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# $H^2$ CONVERGENCE OF SOLUTIONS OF A BIHARMONIC PROBLEM ON A TRUNCATED CONVEX SECTOR NEAR THE ANGLE $\pi$

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*Abstract.* We consider a biharmonic problem $\Delta^2 u_\omega = f_\omega$ with Navier type boundary conditions $u_\omega = \Delta u_\omega = 0$, on a family of truncated sectors $\Omega_\omega$ in $\mathbb{R}^2$ of radius $r$, $0 < r < 1$ and opening angle $\omega$, $\omega \in (2\pi/3, \pi]$ when $\omega$ is close to $\pi$. The family of right-hand sides $(f_\omega)_{\omega \in (2\pi/3,\pi]}$ is assumed to depend smoothly on $\omega$ in $L^2(\Omega_\omega)$. The main result is that $u_\omega$ converges to $u_\pi$ when $\omega \to \pi$ with respect to the $H^2$-norm. We can also show that the $H^2$-topology is optimal for such a convergence result.

*Keywords*: sector; convex; biharmonic; elliptic; singularity; convergence; Sobolev space

*MSC 2020*: 35J25, 35J40, 35J75, 35B45, 35Q99, 35B40

## 1. Introduction

We are interested in a family of Navier boundary value problems of the following type. For $\Omega_\omega$ an open polygonal convex set in $\mathbb{R}^2$ and $f_\omega \in L^2(\Omega_\omega)$ given, we search a (unique) solution $u_\omega \colon \overline{\Omega_\omega} \to \mathbb{R}$ to the problem

$$(P_\omega) \qquad \begin{cases} \Delta^2 u_\omega = f_\omega & \text{in } \Omega_\omega, \\ u = \Delta u_\omega = 0 & \text{on } \partial\Omega_\omega. \end{cases}$$

The proposed problem may describe the behaviour of a hinged plate coming from linear elasticity in planar domains with corner type singularities. The question of existence, uniqueness and regularity of the solution of such problems with different boundary conditions has been addressed by many authors in the literature,

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cf. e.g. Kondratiev [9], Blum and Rannacher [1], Grisvard ([7], [8]), Maz'ya ([11], [10], [12]), Nicaise ([14], [15], [13]), Dauge ([4], [5], [2]), [3], Stylianou [16], Tami ([17], [18]) and others.

Thanks to localization techniques similar to Chapter 7 of [8] or Chapter 2 of [17], one can assume that the given open polygonal set $\Omega_\omega$ is a conic sector of $B_1(0)$ with opening angle $\omega \in (2\pi/3, \pi]$, where $\omega$ is defined uniquely up to rotation. The solution $u_\omega$ of problem $(P_\omega)$ in $\Omega_\omega$ associated to a right-hand side $f_\omega \in L^2(\Omega_\omega)$ exhibits a singularity at the origin 0 whose effect is to limit the regularity of $u_\omega$, expressed in the scale of Sobolev spaces $H^\sigma(\Omega_\omega)$ of order $\sigma < 1 + \pi/\omega$. In contrast, when $\omega = \pi$, the solution $u_\pi$ belongs to $H^4(\Omega_\pi)$. Therefore, there is a jump in Sobolev exponents describing the regularity of solution when $\omega \to \pi$ on the side $\omega < \pi$, i.e. maintaining $\Omega_\omega$ convex at 0.

**Assumption 1.1.** The family $(f_\omega)_{\omega\in(2\pi/3,\pi]}$ is assumed to satisfy the following convergence relation:

$$\lim_{\omega\to\pi} \|f_\omega - f_\pi\|_{L^2(\Omega_\omega)} = 0.$$

The main goal of this paper is to prove under Assumption 1.1 and the convexity of $\Omega_\omega$ the convergence in $H^2$-norm of $u_\omega$ to $u_\pi$ as $\omega$ tends to $\pi$ in the sense that

$$\lim_{\substack{\omega\to\pi\\ <}} \|u_\omega - u_\pi\|_{H^2(\Omega_\omega)} = 0.$$

Since $1 + \pi/\omega$ tends to 2 for $\omega$ close to $\pi$, this topology is expected to be the best possible in that sense. To the best of our knowledge, very few authors have addressed this question in the biharmonic case, while most other works in the literature only consider second order problems.

