4q)}\quad (3<q<6).
\end{aligned}
$$

Testing (1.1) by ϱ^{m-1}, we see that

$$
\begin{aligned}
\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho^m\,\mathrm{d}x &= -\int \operatorname{div}(\varrho u)\varrho^{m-1}\,\mathrm{d}x\\
&= \int \varrho u\nabla\varrho^{m-1}\,\mathrm{d}x = -\frac{m-1}{m}\int \varrho^m \operatorname{div} u\,\mathrm{d}x,
\end{aligned}
$$

which leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{L^m}\leqslant \|\operatorname{div} u\|_{L^\infty}\|\varrho\|_{L^m},
$$

and thus

$$
\begin{aligned}
(2.11)\qquad \|\varrho\|_{L^m} &\leqslant \|\varrho_0\|_{L^m}\exp\Big(\int_0^t \|\operatorname{div} u\|_{L^\infty}\,\mathrm{d}s\Big)\\
&\leqslant \|\varrho_0\|_{L^m}\exp(t^{(6-q)/(4q)}C(M))\quad (2\leqslant m\leqslant\infty).
\end{aligned}
$$

Applying ∇ to (1.1) and testing by $|\nabla\varrho|^{q-2}\nabla\varrho$, we find that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varrho\|_{L^q}\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\varrho\|_{L^q}+C\|\varrho\|_{L^\infty}\|\nabla\operatorname{div} u\|_{L^q},
$$

which implies

$$
\begin{aligned}
(2.12)\quad \|\nabla\varrho\|_{L^q} &\leqslant C\Big(\|\nabla\varrho_0\|_{L^q}+\int_0^t \|\varrho\|_{L^\infty}\|\nabla\operatorname{div} u\|_{L^q}\,\mathrm{d}s\Big)\exp\Big(\int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}s\Big)\\
&\leqslant C(1+C(M)t^{(6-q)/(4q)})\exp(t^{(6-q)/(4q)}C(M))\\
&\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)).
\end{aligned}
$$

Similarly, we have

$$
(2.13)\qquad \|n\|_{W^{1,q}}\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)).
$$

Testing (1.2) by u_t, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2)\,\mathrm{d}x+\int(\varrho+n)|u_t|^2\,\mathrm{d}x\\
&\quad=-\int(\varrho+n)u\cdot\nabla u\cdot u_t\,\mathrm{d}x+\int p\operatorname{div}u_t\,\mathrm{d}x\\
&\qquad-\int\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u_t\,\mathrm{d}x\\
&\quad=: I_1+I_2+I_3.
\end{aligned}\tag{2.14}
$$

We first bound I_1 and I_2 as follows:

$$
\begin{aligned}
|I_1|&\leqslant\|\sqrt{\varrho+n}u_t\|_{L^2}\|\sqrt{\varrho+n}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}\\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}\|u\|_{H^2}^{1/2}\\
&\leqslant\frac{1}{4}\|\sqrt{\varrho+n}u_t\|_{L^2}^2+C(M)\|u\|_{H^2};\\
I_2&=\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div}u\,\mathrm{d}x-\int p_t\operatorname{div}u\,\mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div}u\,\mathrm{d}x+\int(u\cdot\nabla\tilde{p}+\gamma\tilde{p}\operatorname{div}u)\operatorname{div}u\,\mathrm{d}x\\
&\quad+\int(u\cdot\nabla n+n\operatorname{div}u)\operatorname{div}u\,\mathrm{d}x\\
&\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div}u\,\mathrm{d}x+(\|u\|_{L^6}\|\nabla\tilde{p}\|_{L^3}+\gamma\|\tilde{p}\|_{L^\infty}\|\operatorname{div}u\|_{L^2})\|\operatorname{div}u\|_{L^2}\\
&\quad+(\|u\|_{L^6}\|\nabla n\|_{L^3}+\|n\|_{L^\infty}\|\operatorname{div}u\|_{L^2})\|\operatorname{div}u\|_{L^2}\\
&\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div}u\,\mathrm{d}x+C(M).
\end{aligned}
$$

Here we have used the fact: $p:=\tilde{p}+n$ and $\tilde{p}:=a\varrho^\gamma$, and

$$
\partial_t\tilde{p}+u\cdot\nabla\tilde{p}+\gamma\tilde{p}\operatorname{div}u=0.\tag{2.15}
$$

For the term I_3 we have

$$
\begin{aligned}
I_3&=-\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u\,\mathrm{d}x+\int\partial_t\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u\,\mathrm{d}x\\
&\leqslant-\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u\,\mathrm{d}x+C\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u\|_{L^2}\\
&\leqslant-\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u\,\mathrm{d}x+C(M)\|b\|_{H^2}^{1/2}\|b_t\|_{L^2}.
\end{aligned}
$$

Inserting the above estimates into (2.14) and integrating over $(0,t)$, we have

$$
\begin{aligned}
(2.16)\quad & \|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}t \\
&\leqslant C_0 + C(M(t))t + \int_0^t C(M)\|u\|_{H^2}\,\mathrm{d}s + C(M(t))\int_0^t \|b\|_{H^2}^{1/2}\|b_t\|_{L^2}\,\mathrm{d}s \\
&\quad + \int p\,\mathrm{div}\,u\,\mathrm{d}x - \int\Big(b\otimes b - \frac{1}{2}|b|^2\mathbb{I}_3\Big) : \nabla u\,\mathrm{d}x \\
&=: C_0 + C(M(t))t + \sum_{i=1}^4 l_i.
\end{aligned}
$$

