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COHOMOLOGY AND DEFORMATIONS OF 3-DIMENSIONAL HEISENBERG HOM-LIE SUPERALGEBRAS

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Abstract. We study Hom-Lie superalgebras of Heisenberg type. For 3-dimensional Heisenberg Hom-Lie superalgebras we describe their Hom-Lie super structures, compute the cohomology spaces and characterize their infinitesimal deformations.

Keywords: Hom-Lie superalgebra; Lie superalgebra; Heisenberg Hom-Lie superalgebra; cohomology; deformation

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1. Introduction

In recent years, Hom-Lie algebras and other Hom-algebras are widely studied, motivated initially by instances appeared in Physics literature when looking for quantum deformations of some algebras of vector fields. Hom-Lie superalgebras, as a generalization of Hom-Lie algebras, are introduced in [3], [4]. Furthermore, the cohomology and deformation theories of Hom-algebras are studied in [1], [2], [6], [9] and so on, while the two theories of Hom-Lie superalgebras can be seen in [4], [5].

We will follow [7], [8] to define Heisenberg Hom-Lie superalgebras, which are a special case of 2-step nilpotent Hom-Lie superalgebras. The main idea of this paper is to characterize the infinitesimal deformations of Heisenberg Hom-Lie superalgebras using cohomology.

The paper proceeds as follows. In Section 2, we recall the definitions of Hom-Lie superalgebras. Section 3 is dedicated to introduce Heisenberg Hom-Lie superalgebras and classify three-dimensional Heisenberg Hom-Lie superalgebras. In Section 4, we review the cohomology theory and give the 2nd cohomology spaces of Heisenberg

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Hom-Lie superalgebras of dimension three. In the last section, we characterize all the infinitesimal deformations of three-dimensional Heisenberg Hom-Lie superalgebras using cohomology.

2. Preliminaries

Let V be a vector superspace over a field $\mathbb{F}$, that is, a $\mathbb{Z}_2$-graded vector space with a direct sum decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The elements of $V_{\bar{j}}$, $j = 0, 1$, are called *homogeneous of parity* j. The parity of homogenous element x is denoted by $|x|$. Moreover, the superspace $\mathrm{End}(V)$ has a natural direct sum decomposition $\mathrm{End}(V) = \mathrm{End}(V)_{\bar{0}} \oplus \mathrm{End}(V)_{\bar{1}}$, where $\mathrm{End}(V)_{\bar{j}} = \{f\colon f(V_{\bar{i}}) \subseteq V_{\overline{i+j}}\}$, $j = 0, 1$. Elements of $\mathrm{End}(V)_{\bar{j}}$ are homogeneous of parity j.

We review the definition of Hom-Lie superalgebra in [4].

Definition 2.1. A Hom-Lie superalgebra $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ is a triple consisting of a superspace V over a field $\mathbb{F}$, an even bilinear map $[\cdot,\cdot]\colon V \times V \to V$ and an even superspace homomorphism $\alpha\colon V \to V$ satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x] \quad \text{(skew-supersymmetry)}, \tag{2.1}$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\alpha(x), [y, z]] = 0 \quad \text{(hom-Jacobi identity)} \tag{2.2}$$

for all homogenous elements $x, y, z \in V$, where $\circlearrowleft_{x,y,z}$ denotes the cyclic summation over x, y, z.

We denote $\mathfrak{g}_{\bar{0}} = \mathfrak{g}|_{V_{\bar{0}}}$, $\mathfrak{g}_{\bar{1}} = \mathfrak{g}|_{V_{\bar{1}}}$ and then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. It follows that $\mathfrak{g}$ is a Hom-Lie algebra when $\mathfrak{g}_{\bar{1}} = 0$. The classical Lie superalgebra can be obtained when $\alpha = \mathrm{id}$.

A Hom-Lie superalgebra is called *multiplicative* if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all x, y. It is obvious that the classical Lie superalgebras are a special case of multiplicative Hom-Lie superalgebras.

The *center* of Hom-Lie superalgebra $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ is defined by

$$Z(\mathfrak{g}) = \{x \in V\colon\ [x, y] = 0,\ \forall\, y \in V\}.$$

Two Hom-Lie superalgebras $(V, [\cdot,\cdot]_1, \alpha)$ and $(V, [\cdot,\cdot]_2, \beta)$ are said to be *isomorphic* if there exists an even bijective homomorphism $\phi\colon\ (V, [\cdot,\cdot]_1) \to (V, [\cdot,\cdot]_2)$ satisfying

$$\phi([x, y]_1) = [\phi(x), \phi(y)]_2 \quad \forall\, x, y \in V,$$
$$\phi \circ \alpha = \beta \circ \phi.$$

In particular, $(V, [\cdot,\cdot], \alpha)$ and $(V, [\cdot,\cdot], \beta)$ are isomorphic if and only if there exists an even automorphism ϕ such that $\beta = \phi\alpha\phi^{-1}$.

Let V be a vector superspace as before. A bilinear form $\mathcal{B}$ on V is called *homogeneous of parity* j if it satisfies $\mathcal{B}(x,y) = 0$ for all $x, y \in V$, $|x| \neq |y| + j$, skew-supersymmetric if $\mathcal{B}(x,y) = -(-1)^{|x||y|}\mathcal{B}(y,x)$ for all homogenous elements $x, y \in V$, non-degenerate if from $\mathcal{B}(x,y) = 0$ for all $x \in V$ it follows that $y = 0$.

In this paper, we only discuss multiplicative Hom-Lie superalgebras over the complex field $\mathbb{C}$ and the elements mentioned are homogenous.

3. Heisenberg Hom-Lie superalgebras

Let $\mathfrak{g}$ be a finite-dimensional Hom-Lie superalgebra with a 1-dimensional homogenous derived ideal such that $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$. Let $h \in Z(\mathfrak{g})$ be the homogenous generator of $[\mathfrak{g},\mathfrak{g}]$. Then a homogenous skew-supersymmetric bilinear form $\overline{\mathcal{B}}$ can be defined on $\mathfrak{g}$ via $[x,y] = \overline{\mathcal{B}}(x,y)h$ for all $x, y \in \mathfrak{g}$. This induces a homogenous skew-supersymmetric bilinear form $\mathcal{B}$ on $\mathfrak{g}/Z(\mathfrak{g})$ via $\mathcal{B}(x + Z(\mathfrak{g}), y + Z(\mathfrak{g})) = \overline{\mathcal{B}}(x,y)$.