Recently, in [18], we described the singularities of solutions of such problems locally in the vicinity of the corner and we gave uniform estimates with respect to the angle parameter $\omega$ close to $\pi$ which is analogous to a Taylor expansion of $u_\omega$ near 0 that converges to the Taylor expansion of $u_\pi$. If $\omega < \pi$, it is known cf. [18], [17], [1] that the solution of this problem $u_\omega$ decomposes in the vicinity of the origin as $u_\omega = u_{1,\omega} + u_{2,\omega} + u_{3,\omega}$, where $u_{1,\omega}$, $u_{2,\omega}$ are the singular parts of $u_\omega$ and $u_{3,\omega}$ is the regular part. More precisely, in the vicinity of the origin, $u_\omega \in H^\sigma(\Omega_\omega)$ for all $\sigma < 1 + \pi/\omega$, while for $\omega = \pi$, the solution $u_\pi$ enjoys $H^4$ regularity on $\Omega_\pi$.

An unpublished version of the result presented here, the proof of the $H^2$ convergence, was completely achieved *via* a Fourier series technique in Chapter 4 of [17] with the inconvenience of being tedious and quite technical. In addition, the method was based on explicit estimates of Fourier coefficients of the solutions, hence making it specific to the biharmonic operator. In the present paper, we give a different and simpler version of the proof which is possibly applicable to other elliptic operators.

The main tools are based on explicit estimates of the trace $h_\omega$ of $u_\pi$ on the boundary $\partial\Omega_\omega$ which allows one to construct a suitable extension with desirable properties in order to compare $u_\pi$ and $u_\omega$ on the whole of $\Omega_\omega$. The convergence result follows thanks to the control of the maximal regularity of the Laplacian on convex domains.

The paper is organized as follows: In the second section, a problem setting is presented on a truncated sector configuration with Navier type boundary conditions. There, some definitions and a reasonable set of preliminary lemmas are given in order to characterize some useful properties of traces and extension operators on radial boundaries of a sector with explicit estimates with respect to to the opening angle $\omega$ close to $\pi$. Moreover, the extension operator is given explicitly in a very simple way, provided some extra regularity on the traces. The third section is the main one, where we give the proof of convergence theorem in the best expected topological space, based on an additional lemma which describes the behaviour of solutions $u_\omega$ at a nearly flat boundary with respect to the energy norm, highlighting that this method takes partial advantage of the extra $H^4$ regularity of the solution $u_\pi$ on the regular domain $\Omega_\pi$. Concluding remarks and future works are discussed in the last section.

## 2. Problem setting and preliminary results

Let $(\Omega_\omega)_{\omega\in(2\pi/3,\pi]}$ denote a family of conic sectors of $B_1(0)$ in $\mathbb{R}^2$ with an opening angle $\omega$, and define the family of problems $(P_\omega)$ where $f_\omega \in L^2(\Omega_\omega)$ is assumed to depend continuously on $\omega$ according to Assumption 1.1. Since Navier boundary conditions are considered in a convex (Lipschitz) domain, it should be mentioned that the notion of a solution to problem $(P_\omega)$ is understood in the sense of Chapter 2.3 of [17] or equivalently Chapter 2.7 of [6] by solving a system of two Poisson's equations.

In polar coordinates $(x = r\cos\theta, y = r\sin\theta)$, cf. Figure 1, we will use the following notation such that $\partial\Omega_\omega = \overline{\Gamma^+} \cup \overline{C_\omega} \cup \overline{\Gamma^-_\omega}$ designates the closed boundary of $\Omega_\omega$,

$$\begin{aligned} \Omega_\omega &= \{(x,y);\ 0<r<1,\, 0<\theta<\omega\}, & \Gamma^+ &= \{(x,y);\ 0<r<1,\, \theta=0\},\\ C_\omega &= \{(x,y);\ r=1,\, 0<\theta<\omega\}, & \Gamma^-_\omega &= \{(x,y);\ 0<r<1,\, \theta=\omega\}. \end{aligned}$$