Here we have used the nondecreasinguess of $C(M(t))$. Using the definition of $M(t)$, (2.11), (2.13) and (2.2), the terms l_i $(i=1,2,3,4)$ can be estimates as follows:

$$
\begin{aligned}
l_1 &\leqslant \Big(\int_0^t \|u\|_{H^2}^2\,\mathrm{d}s\Big)^{1/2} C(M(t))\Big(\int_0^t 1\,\mathrm{d}s\Big)^{1/2} \leqslant C(M)t, \\
l_2 &\leqslant C(M)\Big(\int_0^t \|b\|_{H^2}^{1/2\cdot 4}\,\mathrm{d}s\Big)^{1/4}\Big(\int_0^t \|b_t\|_{L^2}^2\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t 1\,\mathrm{d}s\Big)^{1/4} \leqslant C(M)t^{1/4}, \\
l_3 &\leqslant C_0\|p\|_{L^2}\|\nabla u\|_{L^2} \leqslant \frac{1}{8}\|\nabla u\|_{L^2}^2 + C_0\|p\|_{L^2}^2 \quad \text{(by (2.11) and (2.13))} \\
&\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)) + \frac{1}{8}\|\nabla u\|_{L^2}^2, \\
l_4 &\leqslant C_0\|b\|_{L^4}^2\|\nabla u\|_{L^2} \leqslant \frac{1}{8}\|\nabla u\|_{L^2} + C_0\|b\|_{L^4}^2 \\
&\leqslant \frac{1}{8}\|\nabla u\|_{L^2} + C_0(M_0) + C(M)t^{1/4} \quad \text{(by (2.2))}.
\end{aligned}
$$

Putting the above estimates into (2.16) and using the facts

$$
(2.17)\qquad t\leqslant 1,\quad \sqrt{t}\leqslant t^{1/4}\leqslant t^{(6-q)/(4q)} \quad (3<q<6),
$$

we finally obtain that

$$
\begin{aligned}
(2.18)\qquad \|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}s &\leqslant C_0(M_0) + C(M)t + C(M)t^{1/4} \\
&\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)).
\end{aligned}
$$

Applying ∂_t to (1.2) and using (1.1), we infer that

$$
\begin{aligned}
(2.19)\qquad (\varrho+n)\partial_t^2 u + (\varrho+n)u\cdot\nabla u_t - \mu\Delta u_t - (\lambda+\mu)\nabla\,\mathrm{div}\,u_t \\
= -\nabla p_t + \mathrm{div}((\varrho+n)u)(u_t + u\cdot\nabla u) \\
- (\varrho+n)u_t\cdot\nabla u + \partial_t\,\mathrm{div}\Big(b\otimes b - \frac{1}{2}|b|^2\mathbb{I}_3\Big).
\end{aligned}
$$

Testing (2.19) by u_t and using (1.1), we have

$$
\begin{aligned}
(2.20)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\varrho+n)|u_t|^2\,\mathrm{d}x+\int(\mu|\nabla u_t|^2+(\lambda+\mu)(\operatorname{div}u_t)^2)\,\mathrm{d}x\\
&=\int p_t\operatorname{div}u_t\,\mathrm{d}x-\int(\varrho+n)u\nabla|u_t|^2\,\mathrm{d}x-\int(\varrho+n)u\cdot\nabla(u\cdot\nabla u\cdot u_t)\,\mathrm{d}x\\
&\quad-\int(\varrho+n)u_t\cdot\nabla u\cdot u_t\,\mathrm{d}x-\int\partial_t\Big(b\otimes b-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u_t\,\mathrm{d}x\\
&=:\sum_{i=4}^{8}I_i.
\end{aligned}
$$

We bound I_i $(i=4,\dots,8)$ as follows:

$$
\begin{aligned}
|I_4|&\leqslant\Big|\int(u\cdot\nabla\tilde p+\gamma\tilde p\operatorname{div}u)\operatorname{div}u_t\,\mathrm{d}x\Big|+\Big|\int(u\cdot\nabla n+n\operatorname{div}u)\operatorname{div}u_t\,\mathrm{d}x\Big|\\
&\leqslant(\|u\|_{L^6}\|\nabla\tilde p\|_{L^3}+\gamma\|\tilde p\|_{L^\infty}\|\operatorname{div}u\|_{L^2})\|\operatorname{div}u_t\|_{L^2}\\
&\quad+(\|u\|_{L^6}\|\nabla n\|_{L^3}+\|n\|_{L^\infty}\|\operatorname{div}u\|_{L^2})\|\operatorname{div}u_t\|_{L^2}\\
&\leqslant C(M)\|\nabla u_t\|_{L^2}\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M);\\
|I_5|&\leqslant\|\sqrt{\varrho+n}\|_{L^\infty}\|\sqrt{\varrho+n}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^{1/2}\|\sqrt{\varrho+n}u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2}\\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{3/2}\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^2;\\
|I_6|&\leqslant\|\varrho+n\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6}+\|\varrho+n\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2u\|_{L^2}\|u_t\|_{L^6}\\
&\quad+\|\varrho+n\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\leqslant C(M)(\|\nabla u\|_{L^3}^2+\|u\|_{H^2})\|\nabla u_t\|_{L^2}\\
&\leqslant C(M)(\|\nabla u\|_{L^2}\|u\|_{H^2}+\|u\|_{H^2})\|\nabla u_t\|_{L^2}\\
&\leqslant C(M)\|u\|_{H^2}\|\nabla u_t\|_{L^2}\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|u\|_{H^2}^2;\\
|I_7|&\leqslant\|\nabla u\|_{L^2}\|\sqrt{\varrho+n}u_t\|_{L^4}^2\leqslant\|\nabla u\|_{L^2}\|\sqrt{\varrho+n}u_t\|_{L^2}^{1/2}\|\sqrt{\varrho+n}u_t\|_{L^6}^{3/2}\\
&\leqslant C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{3/2}\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^2;\\
|I_8|&\leqslant C\|\nabla u_t\|_{L^2}\|b\|_{L^\infty}\|b_t\|_{L^2}\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C\|b\|_{L^\infty}^2\|b_t\|_{L^2}^2\\
&\leqslant\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|b\|_{H^2}\|b_t\|_{L^2}^2.
\end{aligned}
$$

Inserting the above estimates into (2.20) gives

$$
\begin{aligned}
(2.21)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\varrho+n)|u_t|^2\,\mathrm{d}x+\frac{11}{16}\mu\int|\nabla u_t|^2\,\mathrm{d}x\\
&\leqslant C(M)+C(M)\|\sqrt{\varrho+n}u_t\|_{L^2}^2+C(M)\|u\|_{H^2}^2+C(M)\|b\|_{H^2}\|b_t\|_{L^2}^2.
\end{aligned}
$$