Definition 3.1. A Hom-Lie superalgebra $\mathfrak{g}$ is called a *Heisenberg Hom-Lie superalgebra* if the derived ideal $[\mathfrak{g},\mathfrak{g}]$ is generated by a homogenous element $h \in Z(\mathfrak{g})$ and $\mathcal{B}$ is non-degenerate.

From now on, we will also denote a Hom-Lie superalgebra by $(\mathfrak{h},\alpha)$, where $\mathfrak{h} = (V, [\cdot,\cdot]_{\mathfrak{h}})$ is a superalgebra and α is an even linear map. All brackets unmentioned in the following are zero.

Let $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ be a 3-dimensional Heisenberg Hom-Lie superalgebra with a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Let $h \in Z(\mathfrak{g})$ be the homogenous generator of the derived ideal $[\mathfrak{g},\mathfrak{g}]$. We analyze the cases $h \in \mathfrak{g}_{\bar{0}}$ and $h \in \mathfrak{g}_{\bar{1}}$ separately.

Case 1: If $h \in \mathfrak{g}_{\bar{0}}$, we have two subcases:

Subcase 1.1: There are $u_1, u_2 \in \mathfrak{g}_{\bar{0}}$ such that $\{u_1, u_2, h\}$ is a basis of $\mathfrak{g}$ and $[u_1, u_2] = h$, which implies that $\mathfrak{g}$ is a Hom-Lie algebra.

Subcase 1.2: There are $v_1, v_2 \in \mathfrak{g}_{\bar{1}}$ such that $\{h \mid v_1, v_2\}$ is a basis of $\mathfrak{g}$ and $[v_1, v_2] = h$. Then the Hom-Lie superalgebra will be denoted by $(\mathfrak{h}_1, \alpha)$.

Case 2: If $h \in \mathfrak{g}_{\bar{1}}$, there exist $u \in \mathfrak{g}_{\bar{0}}$, $v \in \mathfrak{g}_{\bar{1}}$ such that $\{u \mid v, h\}$ is a basis of $\mathfrak{g}$ and $[u, v] = h$. In this case, we denote the Hom-Lie superalgebra by $(\mathfrak{h}_2, \alpha)$.

Theorem 3.2. *Let $\mathfrak{g}$ be a multiplicative Heisenberg Hom-Lie (non-Lie) superalgebra of dimension three. Then $\mathfrak{g}$ must be isomorphic to one of the following:*

$$(1) \qquad \left(\mathfrak{h}_1, \begin{pmatrix} \mu_{11}\mu_{22} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}\right), \quad \mu_{11}\mu_{22} \neq 0,$$

$$(2) \qquad \left(\mathfrak{h}_1, \begin{pmatrix} \mu_{12}\mu_{21} & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & \mu_{21} & 0 \end{pmatrix}\right), \quad \mu_{12}\mu_{21} \neq 0,$$

$$(3) \qquad \left(\mathfrak{h}_1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$(4) \qquad \left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix}\right),$$

$$(5) \qquad \left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}\right), \quad (\mu_0 - 1)\mu_{11} = 0,$$

where $\mu_0, \mu_{ij} \in \mathbb{C}$, $i, j = 1, 2$.

P r o o f. We analyze the cases $h \in \mathfrak{g}_{\bar{1}}$ and $h \in \mathfrak{g}_{\bar{0}}$ separately.

Case 1: If $h \in \mathfrak{g}_{\bar{1}}$, there exists a basis $\{u \mid v, h\}$ of $\mathfrak{g}$ such that $[u, v] = h$. Suppose that $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & \mu_{21} & \mu_{22} \end{pmatrix}$, $\mu_0, \mu_{ij} \in \mathbb{C}$, $i, j = 1, 2$.

We have that $\mathfrak{g}$ is multiplicative if and only if $\alpha([e_i, e_j]) = [\alpha(e_i), \alpha(e_j)]$ for $i, j = 1, 2, 3$, which implies $\mu_{12} = 0$ and $\mu_{22} = \mu_0\mu_{11}$. Then we obtain that $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & \mu_{21} & \mu_0\mu_{11} \end{pmatrix}$.

(a) If $\mu_{21} = 0$, we obtain a Heisenberg Hom-Lie superalgebra

$$\left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix}\right).$$

(b) If $\mu_{21} \neq 0$, let

$$\phi = \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_{11} & 0 \\ 0 & b_{21} & b_0 b_{11} \end{pmatrix}, \quad \phi^{-1} = \begin{pmatrix} b_0^{-1} & 0 & 0 \\ 0 & b_{11}^{-1} & 0 \\ 0 & -b_0^{-1}b_{11}^{-2}b_{21} & b_0^{-1}b_{11}^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned}\phi\alpha\phi^{-1} &= \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_{11} & 0 \\ 0 & b_{21} & b_0 b_{11} \end{pmatrix} \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & \mu_{21} & \mu_0\mu_{11} \end{pmatrix} \begin{pmatrix} b_0^{-1} & 0 & 0 \\ 0 & b_{11}^{-1} & 0 \\ 0 & -b_0^{-1}b_{11}^{-2}b_{21} & b_0^{-1}b_{11}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & (1-\mu_0)\mu_{11}b_{11}^{-1}b_{21} + \mu_{21}b_0 & \mu_0\mu_{11} \end{pmatrix}.\end{aligned}$$

If $\mu_0 \neq 1$ and $\mu_{11} \neq 0$, then $b_{21} = -(1-\mu_0)^{-1}\mu_{11}^{-1}\mu_{21}b_0b_{11}$ yields

$$\phi\alpha\phi^{-1} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix},$$

which induces a Heisenberg Hom-Lie superalgebra like the one in (a).