Trace spaces such as $H^s(\Gamma^-_\omega)$ are defined straightforwardly as follows. Let us denote by $\delta_\omega := (\cos\omega, \sin\omega)$ the unit vector on $\Gamma^-_\omega$ as in Figure 1 and for any $\phi \in \mathcal{D}'(\Gamma^-_\omega)$ define $\bar\phi \in \mathcal{D}'(0,1)$ by the relation $\bar\phi(r) := \phi(r\delta_\omega)$ for all $r \in (0,1)$. Thus, for any real $s$, we set

$$H^s(\Gamma^-_\omega) := \{\phi \in \mathcal{D}'(\Gamma^-_\omega);\ \bar\phi \in H^s(0,1)\},$$

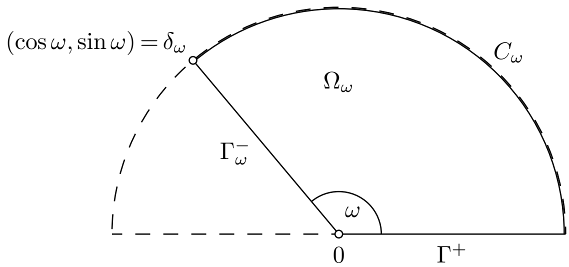


Figure 1. The sector $\Omega_\omega$, $2\pi/3 < \omega \leqslant \pi$.

endowed with norm $\|\phi\|_{H^s(\Gamma_\omega^-)} := \|\bar{\phi}\|_{H^s(0,1)}$. In particular, integer order spaces such as $H^m(\Gamma_\omega^-)$ can be defined straightforwardly, i.e., a function $\phi$ defined on $\Gamma_\omega^-$ belongs to $H^m(\Gamma_\omega^-)$ if and only if $\phi$ and all its tangential derivatives on $\Gamma_\omega^-$ up to order $m$ are in $L^2(\Gamma_\omega^-)$.

Since the trace on $\Gamma_\omega^-$ of a function in $H_0^1(\Omega_\pi)$ is not necessarily zero, we will need to define the following space:

$$H^1_{0,\Gamma_\omega^-}(\Omega_\omega) := \Big\{\phi \in H^1(\Omega_\omega);\ \phi|_{\partial\Omega_\omega\setminus\overline{\Gamma_\omega^-}} = 0\Big\},$$

which can be identified with the space of restrictions to $\Omega_\omega$ of functions in $H_0^1(\Omega_\pi)$. In particular, we will show that the semi-norm defined by $\|\nabla^2 \cdot\|_{L^2(\Omega_\omega)}$ ($\nabla^2$ denotes the Hessian matrix) is a norm on $H^2(\Omega_\omega) \cap H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$ equivalent to the norm $H^2(\Omega_\omega)$ with explicit control on the constant $C(\omega)$ with respect to $\omega \in (0, 2\pi]$. More interestingly, it will follow in the convex case $\omega \in (0,\pi]$ cf. Chapter 2.3 of [16] that the Laplacian of a function is a norm on the space $H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ with namely the same explicit control on the fundamental norm inequality for the Laplacian with respect to $\omega \in (0,\pi]$.

In polar coordinates, we will denote by $\overline{G}(r,\theta) := G(r\cos\theta, r\sin\theta)$, recalling that $H^m(\Omega_\omega, r\,\mathrm{d}r\,\mathrm{d}\theta)$ is defined as the space of functions $\overline{G} \in L^2(\Omega_\omega, r\,\mathrm{d}r\,\mathrm{d}\theta)$ such that

$$\frac{1}{r^{k_2}}\frac{\partial^{k_1+k_2}\overline{G}}{\partial r^{k_1}\partial\theta^{k_2}} \in L^2(\Omega_\omega, r\,\mathrm{d}r\,\mathrm{d}\theta) \quad\text{and}\quad \frac{\partial^{k_1}}{\partial r^{k_1}}\Big(\frac{1}{r^{k_2}}\frac{\partial^{k_2}\overline{G}}{\partial\theta^{k_2}}\Big) \in L^2(\Omega_\omega, r\,\mathrm{d}r\,\mathrm{d}\theta)$$

for all $k_1$, $k_2$ satisfying $0 \leqslant k_1 + k_2 \leqslant m$. It follows that functions $G \in H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$ satisfy in this coordinate system $\overline{G} \in H^1(\Omega_\omega, r\,\mathrm{d}r\,\mathrm{d}\theta)$ and $\overline{G}(r,0) = \overline{G}(1,\theta) = 0$ for all $r \in (0,1]$ and $\theta \in (0,\omega)$.