Multiplying the above inequality by t, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(t\int(\varrho+n)|u_t|^2\,\mathrm{d}x\right)+\frac{11\mu}{16}t\int|\nabla u_t|^2\,\mathrm{d}x \tag{2.22}$$
$$\leqslant\frac{1}{2}\int(\varrho+n)|u_t|^2\,\mathrm{d}x+C(M)t+C(M)t\|\sqrt{\varrho+n}u_t\|_{L^2}^2$$
$$+C(M)t\|u\|_{H^2}^2+C(M)t\|b\|_{H^2}\|b_t\|_{L^2}^2.$$

Integrating (2.22) over $(0,t)$, we get

$$t\int(\varrho+n)|u_t|^2\,\mathrm{d}x+\int_0^t s\|\nabla u_t\|_{L^2}^2\,\mathrm{d}s \tag{2.23}$$
$$\leqslant C_0+C(M(t))t^2+\frac{1}{2}\int_0^t\int(\varrho+n)|u_t|^2\,\mathrm{d}x$$
$$+C(M(t))\int_0^t s\|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}s$$
$$+C(M(t))\int_0^t s\|u\|_{H^2}^2\,\mathrm{d}s+C(M(t))\int_0^t s\|b\|_{H^2}\|b_t\|_{L^2}^2\,\mathrm{d}s$$
$$=:C_0+C(M(t))t^2+\sum_{i=5}^{8}l_i.$$

Now we estimate the terms l_i $(i=5,6,7,8)$. For the term l_5, using (2.18), we have

$$l_5\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)).$$

Similarly, for the term l_6 we have

$$l_6\leqslant C(M(t))t\int_0^t\|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}s\leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)).$$

To estimate l_7 we notice from (2.6) that

$$\begin{aligned}\|u\|_{H^2}&\leqslant C\|f\|_{L^2}\leqslant C\|(\varrho+n)u_t+(\varrho+n)u\cdot\nabla u+\nabla p-\operatorname{rot}b\times b\|_{L^2}\\&\leqslant C\|\sqrt{\varrho+n}\|_{L^\infty}\|\sqrt{\varrho+n}u_t\|_{L^2}+C\|\varrho+n\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\&\quad+C\|\nabla p\|_{L^2}+C\|b\|_{L^\infty}\|\operatorname{rot}b\|_{L^2}\\&\leqslant C\|\sqrt{\varrho+n}\|_{L^\infty}\|\sqrt{\varrho+n}u_t\|_{L^2}+C(M)\|u\|_{H^2}^{1/2}+C(M)+C(M)\|b\|_{H^2}^{1/2},\end{aligned}$$

which yields

$$\|u\|_{H^2}\leqslant C\|\sqrt{\varrho+n}\|_{L^\infty}\|\sqrt{\varrho+n}u_t\|_{L^2}+C(M)+C(M)\|b\|_{H^2}^{1/2},$$

and therefore, by the basic facts (2.17),

$$\|u\|_{L^2(0,t;H^2)} \leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)). \tag{2.24}$$

Now the terms l_7 and l_8 can be estimated as

$$\begin{aligned} l_7 &\leqslant C(M(t))t\int_0^t \|\sqrt{\varrho+n}u_t\|_{L^2}^2\,\mathrm{d}s \leqslant C(M(t))t,\\ l_8 &\leqslant C(M(t))\int_0^t \|b\|_{H^2}^2\,\mathrm{d}s\cdot\max_{[0,t]}\{s\|b_t\|_{L^2}^2\} \leqslant C(M(t))\sqrt{t}. \end{aligned}$$

Putting the above estimates into (2.22) and using the basic facts (2.17) imply that

$$t\int(\varrho+n)|u_t|^2\,\mathrm{d}x + \int_0^t s\|\nabla u_t\|_{L^2}^2\,\mathrm{d}s \leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)). \tag{2.25}$$

Applying ∂_t to (1.3) and testing by b_t, we compute

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|b_t|^2\,\mathrm{d}x + \int|\mathrm{rot}\,b_t|^2\,\mathrm{d}x &= -\int(b_t\times u + b\times u_t)\,\mathrm{rot}\,b_t\,\mathrm{d}x\\ &\leqslant (\|u\|_{L^6}\|b_t\|_{L^3} + \|b\|_{L^6}\|u_t\|_{L^3})\|\mathrm{rot}\,b_t\|_{L^2}\\ &\leqslant (C(M)\|b_t\|_{L^3} + C\|b\|_{L^6}\|u_t\|_{L^3})\|\mathrm{rot}\,b_t\|_{L^2}\\ &\leqslant \frac{1}{2}\|\mathrm{rot}\,b_t\|_{L^2}^2 + C(M)\|b_t\|_{L^2}^2 + C\|b\|_{L^6}^2\|\nabla u_t\|_{L^2}^2. \end{aligned} \tag{2.26}$$

Multiplying the above inequality by t and integrating it over $(0,t)$, and then using the similar arguments as for (2.23), we have

$$t\int|b_t|^2\,\mathrm{d}x + \int_0^t s\|\nabla b_t\|_{L^2}^2\,\mathrm{d}s \leqslant C_0(M_0)\exp(t^{(6-q)/(4q)}C(M)). \tag{2.27}$$

Combining (2.2), (2.11), (2.12), (2.13), (2.18), (2.25), (2.24), and (2.27), we conclude that (1.12) holds true. This completes the proof of Theorem 1.2. □

3. Appendix A. Local well-posedness of strong solutions to problem (1.1)–(1.7) with positive and bounded ϱ_0 and n_0

In this appendix, we shall prove the local well-posedness of strong solutions to problem (1.1)–(1.7) when the initial data ϱ_0 and n_0 are positive and bounded. We have:

Theorem A.1. *Let $1/C \leqslant \varrho_0,\ n_0 \leqslant C$ for some $C > 0$, $\varrho_0, n_0, u_0, b_0 \in H^2$ with $u_0 = 0$, $b_0 \cdot \nu = 0$, $\operatorname{rot} b_0 \times \nu = 0$ on $\partial\Omega$. Then problem (1.1)–(1.7) has a unique strong solution (ϱ, n, u, b) satisfying*

$$\text{(A.1)} \qquad \begin{aligned} &1/C_1 \leqslant \varrho, n \leqslant C_1, \quad \varrho, n \in C([0,T]; H^2), \quad \varrho_t, n_t \in C([0,T]; H^1);\\ &u, b \in C([0,T]; H^2) \cap \in L^2([0,T; H^3);\\ &u_t, b_t \in C([0,T]; L^2) \cap \in L^2([0,T; H^1) \end{aligned}$$

for some $C_1 > 0$ and $0 < T \leqslant \infty$.