Otherwise, i.e., $\mu_0 = 1$ or $\mu_{11} = 0$, then $b_0 = \mu_{21}^{-1}$ yields $\phi\alpha\phi^{-1} = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}$. We can obtain a new Heisenberg Hom-Lie superalgebra

$$\left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}\right), \quad (\mu_0 - 1)\mu_{11} = 0.$$

Case 2: If $h \in \mathfrak{g}_{\bar{0}}$, there exist $v_1, v_2 \in \mathfrak{g}_{\bar{1}}$ such that $\{h \mid v_1, v_2\}$ is a basis of $\mathfrak{g}$ and $[v_1, v_2] = h$. In this case, we can get three Heisenberg Hom-Lie superalgebras:

$$\left(\mathfrak{h}_1, \begin{pmatrix} \mu_{11}\mu_{22} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}\right), \quad \mu_{11}\mu_{22} \neq 0,$$

$$\left(\mathfrak{h}_1, \begin{pmatrix} \mu_{12}\mu_{21} & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & \mu_{21} & 0 \end{pmatrix}\right), \quad \mu_{12}\mu_{21} \neq 0,$$

$$\left(\mathfrak{h}_1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}\right).$$

□

4. The adjoint cohomology of Heisenberg Hom-Lie superalgebras

Let $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra. Let $x_1, \ldots, x_k$ be k homogeneous elements of V and $(x_1, \ldots, x_k) \in \wedge^k V$. Then we denote by $|(x_1, \ldots, x_k)| = |x_1| + \ldots + |x_k|$ the parity of $(x_1, \ldots, x_k)$. The set $C_\alpha^k(\mathfrak{g}, \mathfrak{g})$ of k-hom-cochains of $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ is the set of k-linear maps $\varphi\colon \wedge^k V \to V$ satisfying

$$\varphi(x_1, \ldots, x_{i+1}, x_i \ldots, x_k) = -(-1)^{|x_i||x_{i+1}|}\varphi(x_1, \ldots, x_i, x_{i+1} \ldots, x_k), \tag{4.1}$$

$$\alpha(\varphi(x_1, \ldots, x_k)) = \varphi(\alpha(x_1), \ldots, \alpha(x_k)) \tag{4.2}$$

for $x_1, \ldots, x_k \in V$, $1 \leqslant i \leqslant k-1$. In particular, $C_\alpha^0(\mathfrak{g}, \mathfrak{g}) = \{x \in \mathfrak{g}\colon \alpha(x) = x\}$. Denote by $|\varphi|$ the parity of φ and $|\varphi(x_1, \ldots, x_k)| = |(x_1, \ldots, x_k)| + |\varphi|$. We immediately get a direct sum decomposition $C_\alpha^k(\mathfrak{g}, \mathfrak{g}) = C_\alpha^k(\mathfrak{g}, \mathfrak{g})_{\bar{0}} \oplus C_\alpha^k(\mathfrak{g}, \mathfrak{g})_{\bar{1}}$.

A k-coboundary operator $\delta^k(\varphi)\colon C^k_\alpha(\mathfrak{g},\mathfrak{g}) \to C^{k+1}_\alpha(\mathfrak{g},\mathfrak{g})$ is defined by

$$\begin{aligned}\delta^k(\varphi)(x_0,\dots,x_k) = &\sum_{0\leqslant s<t\leqslant k}(-1)^{t+|x_t|(|x_{s+1}|+\dots+|x_{t-1}|)}\\ &\times \varphi(\alpha(x_0),\dots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\dots,\hat{x}_t,\dots,\alpha(x_k))\\ &+\sum_{s=1}^{k}(-1)^{s+|x_s|(|\varphi|+|x_0|+\dots+|x_{s-1}|)}\\ &\times [\alpha^{k-1}(x_s),\varphi(x_0,\dots,\hat{x}_s,\dots,x_k)],\end{aligned}$$

where $\hat{x}_i$ means that x_i is omitted.

The k-cocycles space, k-coboundaries space and kth cohomology space are defined as:

(1) $Z^k(\mathfrak{g},\mathfrak{g}) = \ker\delta^k$, $Z^k(\mathfrak{g},\mathfrak{g})_{\bar{j}} = Z^k(\mathfrak{g},\mathfrak{g}) \cap C^k_\alpha(\mathfrak{g},\mathfrak{g})_{\bar{j}}$, $j=0,1$,

(2) $B^k(\mathfrak{g},\mathfrak{g}) = \operatorname{Im}\delta^{k-1}$, $B^k(\mathfrak{g},\mathfrak{g})_{\bar{j}} = B^k(\mathfrak{g},\mathfrak{g}) \cap C^k_\alpha(\mathfrak{g},\mathfrak{g})_{\bar{j}}$, $j=0,1$,

(3) $H^k(\mathfrak{g},\mathfrak{g}) = Z^k(\mathfrak{g},\mathfrak{g})/B^k(\mathfrak{g},\mathfrak{g}) = H^k(\mathfrak{g},\mathfrak{g})_{\bar{0}} \oplus H^k(\mathfrak{g},\mathfrak{g})_{\bar{1}}$, where $H^k(\mathfrak{g},\mathfrak{g})_{\bar{j}} = Z^k(\mathfrak{g},\mathfrak{g})_{\bar{j}}/B^k(\mathfrak{g},\mathfrak{g})_{\bar{j}}$, $j=0,1$.

Theorem 4.1. *The cohomology spaces of Heisenberg Hom-Lie superalgebras are:*

(1) $$H^1(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} a_{22}+a_{33} & a_{12}\delta_{\mu_{22},1} & a_{13}\delta_{\mu_{11},1} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33}\end{pmatrix}\right\rangle,$$

$$H^2(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2}\mu_{22}a_{22}\delta_{\mu_{11},1} & -\frac{1}{2}\mu_{11}a_{34}\delta_{\mu_{22},1} \\ 0 & \delta_{\mu_{11},1}a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{\mu_{22},1}a_{34} & 0 & 0 \end{pmatrix}\right\rangle$$

for $\mathfrak{g} = \left(\mathfrak{h}_1, \begin{pmatrix} \mu_{11}\mu_{22} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22}\end{pmatrix}\right)$, $\mu_{11}\mu_{22}\neq 0$.

(2) $$H^1(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} 2a_{22} & a_{12}\delta_{\mu_{12}\mu_{21},1} & a_{12}\delta_{\mu_{12}\mu_{21},1} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{22}\end{pmatrix}\right\rangle,$$

$$H^2(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_{12}a_{23}\delta_{\mu_{12}\mu_{21},1} & -2\mu_{12}^2a_{23}\delta_{\mu_{12}\mu_{21},1} & 0 & 0 \\ 0 & -2\mu_{21}a_{23}\delta_{\mu_{12}\mu_{21},1} & a_{23}\delta_{\mu_{12}\mu_{21},1} & 0 & 0 & 0 \end{pmatrix}\right\rangle$$

for $\mathfrak{g} = \left(\mathfrak{h}_1, \begin{pmatrix} \mu_{12}\mu_{21} & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & \mu_{21} & 0 \end{pmatrix}\right)$, $\mu_{12}\mu_{21} \neq 0$.