We will prove Theorem A.1 by the classical Banach fixed-point theorem; see also Fan and Yu [6] for the local well-posedness on compressible magnetohydrodynamic equations. We denote the following nonempty closed convex set by

$$\mathscr{A} := \{\tilde{u} \in \mathscr{A};\ \tilde{u}(\cdot, 0) = u_0,\ \tilde{u}|_{\partial\Omega} = 0,\ \|\tilde{u}\|_{\mathscr{A}} \leqslant A\}$$

with the norm

$$\|\tilde{u}\|_{\mathscr{A}} := \|\tilde{u}\|_{C([0,T];H^2)} + \|\tilde{u}\|_{L^2(0,T;H^3)} + \|\partial_t \tilde{u}\|_{C([0,T];L^2)} + \|\partial_t \tilde{u}\|_{L^2(0,T;H^1)}.$$

Let $\tilde{u} \in \mathscr{A}$ be given, we consider the following linear problems:

$$\begin{aligned}
&\text{(A.2a)} && \partial_t \varrho + \operatorname{div}(\varrho \tilde{u}) = 0 && \text{in } \Omega \times (0,T),\\
&\text{(A.2b)} && \varrho(\cdot, 0) = \varrho_0 && \text{in } \Omega,\\
&\text{(A.3a)} && \partial_t n + \operatorname{div}(n \tilde{u}) = 0 && \text{in } \Omega \times (0,T),\\
&\text{(A.3b)} && n(\cdot, 0) = n_0 && \text{in } \Omega,\\
&\text{(A.4a)} && \partial_t b + \operatorname{rot}(b \times \tilde{u}) = \Delta b && \text{in } \Omega \times (0,T),\\
&\text{(A.4b)} && \operatorname{div} b(\cdot, 0) = 0 && \text{in } \Omega \times (0,T),\\
&\text{(A.4c)} && b \cdot \nu = 0,\ \operatorname{rot} b \times \nu = 0 && \text{on } \partial\Omega \times (0,T),\\
&\text{(A.4d)} && b(\cdot, 0) = b_0 && \text{in } \Omega,\\
&\text{(A.5a)} && \partial_t((\varrho + n)u) + \operatorname{div}((\varrho + n)\tilde{u} \otimes u) + \nabla p(\varrho, n) && \\
& && \qquad - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = \operatorname{rot} b \times b && \text{in } \Omega \times (0,T),\\
&\text{(A.5b)} && u = 0 && \text{on } \partial\Omega \times (0,T),\\
&\text{(A.5c)} && u(\cdot, 0) = u_0 && \text{in } \Omega.
\end{aligned}$$

We define the fixed point map:

$$F\colon\ \tilde{u} \in \mathscr{A} \to u \in \mathscr{A}.$$

We will prove that the map F maps $\mathscr{A}$ in $\mathscr{A}$ for suitable constant A and small T, and F is a contraction mapping on $\mathscr{A}$ and thus F has a unique fixed point in $\mathscr{A}$.

Lemma A.1. *Let $\tilde{u} \in \mathscr{A}$ be given. Then problem* (A.2) *has a unique solution ϱ satisfying*

$$1/C \leqslant \varrho \leqslant C, \quad \|\varrho\|_{L^\infty(0,T;H^2)} \leqslant C,$$
$$\|\varrho_t\|_{L^\infty(0,T;L^2)} \leqslant C, \quad \|\varrho_t\|_{L^\infty(0,T;H^1)} \leqslant CA. \tag{A.6}$$

Here and later on, the letter C will denote a constant independent of T and A.

Proof. Since the equation (A.2a) is linear with regular $\tilde{u}$, the existence and uniqueness are well-known, we only need to show the a priori estimates.

Let

$$\frac{\mathrm{d}x(X,t)}{\mathrm{d}t} = \tilde{u}(x(X,t),t) \quad \text{and} \quad x(X,0) = X.$$

Then we see that

$$\frac{\mathrm{d}\varrho(x(X,t),t)}{\mathrm{d}t} = -\varrho(x(X,t),t)\tilde{u}(x(X,t),t),$$

whence

$$\varrho(x,t) = \varrho_0 \exp\Big\{-\int_0^t \operatorname{div}\tilde{u}\,\mathrm{d}s\Big\},$$

which gives

$$\varrho \leqslant \varrho_0 \exp\Big\{\int_0^t \|\operatorname{div}\tilde{u}\|_{L^\infty}\,\mathrm{d}s\Big\} \leqslant \varrho_0 \exp\Big\{\int_0^t \|\operatorname{div}\tilde{u}\|_{H^3}\,\mathrm{d}s\Big\}$$
$$\leqslant \varrho_0 \exp\{CA\sqrt{T}\} \leqslant C\|\varrho_0\|_{L^\infty} \tag{A.7}$$

if $A\sqrt{T} \leqslant 1$; and

$$\varrho \geqslant \inf \varrho_0 \exp\Big\{\int_0^t \|\operatorname{div}\tilde{u}\|_{L^\infty}\,\mathrm{d}s\Big\} \geqslant \inf \varrho_0 \exp\{-CA\sqrt{T}\} \geqslant C\inf \varrho_0 \tag{A.8}$$

if $A\sqrt{T} \leqslant 1$.