(3)
$$H^1(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} (a_{22}+a_{33})\delta_{\mu_{12},0} & 0 & a_{13} \\ 0 & a_{22}\delta_{\mu_{12},0} & 0 \\ 0 & 0 & a_{33}\delta_{\mu_{12},0} \end{pmatrix} \right\rangle,$$
$$H^2(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} 0 & 0 & 0 & \delta_{\mu_{12},0}a_{14} & 0 & a_{16} \\ 0 & \mathcal{A} & \mathcal{B} & \mathcal{C} & 0 & \delta_{\mu_{11},0}a_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

for $\mathfrak{g} = \left(\mathfrak{h}_1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}\right)$, where $\mathcal{A} = a_{22}\delta_{\mu_{11}(\mu_{11}-1),0}$, $\mathcal{B} = a_{23}\delta_{\mu_{11},0} + \mu_{12}a_{22}\delta_{\mu_{11},1}$, $\mathcal{C} = a_{24}\delta_{\mu_{11},0} + \mu_{11}^{-1}\mu_{12}^2 a_{22}\delta_{\mu_{11},1}$.

(4)
$$H^1(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21}\delta_{\mu_0,\mu_{11}} & a_{22} & 0 \\ a_{31}\delta_{\mu_0(\mu_{11}-1),0} & a_{32}\delta_{(\mu_0-1)\mu_{11},0} & a_{11}+a_{22} \end{pmatrix} \right\rangle,$$
$$H^2(\mathfrak{g},\mathfrak{g}) =$$
$$\left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}\mu_0 a_{22}\delta_{\mu_{11},1} & -\mu_0 a_{34}\delta_{\mu_0\mu_{11},1} \\ 0 & a_{22}\delta_{\mu_{11},1} & 0 & 0 & 0 & a_{26}\delta_{(\mu_0{}^2-1)\mu_{11},0} \\ 0 & a_{32}\delta_{\mu_{11},0} & a_{33}\delta_{\mu_{11},0} & a_{34}\delta_{\mu_0\mu_{11}(\mu_0\mu_{11}-1),0} & 0 & a_{36}\delta_{\mu_0(\mu_0-1)\mu_{11},0} \end{pmatrix} \right\rangle$$

for $\mathfrak{g} = \left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix}\right)$.

(5)
$$H^1(\mathfrak{g},\mathfrak{g}) = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{33} & 0 \\ a_{31}\delta_{\mu_0,\mu_{11}} & a_{32}\delta_{(\mu_0-1)\mu_{11},0} & a_{33} \end{pmatrix} \right\rangle,$$
$$H^2(\mathfrak{g},\mathfrak{g}) =$$
$$\left\langle \begin{pmatrix} 0 & a_{12}\delta_{\mu_0,0}\delta_{\mu_{11},0} & a_{13}\delta_{\mu_0,0}\delta_{\mu_{11},0} & 0 & 0 & a_{16}\delta_{\mu_0,0} \\ 0 & \mathcal{D} & 0 & 0 & \mu_0 a_{36} & 0 \\ 0 & a_{32}\delta_{\mu_{11}(\mu_{11}-1),0} & a_{33}\delta_{\mu_{11}(\mu_{11}-1),0} & a_{34}\delta_{\mu_0,2}\delta_{\mu_{11},0} & 0 & a_{36} \end{pmatrix} \right\rangle$$

for $\mathfrak{g} = \left(\mathfrak{h}_2, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}\right)$, $(\mu_0-1)\mu_{11} = 0$, where $\mathcal{D} = 2a_{33}\delta_{\mu_{11},1} + a_{34}\delta_{\mu_{11},0}$.

P r o o f. It is easy to obtain $C^k_\alpha(\mathfrak{g},\mathfrak{g})$ for $k=1,2$ by (4.1) and (4.2). Taking $\mathfrak{g}=\left(\mathfrak{h}_1,\begin{pmatrix}\mu_{11}\mu_{22} & 0 & 0\\ 0 & \mu_{11} & 0\\ 0 & 0 & \mu_{22}\end{pmatrix}\right)$, $\mu_{11}\mu_{22}\neq 0$ for example,

$$C^1_\alpha(\mathfrak{g},\mathfrak{g})=\begin{pmatrix}a_{11} & 0 & 0\\ 0 & a_{22} & a_{23}\delta_{\mu_{11},\mu_{22}}\\ 0 & a_{32}\delta_{\mu_{11},\mu_{22}} & a_{33}\end{pmatrix},$$

$$C^2_\alpha(\mathfrak{g},\mathfrak{g})=\begin{pmatrix}0 & a_{12}\delta_{\mu_{11},\mu_{22}} & a_{13} & a_{14}\delta_{\mu_{11},\mu_{22}} & 0 & 0\\ 0 & 0 & 0 & 0 & a_{25}\delta_{\mu_{11}\mu_{22},1} & a_{26}\delta_{\mu_{22}{}^2,1}\\ 0 & 0 & 0 & 0 & a_{35}\delta_{\mu_{11}{}^2,1} & a_{36}\delta_{\mu_{11}\mu_{22},1}\end{pmatrix}.$$

Let $\varphi_0=\begin{pmatrix}a_{11} & 0 & 0\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{pmatrix}\in C^1_\alpha(\mathfrak{g},\mathfrak{g})_{\overline{0}}$. Then

$$B^2(\mathfrak{g},\mathfrak{g})_{\bar 0}=\{\delta^1\varphi_0\colon \varphi_0\in C^1_\alpha(\mathfrak{g},\mathfrak{g})_{\bar 0}\}=\left\langle\begin{pmatrix}0 & 2a_{32} & a_{22}+a_{33}-a_{11} & 2a_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\right\rangle.$$

Moreover, we have $\varphi_0\in Z^1(\mathfrak{g},\mathfrak{g})_{\bar 0}$ if and only if $\delta^1(\varphi_0)=0$.

In the same way, we suppose $\varphi_1=\begin{pmatrix}0 & a_{12} & a_{13}\\ a_{21} & 0 & 0\\ a_{31} & 0 & 0\end{pmatrix}\in C^1(\mathfrak{g},\mathfrak{g})_{\bar 1}$ and immediately get $Z^1(\mathfrak{g},\mathfrak{g})_{\bar 1}$ and $B^2(\mathfrak{g},\mathfrak{g})_{\bar 1}$.

Now suppose $\alpha=\begin{pmatrix}\mu_0 & 0 & 0\\ 0 & \mu_{11} & \mu_{12}\\ 0 & \mu_{21} & \mu_{22}\end{pmatrix}$, $\psi_0=\begin{pmatrix}0 & a_{12} & a_{13} & a_{14} & 0 & 0\\ 0 & 0 & 0 & 0 & a_{25} & a_{26}\\ 0 & 0 & 0 & 0 & a_{35} & b_{36}\end{pmatrix}\in C^2_\alpha(\mathfrak{g},\mathfrak{g})_{\bar 0}$, $\psi_1=\begin{pmatrix}0 & 0 & 0 & 0 & a_{15} & a_{16}\\ 0 & a_{22} & a_{23} & a_{24} & 0 & 0\\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0\end{pmatrix}\in C^2_\alpha(\mathfrak{g},\mathfrak{g})_{\bar 1}$. We know that $\psi_0\in Z^2_\alpha(g,g)_{\bar 0}$ (or $\psi_1\in Z^2_\alpha(g,g)_{\bar 1}$) if and only if $\delta^1(\psi_0)=0$ (or $\delta^1(\psi_1)=0$). □

5. Infinitesimal deformations of Heisenberg Hom-Lie superalgebras

Let $\mathfrak{g}=(V,[\cdot,\cdot]_0,\alpha)$ be a Hom-Lie superalgebra and $\varphi\colon V\times V\to V$ be an even bilinear map commuting with α. A bilinear map $[\cdot,\cdot]_t=[\cdot,\cdot]_0+t\varphi(\cdot,\cdot)$ is called an *infinitesimal deformation* of $\mathfrak{g}$ if φ satisfies

$$[x,y]_t=-[y,x]_t, \tag{5.1}$$

$$\circlearrowleft_{x,y,z}(-1)^{|x||z|}[\alpha(x),[y,z]_t]_t=0 \tag{5.2}$$

for $x,y,z\in V$. The previous equation (5.1) implies φ is skew-supersymmetric. We denote

$$\varphi\circ\psi(x,y,z)=\circlearrowleft_{x,y,z}(-1)^{|x||z|}\varphi(\alpha(x),\psi(y,z)),$$

and then equation (5.2) can be denoted by $[\cdot,\cdot]_t\circ[\cdot,\cdot]_t=0$.

Lemma 5.1. *Let* $\mathfrak{g}=(V,[\cdot,\cdot]_0,\alpha)$ *be a Hom-Lie superalgebra and* $[\cdot,\cdot]_t=[\cdot,\cdot]_0+t\varphi(\cdot,\cdot)$ *be an infinitesimal deformation of* $g=(V,[\cdot,\cdot]_0,\alpha)$. *Then* $\varphi\in Z^2(\mathfrak{g},\mathfrak{g})_{\bar{0}}$.

Proof. By (5.2) we have

$$\begin{aligned}(5.3)\qquad 0&=[\cdot,\cdot]_t\circ[\cdot,\cdot]_t\\ &=\circlearrowleft_{x,y,z}(-1)^{|x||z|}([\alpha(x),[y,z]_t]_0+t\varphi(\alpha(x),[y,z]_t))\\ &=\circlearrowleft_{x,y,z}(-1)^{|x||z|}[t([\alpha(x),\varphi(y,z)]_0+\varphi(\alpha(x),[y,z]_0))+t^2\varphi(\alpha(x),\varphi(y,z))].\end{aligned}$$

Note that

$$\circlearrowleft_{x,y,z}(-1)^{|x||z|}([\alpha(x),\varphi(y,z)]_0+\varphi(\alpha(x),[y,z]_0))=(-1)^{|x||z|}\delta^2\varphi.$$

Hence, $\varphi\in Z^2(\mathfrak{g},\mathfrak{g})_{\bar{0}}$. □

By (5.3), we can see that $[\cdot,\cdot]_t$ is an infinitesimal deformation if and only if $\varphi\circ\varphi=0$. Let $\mathfrak{g}_t=(V,[\cdot,\cdot]_t,\alpha)$ and $\mathfrak{g}'_t=(V,[\cdot,\cdot]'_t,\alpha')$ be two deformations of $\mathfrak{g}$, where $[\cdot,\cdot]_t=[\cdot,\cdot]_0+t\varphi(\cdot,\cdot)$ and $[\cdot,\cdot]'_t=[\cdot,\cdot]_0+t\psi(\cdot,\cdot)$. If there exists a linear automorphism $\Phi_t=\mathrm{id}+t\phi$, $\phi\in C^1_\alpha(\mathfrak{g},\mathfrak{g})_{\bar{0}}$ satisfying

$$\Phi_t([x,y]_t)=[\Phi_t(x),\Phi_t(y)]'_t\quad\forall\, x,y\in V,$$

we say that the deformations $\mathfrak{g}_t$ and $\mathfrak{g}'_t$ are equivalent. It is obvious that $\mathfrak{g}_t$ and $\mathfrak{g}'_t$ are equivalent if and only if $\varphi-\psi\in B^2(\mathfrak{g},\mathfrak{g})_{\bar{0}}$. Therefore, the set of infinitesimal deformations of $\mathfrak{g}$ can be parameterized by $H^2(\mathfrak{g},\mathfrak{g})_{\bar{0}}$. A deformation $\mathfrak{g}_t$ of Hom-Lie superalgebras $\mathfrak{g}$ is called *trivial* if it is equivalent to $\mathfrak{g}$.