Applying ∇^2 to (A.2a), test the result by $\nabla^2\varrho$, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla^2\varrho|^2\,\mathrm{d}x = &-\int(\nabla^2(\tilde{u}\cdot\nabla\varrho) - (\tilde{u}\cdot\nabla)\nabla^2\varrho)\cdot\nabla^2\varrho\,\mathrm{d}x\\
&-\int(\tilde{u}\cdot\nabla)\nabla^2\varrho\cdot\nabla^2\varrho\,\mathrm{d}x - \int\nabla^2(\varrho\operatorname{div}\tilde{u})\cdot\nabla^2\varrho\,\mathrm{d}x\\
\leqslant\ & C\|\nabla\tilde{u}\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}^2 + C\|\nabla\varrho\|_{L^6}\|\nabla^2\tilde{u}\|_{L^3}\|\nabla^2\varrho\|_{L^2}\\
&+ C\|\varrho\|_{L^\infty}\|\nabla^3\tilde{u}\|_{L^2}\|\nabla^2\varrho\|_{L^2},\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\varrho\|_{L^2} \leqslant C\|\nabla\tilde{u}\|_{L^\infty}\|\nabla^2\varrho\|_{L^2} + C\|\tilde{u}\|_{H^3}(1 + \|\nabla^2\varrho\|_{L^2}). \tag{A.9}$$

Using the Gronwall inequality, one has

$$\begin{aligned}
(\text{A.10}) \qquad \|\nabla^2\varrho\|_{L^2} &\leqslant \left(\|\nabla^2\varrho_0\|_{L^2} + C\int_0^T \|\tilde{u}\|_{H^3}\,\mathrm{d}t\right)\exp\left\{\int_0^T \|\tilde{u}\|_{H^3}\,\mathrm{d}t\right\} \\
&\leqslant (\|\nabla^2\varrho_0\|_{L^2} + CA\sqrt{T})\exp\{CA\sqrt{T}\} \leqslant C
\end{aligned}$$

if $A\sqrt{T} \leqslant 1$. It is clear that

$$\tilde{u} = u_0 + \int_0^t \partial_t\tilde{u}\,\mathrm{d}s$$

and

$$(\text{A.11}) \qquad \|\nabla\tilde{u}\|_{L^2} \leqslant \|\nabla u_0\|_{L^2} + \int_0^T \|\nabla\tilde{u}_t\|_{L^2}\,\mathrm{d}t \leqslant C + CA\sqrt{T} \leqslant C$$

if $A\sqrt{T} \leqslant 1$. Thus

$$(\text{A.12}) \qquad \|\varrho_t\|_{L^2} = \|\tilde{u}\cdot\nabla\varrho + \varrho\,\mathrm{div}\,\tilde{u}\|_{L^2} \leqslant \|\tilde{u}\|_{L^6}\|\nabla\varrho\|_{L^3} + \|\varrho\|_{L^\infty}\|\mathrm{div}\,\tilde{u}\|_{L^2} \leqslant C$$

and

$$(\text{A.13}) \qquad \|\varrho_t\|_{H^1} \leqslant \|\mathrm{div}(\varrho\tilde{u})\|_{H^1} \leqslant \|\varrho\|_{H^2}\|\tilde{u}\|_{H^2} \leqslant CA.$$

This completes the proof. □

Similarly, one has the following lemma.

Lemma A.2. *Let* $\tilde{u} \in \mathscr{A}$ *be given. Then problem* (A.3) *has a unique solution* n *satisfying*

$$\begin{aligned}
(\text{A.14}) \qquad &1/C \leqslant n \leqslant C, \quad \|n\|_{L^\infty(0,T;H^2)} \leqslant C, \\
&\|n_t\|_{L^\infty(0,T;L^2)} \leqslant C, \quad \|n_t\|_{L^\infty(0,T;H^1)} \leqslant CA.
\end{aligned}$$

Lemma A.3. *Let* $\tilde{u} \in \mathscr{A}$ *be given. Then problem* (A.4) *has a unique solution* b *satisfying*

$$(\text{A.15}) \qquad \|b\|_{L^\infty(0,T;H^2)} + \|b\|_{L^2(0,T;H^3)} + \|b_t\|_{L^\infty(0,T;L^2)} + \|b_t\|_{L^2(0,T;H^1)} \leqslant C.$$

P r o o f. Testing (A.4a) by b and using (A.11), one has

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\,\mathrm{d}x+\int |\mathrm{rot}\, b|^2\,\mathrm{d}x &= \int b\,\mathrm{rot}(\tilde u\times b)\,\mathrm{d}x\\ &= \int b(b\cdot\nabla\tilde u-b\,\mathrm{div}\,\tilde u-\tilde u\cdot\nabla b)\,\mathrm{d}x\\ &\leqslant C\|b\|_{L^4}^2\|\nabla\tilde u\|_{L^2}+\|\tilde u\|_{L^6}\|\nabla b\|_{L^2}\|b\|_{L^3}\\ &\leqslant C\|b\|_{L^4}^2+C\|\nabla b\|_{L^2}\|b\|_{L^3}\\ &\leqslant \frac{1}{2}\|\mathrm{rot}\, b\|_{L^2}^2+C\|b\|_{L^2}^2,\end{aligned}$$

which gives

$$\|b\|_{L^\infty(0,T;L^2)}+\|b\|_{L^2(0,T;H^1)}\leqslant C. \tag{A.16}$$

Here we have used the inequalities

$$\|b\|_{H^1}\leqslant C\|\mathrm{rot}\, b\|_{L^2}, \tag{A.17}$$
$$\|b\|_{L^4}^2\leqslant C\|b\|_{L^2}^{1/2}\|\mathrm{rot}\, b\|_{L^2}^{3/2}, \tag{A.18}$$
$$\|b\|_{L^3}^2\leqslant C\|b\|_{L^2}\|\mathrm{rot}\, b\|_{L^2}. \tag{A.19}$$

Testing (A.4a) by $-\Delta b$ and using (A.11), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathrm{rot}\, b|^2\,\mathrm{d}x+\int |\Delta b|^2\,\mathrm{d}x &= \int(\tilde u\cdot\nabla b-b\cdot\nabla\tilde u+b\,\mathrm{div}\,\tilde u)\Delta b\,\mathrm{d}x\\ &\leqslant (\|\tilde u\|_{L^6}\|\nabla b\|_{L^3}+C\|b\|_{L^\infty}\|\nabla\tilde u\|_{L^2})\|\Delta b\|_{L^2}\\ &\leqslant C(\|\nabla b\|_{L^3}+\|b\|_{L^\infty})\|\Delta b\|_{L^2}\\ &\leqslant C\|\nabla b\|_{L^2}^{1/2}\|\Delta b\|_{L^2}^{1/2}\|\Delta b\|_{L^2}\\ &\leqslant \frac{1}{2}\|\Delta b\|_{L^2}^2+C\|\mathrm{rot}\, b\|_{L^2}^2,\end{aligned}$$

which gives

$$\|b\|_{L^\infty(0,T;H^1)}+\|b\|_{L^2(0,T;H^2)}\leqslant C. \tag{A.20}$$

Here we have used the inequalities

$$\|b\|_{H^2}\leqslant C\|\Delta b\|_{L^2}, \tag{A.21}$$
$$\|\nabla b\|_{L^3}+\|b\|_{L^\infty}\leqslant C\|\nabla b\|_{L^2}^{1/2}\|b\|_{H^2}^{1/2}\leqslant C\|\mathrm{rot}\, b\|_{L^2}^{1/2}\|b\|_{H^2}^{1/2}. \tag{A.22}$$

Applying ∂_t to (A.4a), testing the result by b_t, and using (A.11), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b_t|^2\,\mathrm{d}x + \int |\mathrm{rot}\, b_t|^2\,\mathrm{d}x &= \int (\tilde{u}_t \times b + \tilde{u} \times b_t)\,\mathrm{rot}\, b_t\,\mathrm{d}x \\ &\leqslant (\|\tilde{u}_t\|_{L^2}\|b\|_{L^\infty} + \|\tilde{u}\|_{L^6}\|b_t\|_{L^3})\|\mathrm{rot}\, b_t\|_{L^2} \\ &\leqslant (A\|b\|_{L^\infty} + C\|b_t\|_{L^3})\|\mathrm{rot}\, b_t\|_{L^2} \\ &\leqslant \frac{1}{2}\|\,\mathrm{rot}\, b_t\|_{L^2}^2 + C\|b_t\|_{L^2}^2 + CA^2\|b\|_{H^2},\end{aligned}$$

which leads to

$$\text{(A.23)} \qquad \|b_t\|_{L^\infty(0,T;L^2)} + \|b_t\|_{L^2(0,T;H^1)} \leqslant C$$

if $T \leqslant 1$ and $A^2\sqrt{T} \leqslant 1$.

On the other hand, it is clear that

$$\begin{aligned}\|b\|_{H^2} &\leqslant C\|\Delta b\|_{L^2} \leqslant C\|b_t + \mathrm{rot}(b \times \tilde{u})\|_{L^2} \\ &= C\|b_t + \tilde{u}\cdot\nabla b - b\cdot\nabla\tilde{u} + b\,\mathrm{div}\,\tilde{u}\|_{L^2} \\ &\leqslant C\|b_t\|_{L^2} + C\|\tilde{u}\|_{L^6}\|\nabla b\|_{L^3} + C\|b\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2} \\ &\leqslant C + C\|\nabla b\|_{L^3} + C\|b\|_{L^\infty} \\ &\leqslant C + C\|\nabla b\|_{L^2}^{1/2}\|b\|_{H^2}^{1/2} \\ &\leqslant C + \frac{1}{2}\|b\|_{H^2},\end{aligned}$$

which implies

$$\text{(A.24)} \qquad \|b\|_{H^2} \leqslant C.$$

Similarly, we have

$$\begin{aligned}\|b\|_{H^3} &\leqslant C\|\Delta b\|_{H^1} \leqslant C\|\Delta b\|_{L^2} + C\|\nabla\Delta b\|_{L^2} \\ &\leqslant C + C\|\nabla(b_t + \mathrm{rot}(b \times u))\|_{L^2} \\ &\leqslant C + C\|\nabla b_t\|_{L^2} + C\|b\|_{H^2}\|\tilde{u}\|_{H^2} \\ &\leqslant C + C\|\nabla b_t\|_{L^2} + CA,\end{aligned}$$

which yields

$$\text{(A.25)} \qquad \|b\|_{L^2(0,T;H^3)} \leqslant C$$

if $T \leqslant 1$ and $A\sqrt{T} \leqslant 1$. This completes the proof. □

Lemma A.4. *Let $\tilde u \in \mathscr{A}$ be given. Then problem* (A.5) *has a unique solution u satisfying*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;H^3)} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \leqslant C \tag{A.26}$$

if T is small enough.

P r o o f. Since the equation (A.5a) is linear with regular $(\varrho, n, \tilde u, b)$, the existence and uniqueness are well-known, we only need to prove the a priori estimates.

First, testing (A.5a) by u and using (A.2a) and (A.3a), we derive that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm d}{\mathrm dt}\int(\varrho+n)|u|^2\,\mathrm dx + \mu\int|\nabla u|^2\,\mathrm dx + (\lambda+\mu)\int(\operatorname{div}u)^2\,\mathrm dx&\\ = \int(\operatorname{rot}b\times b)\cdot u\,\mathrm dx + \int p\operatorname{div}u\,\mathrm dx&\\ \leqslant \|\operatorname{rot}b\|_{L^2}\|b\|_{L^\infty}\|u\|_{L^2} + \|p\|_{L^2}\|\operatorname{div}u\|_{L^2}&\\ \leqslant \frac{\lambda+\mu}{2}\|\operatorname{div}u\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + C,&\end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leqslant C. \tag{A.27}$$

Testing (A.5a) by u_t, we obtain

$$\begin{aligned}&\frac{\mu}{2}\frac{\mathrm d}{\mathrm dt}\int|\nabla u|^2\,\mathrm dx + \frac{\lambda+\mu}{2}\frac{\mathrm d}{\mathrm dt}\int(\operatorname{div}u)^2\,\mathrm dx + \int(\varrho+n)|u_t|^2\,\mathrm dx\\ &\quad= -\int(\varrho+n)(\tilde u\cdot\nabla)u\cdot u_t\,\mathrm dx - \int\nabla p\cdot u_t\,\mathrm dx + \int(\operatorname{rot}b\times b)u_t\,\mathrm dx\\ &\quad\leqslant \|\varrho+n\|_{L^\infty}\|\tilde u\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^2} + \|\nabla p\|_{L^2}\|u_t\|_{L^2} + \|\operatorname{rot}b\|_{L^2}\|b\|_{L^\infty}\|u_t\|_{L^2}\\ &\quad\leqslant \frac{1}{2}\int(\varrho+n)|u_t|^2\,\mathrm dx + C\|\tilde u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C,\end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;H^1)} + \|u_t\|_{L^2(0,T;L^2)} \leqslant C \tag{A.28}$$