Corollary 5.2. *All the infinitesimal deformations of the following Heisenberg Hom-Lie superalgebras are trivial:*

$$(1)\qquad \left(\mathfrak{h}_1,\begin{pmatrix}\mu_{11}\mu_{22}&0&0\\0&\mu_{11}&0\\0&&\mu_{22}\end{pmatrix}\right),\quad \mu_{11}\mu_{22}\neq0,$$

$$(2)\qquad \left(\mathfrak{h}_1,\begin{pmatrix}\mu_{12}\mu_{21}&0&0\\0&0&\mu_{12}\\0&\mu_{21}&0\end{pmatrix}\right),\quad \mu_{12}\mu_{21}\neq0,$$

$$(3)\qquad \left(\mathfrak{h}_2,\begin{pmatrix}\mu_0&0&0\\0&\mu_{11}&0\\0&0&\mu_0\mu_{11}\end{pmatrix}\right),\quad \mu_0({\mu_0}^2-1)\mu_{11}\neq0.$$

P r o o f. All the infinitesimal deformations of a Heisenberg Hom-Lie superalgebras are trivial if and only if $H^2(\mathfrak{g},\mathfrak{g})_{\bar 0} = 0$. □

In the following, we discuss the nontrivial infinitesimal deformations of Heisenberg Hom-Lie superalgebras. We will distinguish two separate cases: the ones that are also Lie superalgebras and those that are not.

We recall the classification of three-dimensional Lie superalgebras, see [11].

Theorem 5.3. *Let $L = (V, [\cdot,\cdot])$ be a Lie superalgebras with a direct sum decomposition $V = V_{\bar 0} \oplus V_{\bar 1}$, where $\dim V_{\bar 0} = 1$ and $\dim V_{\bar 1} = 2$. There are $e_1 \in V_{\bar 0}$ and $e_2, e_3 \in V_{\bar 1}$ such that $\{e_1 \mid e_2, e_3\}$ is a basis of V. Then L must be isomorphic to one of the following:*

$$\begin{aligned}
&L_1\colon [e_1, V_{\bar 1}] = 0, && [e_2, e_2] = e_1, && [e_3, e_3] = [e_2, e_3] = 0;\\
&L_2\colon [e_1, V_{\bar 1}] = 0, && [e_2, e_2] = [e_3, e_3] = e_1, && [e_2, e_3] = 0;\\
&L_3^{\lambda}\colon [e_1, e_2] = e_2, && [e_1, e_3] = \lambda e_3, && [V_{\bar 1}, V_{\bar 1}] = 0;\\
&L_4\colon [e_1, e_2] = e_2, && [e_1, e_3] = e_2 + e_3, && [V_{\bar 1}, V_{\bar 1}] = 0;\\
&L_5\colon [e_1, e_2] = 0, && [e_1, e_3] = e_2, && [V_{\bar 1}, V_{\bar 1}] = 0.
\end{aligned}$$

We construct a new Lie superalgebra ${L_2}' = (V, [\cdot,\cdot]_0)$. Let $\{h \mid v_1, v_2\}$ be a basis of V satisfying $[v_1, v_2]_0 = h$. There is an even bijective morphism $\phi\colon (V, [\cdot,\cdot]_0) \to L_2$

$$\phi(h) = e_1, \quad \phi(v_1) = e_2 + \mathrm{i}e_3, \quad \phi(v_2) = \frac{1}{2}e_2 - \frac{1}{2}\mathrm{i}e_3$$

such that $\phi([x, y]_0) = [\phi(x), \phi(y)]$ for all $x, y \in V$. Then L_2' is isomorphic to L_2 and we shall replace L_2 with it in Theorem 5.3.

Proposition 5.4. *A nontrivial infinitesimal deformation of Heisenberg Hom-Lie superalgebra $(\mathfrak{h}_1, \alpha)$, which is also a Lie superalgebra, is isomorphic to*

$$\left(L_2', \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}\right).$$

P r o o f. Denote by $(\mathfrak{h}_1, \alpha) = (V, [\cdot,\cdot]_0, \alpha)$. There is a basis $\{h \mid v_1, v_2\}$ such that $[v_1, v_2]_0 = h$ and others are zero. If $\alpha = \begin{pmatrix} \mu_{11}\mu_{22} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}$, $\mu_{11}\mu_{22} \neq 0$ or

$\alpha = \begin{pmatrix} \mu_{12}\mu_{21} & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & \mu_{21} & 0 \end{pmatrix}$, $\mu_{12}\mu_{21} \neq 0$, all the infinitesimal deformations of $(\mathfrak{g}_1, \alpha)$ are trivial.

Consider $\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}$. Let $\varphi = \begin{pmatrix} 0 & 0 & 0 & a_{14}\delta_{\mu_{12},0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{26}\delta_{\mu_{11},0} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ be an even 2-cocycle and $[\cdot,\cdot]_t = [\cdot,\cdot]_0 + t\varphi(\cdot,\cdot)$. Then $\varphi\circ\varphi = 0$ and $\mathfrak{g}_t = (V, [\cdot,\cdot]_t, \alpha')$ is an infinitesimal deformation of $(\mathfrak{h}_1, \alpha)$. Moreover, $\mathfrak{g}_t$ is a Lie superalgebra if and only if $[\cdot,\cdot]_t \circ [\cdot,\cdot]_t = 0$, i.e., $a_{26} = 0$. All deformations are given in Table 1. □

μ_{11}	μ_{12}	$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$\neq 0$	0	$[v_1, v_2]_t = h$ $[v_2, v_2]_t = a_{14}h$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2}a_{14} \\ 0 & 0 & 1 \end{pmatrix}$	$\left(L_2', \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_{11} & \mu_{12}' \\ 0 & 0 & 0 \end{pmatrix}\right)$
		$a_{14} \neq 0$	$a_{14} \neq 0$	${\mu_{12}}' = -\frac{1}{2}a_{14}\mu_{11}$

Table 1.

Proposition 5.5. *The nontrivial infinitesimal deformations of* $(\mathfrak{h}_2, \alpha)$*, which are also Lie superalgebras, are isomorphic to:*

(1a)
$$\left(L_3^\lambda, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right), \quad (\mu_0 - 1)\mu_{11} = 0,$$

(1b)
$$\left(L_3^0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mu_{11} & -\mu_{11} \\ 0 & 0 & -2\mu_{11} \end{pmatrix}\right),$$

(1c)
$$\left(L_3^{-1}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\mu_{11} & \frac{1}{2}\xi\mu_{11} \\ 0 & \frac{1}{2}\xi^{-1}\mu_{11} & \frac{1}{2}\mu_{11} \end{pmatrix}\right)$$

for $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix}$, $\mu_0({\mu_0}^2 - 1)\mu_{11} = 0.$

(2a)
$$\left(L_3^0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta \end{pmatrix}\right),$$

(2b) $$\left(L_4, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right),$$

(2c) $$\left(L_3^{\mu_0}, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & 0 & {\mu_0}^{-1} \\ 0 & 0 & 0 \end{pmatrix}\right), \quad \mu_0 \neq 0, 1$$

for $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}$.

P r o o f. Denote by $(\mathfrak{h}_2, \alpha) = (V, [\cdot,\cdot]_0, \alpha)$. There is a basis $\{u \mid v, h\}$ such that $[u, v]_0 = h$ and others are zero.

Consider $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_0\mu_{11} \end{pmatrix}$, $\mu_0({\mu_0}^2 - 1)\mu_{11} = 0$. Let $\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{E} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{F} \end{pmatrix}$, where $\mathcal{E} = a_{26}\delta_{({\mu_0}^2-1)\mu_{11},0}$, $\mathcal{F} = a_{36}\delta_{\mu_0(\mu_0-1)\mu_{11},0}$, be an even 2-cocycle. Then $\varphi \circ \varphi = 0$ and we obtain an infinitesimal deformation $(V, [\cdot,\cdot]_t, \alpha')$ where $[\cdot,\cdot]_t = [\cdot,\cdot]_0 + t\varphi(\cdot,\cdot)$. It is easy to see that it is also a Lie superalgebra. We analyze the cases (1a) $(\mu_0 - 1)\mu_{11} = 0$, (1b) $\mu_0 = -1$ and (1c) $\mu_0 = 0$, which are given in Tables 2a–2c, separately.

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = h$ $[u,h]_t = a_{26}v + a_{36}h$ $a_{26}a_{36} \neq 0$	$\begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & \tau a_{26}^{-1} & -\tau k_2 \\ 0 & -\tau a_{36}^{-1} & \tau k_1 \end{pmatrix}$ $\tau = (k_1 - k_2)^{-1}$ $k_1 \neq k_2, k_1k_2 = -a_{26}^{-1}$ $k_1 + k_2 = -a_{36}a_{26}^{-1}$	$\left(L_3^{\lambda}, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right)$ $\lambda = k_1k_2^{-1}, \lambda \neq 0, -1$ $(\mu_0 - 1)\mu_{11} = 0$
$[u,v]_t = h$ $[u,v]_t = a_{36}h$ $a_{36} \neq 0$	$\begin{pmatrix} a_{36} & 0 & 0 \\ 0 & a_{36}^{-1} & 1 \\ 0 & a_{36}^{-1} & 0 \end{pmatrix}$ $a_{36} \neq 0$	$\left(L_3^{0}, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right)$ $(\mu_0 - 1)\mu_{11} = 0$
$[u,v]_t = h$ $[u,v]_t = a_{26}$ $a_{26} \neq 0$	$\begin{pmatrix} a_{26}^{1/2} & 0 & 0 \\ 0 & \kappa & \kappa a_{26}^{3/2} \\ 0 & -\kappa a_{26}^{1/2} & \kappa a_{26} \end{pmatrix}$ $\kappa = (1 + a_{26})^{-1}$ $a_{26} \neq 0$	$\left(L_3^{-1}, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right)$ $(\mu_0 - 1)\mu_{11} = 0$

Table 2a. The case (1a) $(\mu_0 - 1)\mu_{11} = 0$.

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = h$ $[u,v]_t = a_{36}h$	$\begin{pmatrix} a_{36} & 0 & 0 \\ 0 & {a_{36}}^{-1} & 1 \\ 0 & {a_{36}}^{-1} & 0 \end{pmatrix}$	$\left(L_3^0, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mu_{11} & \mu_{11} \\ 0 & 0 & 2\mu_{11} \end{pmatrix}\right)$
$a_{36} \neq 0$	$a_{36} \neq 0$	

Table 2b. The case (1b) $\mu_0 = -1$.

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = h$ $[u,v]_t = a_{26}$	$\begin{pmatrix} a_{26}^{1/2} & 0 & 0 \\ 0 & \varrho & \varrho {a_{26}}^{3/2} \\ 0 & -\varrho {a_{26}}^{1/2} & \varrho a_{26} \end{pmatrix}$	$\left(L_3^0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\mu_{11} & \frac{1}{2}\xi\mu_{11} \\ 0 & \frac{1}{2}\xi^{-1}\mu_{11} & \frac{1}{2}\mu_{11} \end{pmatrix}\right)$
$a_{26} \neq 0$	$\varrho = (1 + a_{26})^{-1}$ $a_{26} \neq 0$	$\xi = -{a_{26}^{-1/2}}$

Table 2c. The case (1c) $\mu_0 = 0$.

Consider $\alpha = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 1 & \mu_{11} \end{pmatrix}$, $(\mu_0 - 1)\mu_{11} = 0$. Let $\varphi = \begin{pmatrix} 0 & \mathcal{G} & \mathcal{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0 a_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{36} \end{pmatrix}$, where $\mathcal{G} = a_{12}\delta_{\mu_0,0}\delta_{\mu_{11},0}$, $\mathcal{H} = a_{13}\delta_{\mu_0,0}\delta_{\mu_{11},0}$, be an even 2-cocycle. Then $(V, [\cdot,\cdot]_t, \alpha')$, where $[\cdot,\cdot]_t = [\cdot,\cdot]_0 + t\varphi(\cdot,\cdot)$, if and only if $\varphi \circ \varphi = 0$, is an infinitesimal deformation. We analyze the cases (2a) $\mu_0 = \mu_{11} = 0$, (2b) $\mu_0 = 1$ and (2c) $\mu_0 \neq 0, 1$, $\mu_{11} = 0$, separately.

For case (2a), $\varphi \circ \varphi = 0$ implies $a_{12} = a_{13} = 0$ or $a_{36} = 0$. Furthermore, if $a_{12} = a_{13} = 0$, the deformation $\mathfrak{g}_t$ is also a Lie superalgebra.

For cases (2b) and (2c), $\mathfrak{g}_t$ is an infinitesimal deformation for all φ. The deformation is also a Lie superalgebra if $a_{12} = a_{13} = 0$.

The deformations of cases (2a), (2b) and (2c) are given in Tables 3a–3c.