if $T \leqslant 1$ and $A^2T \leqslant 1$. This proves that

$$\|u\|_{L^\infty(0,T;H^2)} \leqslant C. \tag{A.29}$$

Applying ∂_t to (A.5a), testing the result by u_t, and using Lemmas A.1, A.2 and A.3, we find that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\varrho+n)|u_t|^2\,\mathrm{d}x+\int(\mu|\nabla u_t|^2+(\lambda+\mu)(\operatorname{div}u_t)^2)\,\mathrm{d}x\\
&\quad=\int p_t\operatorname{div}u_t\,\mathrm{d}x-\int(\varrho_t+n_t)(u_t+\tilde{u}\cdot\nabla u)\cdot u_t\,\mathrm{d}x-\int(\varrho+n)\tilde{u}_t\cdot\nabla u\cdot u_t\,\mathrm{d}x\\
&\qquad-\int\partial_t\Big((b\otimes b)-\frac{1}{2}|b|^2\mathbb{I}_3\Big):\nabla u_t\,\mathrm{d}x\\
&\quad\leqslant\|p_t\|_{L^2}\|\operatorname{div}u_t\|_{L^2}+\|\varrho_t+n_t\|_{L^2}\|u_t\|_{L^4}^2+\|\varrho_t+n_t\|_{L^6}\|\tilde{u}\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^3}\\
&\qquad+\|\varrho+n\|_{L^\infty}\|\tilde{u}_t\|_{L^2}\|\nabla u\|_{L^3}\|u_t\|_{L^6}+C\|b\|_{L^\infty}\|b_t\|_{L^2}\|\nabla u_t\|_{L^3}\\
&\quad\leqslant C\|\nabla u_t\|_{L^2}+C\|u_t\|_{L^4}^2+CA^2\|u_t\|_{L^3}+CA\|\nabla u\|_{L^6}\|u_t\|_{L^3}\\
&\quad\leqslant\frac{\mu}{2}\|\nabla u_t\|_{L^2}^2+C+C\int(\varrho+n)|u_t|^2\,\mathrm{d}x+CA^2\|\nabla u\|_{L^3}^2\\
&\quad\leqslant\frac{\mu}{2}\|\nabla u_t\|_{L^2}^2+C+C\int(\varrho+n)|u_t|^2\,\mathrm{d}x+CA^2\|\nabla u\|_{H^2},
\end{aligned}
$$

which implies that

$$\|u_t\|_{L^\infty(0,T;L^2)}+\|u_t\|_{L^\infty(0,T;H^1)}\leqslant C \tag{A.30}$$

if $T\leqslant 1$ and $A^2\sqrt{T}\leqslant 1$.

By the standard elliptic theory, it follows from (A.5) that

$$
\begin{aligned}
\|u\|_{H^2}&\leqslant C\|\operatorname{rot}b\times b-\nabla p-(\varrho+n)u_t-(\varrho+n)\tilde{u}\cdot\nabla u\|_{L^2}\\
&\leqslant C+C\|u_t\|_{L^2}+C\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^3}\leqslant C+C\|u_t\|_{L^2}+C\|u\|_{H^2}^{1/2},
\end{aligned}
$$

which gives

$$\|u\|_{H^2}\leqslant C. \tag{A.31}$$

Similarly,

$$
\begin{aligned}
\|u\|_{H^3}&\leqslant C+C\|\nabla(\operatorname{rot}b\times b-\nabla p-(\varrho+n)u_t-(\varrho+n)\tilde{u}\cdot\nabla u)\|_{L^2}\\
&\leqslant C+C\|\nabla u_t\|_{L^2}+C\|\nabla(\varrho+n)\|_{L^6}\|u_t\|_{L^3}+C\|\nabla(\varrho+n)\|_{L^6}\|\tilde{u}\|_{L^6}\|\nabla u\|_{L^6}\\
&\qquad+C\|\nabla\tilde{u}\|_{L^2}\|\nabla u\|_{L^\infty}+C\|\tilde{u}\|_{L^6}\|\nabla^2u\|_{L^3}\\
&\leqslant C+C\|\nabla u_t\|_{L^2}+C\|u_t\|_{L^3}+C\|\nabla u\|_{L^\infty}+C\|\nabla^2u\|_{L^3}\\
&\leqslant C+C\|\nabla u_t\|_{L^2}+C\|u\|_{H^3}^{1/2},
\end{aligned}
$$

which gives

$$\|u\|_{L^2(0,T;H^3)}\leqslant C. \tag{A.32}$$

This completes the proof. □

Thanks to the above Lemmas A.1–A.4, we can take $A := C$ and thus F maps $\mathscr{A}$ into $\mathscr{A}$. The following lemma tells us that F is a contraction mapping in the sense of weaker norm.

Lemma A.5. *There is a constant* $0 < \eta < 1$ *such that for any* $\tilde{u}_i$ $(i = 1, 2)$,

$$\|F(\tilde{u}_1) - F(\tilde{u}_2)\|_{L^2(0,T;H^1)} \leqslant \eta \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;H^1)} \tag{A.33}$$

for some $0 < T \leqslant 1$.