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = h$ $[u,v]_t = a_{36}h$	$\begin{pmatrix} a_{36} & 0 & 0 \\ 0 & 0 & a_{36}^{-1} \\ 0 & 1 & a_{36}^{-1} \end{pmatrix}$	$\left(L_3^0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}\right)$
$a_{36} \neq 0$	$a_{36} \neq 0$	$\mu_{22} = a_{36}$

Table 3a. The case (2a): $\mu_0 = \mu_{11} = 0$

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = a_{36}v + h$ $[u,v]_t = a_{36}h$	$\begin{pmatrix} a_{36} & 0 & 0 \\ 0 & 0 & a_{36} \\ 0 & 1 & 0 \end{pmatrix}$	$\left(L_4, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{11} \end{pmatrix}\right)$
$a_{36} \neq 0$	$a_{36} \neq 0$	

Table 3b. The case (2b): $\mu_0 = 1$

$[\cdot,\cdot]_t$	base change	Hom-Lie superalgebra
$[u,v]_t = \mu_0 a_{36}v + h$ $[u,v]_t = a_{36}$	$\begin{pmatrix} a_{36} & 0 & 0 \\ 0 & (\mu_0 - 1)a_{36} & \mu_0^{-1} \\ 0 & 0 & 0 \end{pmatrix}$	$\left(L_3^{\lambda}, \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & 0 & \mu_0^{-1} \\ 0 & 0 & 0 \end{pmatrix}\right)$
$a_{36} \neq 0$	$a_{36} \neq 0$	$\lambda \neq 0, 1$

Table 3c. The case (2c): $\mu_0 \neq 0, 1$, $\mu_{11} = 0$

□

Propositions 5.4 and 5.5 give the infinitesimal deformations of Heisenberg Hom-Lie superalgebras that are also Lie superalgebras. Before discussing the rest deformations, we will recall some multiplicative Hom-Lie superalgebras and those can be find in the classification of multiplicative Hom-Lie superalgebras of [10]. Let V be a superspace with a direct sum decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$, $[\cdot,\cdot]$ be an even bilinear map and σ be an even linear map on V. Let $\{e_1 \mid e_2, e_3\}$ be a basis of V. The following are three Hom-Lie superalgebras on V:

$$L_{1,2}^{43,a}\colon\ [e_1,e_2]=0,\ [e_1,e_3]=\beta e_2,\ [e_2,e_2]=0,\ [e_2,e_3]=0,\ [e_3,e_3]=\gamma e_1,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0,$$
$$L_{1,2}^{45,a}\colon\ [e_1,e_2]=0,\ [e_1,e_3]=\beta e_2,\ [e_2,e_2]=0,\ [e_2,e_3]=\nu e_1,\ [e_3,e_3]=\gamma e_1,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0,$$
$$L_{1,2}^{46,a,b}\colon\ [e_1,e_2]=0,\ [e_1,e_3]=\beta e_2,\ [e_2,e_2]=0,\ [e_2,e_3]=\mu e_1,\ [e_3,e_3]=0,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & b \end{pmatrix}, \quad a^2 + b^2 \neq 0.$$

Propositions 5.6 and 5.7 will characterize the infinitesimal deformations of Heisenberg Hom-Lie superalgebras that are not Lie superalgebras.

Proposition 5.6. *An infinitesimal deformation of Heisenberg Hom-Lie superalgebra* $(\mathfrak{h}_1, \alpha)$, *which is not a Lie superalgebra, is isomorphic to* $L^{46,a,0}_{1,2}$, $a \neq 0$.

Proof. By the proof of Proposition 5.4, we know that $(\mathfrak{h}_1, \alpha)$ have an infinitesimal deformation (not a Lie superalgebra) if and only if

$$\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}, \ \mu_{12} \neq 0, \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ a_{26} \neq 0.$$

There is a basis $\{h \mid v_1, v_2\}$ of V such that $[v_1, v_2]_0 = h$. Therefore $\mathfrak{g}_t = (V, [\cdot,\cdot]_t, \alpha')$ is an infinitesimal deformation, where $[\cdot,\cdot]_t = [\cdot,\cdot]_0 + t\varphi$ and $[v_1, v_2]_t = h$, $[h, v_2]_t = a_{26} v_1$, $a_{26} \neq 0$. Then the deformation $\mathfrak{g}_t$ is isomorphic to

$$L^{46,\mu_{12},0}_{1,2}\colon\ [e_1,e_2]=0,\ [e_1,e_3]=a_{26}e_2,\ [e_2,e_2]=0,\ [e_2,e_3]=e_1,\ [e_3,e_3]=0,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu_{12} \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Proposition 5.7. *An infinitesimal deformation of* $(\mathfrak{h}_2, \alpha)$, *which is not a Lie superalgebra, is isomorphic to* $L^{43,1}_{1,2}$, $L^{45,1}_{1,2}$, *or* $L^{46,1,0}_{1,2}$.

Proof. By the proof of Proposition 5.5, Heisenberg Hom-Lie superalgebra $(\mathfrak{h}_2, \alpha)$ has infinitesimal deformations if and only if

$$\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is a basis $\{u \mid v, h\}$ of V such that $[u, v]_0 = h$. Therefore $\mathfrak{g}_t = (V, [\cdot,\cdot]_t, \alpha')$ is an infinitesimal deformation, where $[\cdot,\cdot]_t = [\cdot,\cdot]_0 + t\varphi$ and $[u, v]_t = h$, $[v, v]_t = a_{12} u$, $[v, h]_t = a_{13} u$. We analyze in three cases.

(a) If $a_{12} \neq 0$ and $a_{13} \neq 0$, the infinitesimal deformation $\mathfrak{g}_t$ is isomorphic to

$$L^{45,1}_{1,2}\colon\ [e_1,e_2]=0,\ [e_1,e_3]=e_2,\ [e_2,e_2]=0,\ [e_2,e_3]=a_{13}e_1,\ [e_3,e_3]=a_{12}e_3,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) If $a_{12} = 0$ and $a_{13} \neq 0$, $\mathfrak{g}_t$ is isomorphic to

$$L^{46,1,0}_{1,2}\colon\ [e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_2] = 0,\ [e_2, e_3] = a_{13}e_1,\ [e_3, e_3] = 0,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) If $a_{12} \neq 0$ and $a_{13} = 0$, $\mathfrak{g}_t$ is isomorphic to

$$L^{43,1}_{1,2}\colon\ [e_1, e_2] = 0,\ [e_1, e_3] = e_2,\ [e_2, e_2] = 0,\ [e_2, e_3] = 0,\ [e_3, e_3] = a_{12}e_1,$$
$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

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