Proof. Suppose that $(\varrho_i, n_i, u_i, b_i)$ $(i = 1, 2)$ are the solutions to problems (A.2)–(A.5) corresponding to $\tilde{u}_i$ $(i = 1, 2)$. Denote

$$(\varrho, n, u, b, \tilde{u}) := (\varrho_1 - \varrho_2, n_1 - n_2, u_1 - u_2, b_1 - b_2, \tilde{u}_1 - \tilde{u}_2).$$

Then we have

$$\varrho_t + \operatorname{div}(\varrho\tilde{u}_1) = -\operatorname{div}(\varrho_2\tilde{u}), \quad \varrho(\cdot, 0) = \varrho_0, \tag{A.34}$$

$$n_t + \operatorname{div}(n\tilde{u}_1) = -\operatorname{div}(n_2\tilde{u}), \quad n(\cdot, 0) = n_0, \tag{A.35}$$

$$\begin{aligned} &b_t + \operatorname{rot}(b \times \tilde{u}_1 + b_2 \times \tilde{u}) = \Delta b, \quad \operatorname{div} b = 0, \\ &b \cdot \nu|_{\Omega\times(0,T)} = 0, \quad \operatorname{rot} b \times \nu|_{\Omega\times(0,T)} = 0, \quad b(\cdot, 0) = b_0, \end{aligned} \tag{A.36}$$

$$\begin{aligned} &(\varrho_1 + n_1)u_t + (\varrho_1 + n_1)\tilde{u}_1 \cdot \nabla u + \nabla(p(\varrho_1, n_1) - p(\varrho_2, n_2)) \\ &\qquad - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u) \\ &\quad = -((\varrho + n)\partial_t u_2 - (\varrho\tilde{u}_1 + n\tilde{u}_1 + n_2\tilde{u} + \varrho_2\tilde{u}) \cdot \nabla u_2 \\ &\qquad + \operatorname{div}\Big(b_1 \otimes b_1 - b_2 \otimes b_1 - \frac{1}{2}|b_1|^2\mathbb{I}_3 + \frac{1}{2}|b_2|^2\mathbb{I}_3\Big), \\ &u|_{\Omega\times(0,T)} = 0, \quad u(\cdot, 0) = u_0. \end{aligned} \tag{A.37}$$

Testing (A.34) by ϱ, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho^2 \,\mathrm{d}x &\leqslant C\|\nabla\tilde{u}_1\|_{L^\infty}\|\varrho\|_{L^2}^2 + C(\|\varrho_2\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2} + \|\nabla\varrho_2\|_{L^6}\|\tilde{u}\|_{L^3})\|\varrho\|_{L^2} \\ &\leqslant C\|\nabla\tilde{u}_1\|_{L^\infty}\|\varrho\|_{L^2}^2 + \eta_1\|\nabla\tilde{u}\|_{L^2}^2 + C\|\varrho\|_{L^2}^2 \end{aligned} \tag{A.38}$$

for any $0 < \eta_1 < 1$.

Similarly,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int n^2 \,\mathrm{d}x \leqslant C\|\nabla\tilde{u}_1\|_{L^\infty}\|n\|_{L^2}^2 + \eta_2\|\nabla\tilde{u}\|_{L^2}^2 + C\|n\|_{L^2}^2 \tag{A.39}$$

for any $0 < \eta_2 < 1$.

Testing (A.36) by b, we get

$$
\begin{aligned}
\text{(A.40)}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\,\mathrm{d}x + \int |\operatorname{rot} b|^2\,\mathrm{d}x
&= -\int (b\times\tilde{u}_1 + b_2\times\tilde{u})\operatorname{rot} b\,\mathrm{d}x\\
&= -\int (b\times\tilde{u}_1)\operatorname{rot} b\,\mathrm{d}x + \int \operatorname{rot}(b_2\times\tilde{u}_1)b\,\mathrm{d}x\\
&\leqslant \|b\|_{L^2}\|\tilde{u}_1\|_{L^\infty}\|\operatorname{rot} b\|_{L^2} + C(\|b_2\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^6}\|\nabla b_2\|_{L^3})\|b\|_{L^2}\\
&\leqslant \frac{1}{2}\|\operatorname{rot} b\|_{L^2}^2 + C\|b\|_{L^2}^2 + \eta_3\|\nabla\tilde{u}\|_{L^2}^2
\end{aligned}
$$

for any $0<\eta_3<1$.

Testing (A.37) by u and using $\partial_t(\varrho_1+n_1)+\operatorname{div}((\varrho_1+n_1)\tilde{u}_1)=0$, we get

$$
\begin{aligned}
\text{(A.41)}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\varrho_1+n_1)|u|^2\,\mathrm{d}x + \int (\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2)\,\mathrm{d}x\\
\leqslant C\int (|\varrho|+|n|)|\operatorname{div} u|\,\mathrm{d}x + C\|\partial_t u_2\|_{L^6}(\|\varrho\|_{L^2}+\|n\|_{L^2})\|u\|_{L^3}\\
+ C\big\{\|\tilde{u}_1\|_{L^\infty}\|\varrho\|_{L^2} + \|\tilde{u}_1\|_{L^\infty}\|n\|_{L^2} + \|n_2\|_{L^\infty}\|\tilde{u}\|_{L^2}\\
+ \|\varrho_2\|_{L^\infty}\|\tilde{u}\|_{L^2}\big\}\|\nabla u_2\|_{L^\infty}\|u\|_{L^2}\\
+ C\|b\|_{L^2}(\|b_1\|_{L^\infty}+\|b_2\|_{L^\infty})\|\nabla u\|_{L^2}\\
\leqslant C(\|\varrho\|_{L^2}+\|n\|_{L^2})\|\nabla u\|_{L^2} + C\|\partial_t u_2\|_{L^6}(\|\varrho\|_{L^2}+\|n_2\|_{L^2})\|u\|_{L^3}\\
+ C(\|\varrho\|_{L^2}+\|n\|_{L^2}+\|\tilde{u}\|_{L^2})\|\nabla u_2\|_{L^\infty}\|u\|_{L^2} + C\|b\|_{L^2}\|\nabla u\|_{L^2}\\
\leqslant \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \eta_4\|\tilde{u}\|_{L^2}^2 + C\|\varrho\|_{L^2}^2 + C\|n\|_{L^2}^2 + C\|u\|_{L^2}^2\\
+ C\|\partial_t u_2\|_{L^6}^2(\|\varrho\|_{L^2}^2+\|n\|_{L^2}^2) + C\|\nabla u_2\|_{L^\infty}^2\|u\|_{L^2}^2 + C\|b\|_{L^2}^2
\end{aligned}
$$

for any $0<\eta_4<1$.

Summing up (A.38), (A.39), (A.40) and (A.41), using the Gronwall inequality, and taking η_i $(i=1,\dots,4)$ small enough, we conclude that (A.33) holds true. This completes the proof. □

Proof of Theorem A.1. By Lemmas A.1–A.5 and the Banach fixed-point theorem, we finish the proof. □

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