**Lemma 2.1.** *Let $\omega \in (0,2\pi]$. Then for all $u \in H^2(\Omega_\omega) \cap H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$,*

$$\|u\|_{H^2(\Omega_\omega)} \leqslant C(\omega)\|\nabla^2 u\|_{L^2(\Omega_\omega)}. \tag{2.1}$$

*If $\omega \in (0, \pi]$, then for all $u \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$,*

$$\|u\|_{H^2(\Omega_\omega)} \leqslant C(\omega)\|\Delta u\|_{L^2(\Omega_\omega)}, \tag{2.2}$$

*with $C(\omega) = \sqrt{1 + (1 + \omega)^2}$.*

P r o o f of Lemma 2.1. Let $u \in H^2(\Omega_\omega) \cap H^1_{0,\Gamma^-_\omega}(\Omega_\omega)$. Then in polar coordinates the trace $\bar{u}(\cdot, 0)$ lies in $H^{3/2}(0,1) \hookrightarrow \mathcal{C}^1([0,1])$ and satisfies, for all $r \in (0,1)$,

$$\bar{u}(r,0) = \frac{\partial \bar{u}}{\partial r}(r,0) = 0.$$

Hence,

$$\bar{u}(r,\theta) = \int_0^\theta \frac{\partial \bar{u}}{\partial \alpha}(r,\alpha)\,\mathrm{d}\alpha,$$

$$\frac{\partial \bar{u}}{\partial r}(r,\theta) = \int_0^\theta \frac{\partial^2 \bar{u}}{\partial \alpha \partial r}(r,\alpha)\,\mathrm{d}\alpha,$$

and with the help of Cauchy-Schwarz inequality one deduces the following inequalities:

$$\int_0^\omega \int_0^1 |\bar{u}(r,\theta)|^2 r\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \omega \int_0^\omega \int_0^1 \Big|\frac{\partial \bar{u}}{\partial \theta}(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{2.3}$$

$$\int_0^\omega \int_0^1 \Big|\frac{\partial \bar{u}}{\partial r}(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \omega \int_0^\omega \int_0^1 \Big|\frac{\partial^2 \bar{u}}{\partial \theta \partial r}(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.4}$$

The first inequality (2.3) yields the classical Poincaré's inequality

$$\|u\|^2_{L^2(\Omega_\omega)} \leqslant \omega \|\nabla u\|^2_{L^2(\Omega_\omega)}. \tag{2.5}$$

On the other hand, the trace $\bar{u}(1, \cdot)$ lies in $H^{3/2}(0,\omega) \hookrightarrow \mathcal{C}^1([0,\omega])$ and satisfies for all $\theta \in (0,\omega)$,

$$\bar{u}(1,\theta) = \frac{\partial \bar{u}}{\partial \theta}(1,\theta) = 0.$$

It follows that

$$\frac{1}{r}\frac{\partial \bar{u}}{\partial \theta}(r,\theta) = -\int_r^1 \frac{\partial}{\partial r'}\Big(\frac{1}{r'}\frac{\partial \bar{u}}{\partial \theta}\Big)(r',\theta)\,\mathrm{d}r',$$

and by the same argument of the Cauchy-Schwarz inequality one obtains

$$\int_0^\omega \int_0^1 \Big|\frac{1}{r}\frac{\partial \bar{u}}{\partial \theta}(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \int_0^\omega \int_0^1 \Big|\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial \bar{u}}{\partial \theta}\Big)(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.6}$$

Taking the sum of (2.4) and (2.6), one obtains the second Poincaré's inequality of higher order type

$$\|\nabla u\|^2_{L^2(\Omega_\omega)} \leqslant (\omega+1)\|\nabla^2 u\|^2_{L^2(\Omega_\omega)}. \tag{2.7}$$

It follows from inequalities (2.5) and (2.7) that

$$\begin{aligned}\|u\|^2_{H^2(\Omega_\omega)} &= \|u\|^2_{L^2(\Omega_\omega)} + \|\nabla u\|^2_{L^2(\Omega_\omega)} + \|\nabla^2 u\|^2_{L^2(\Omega_\omega)}\\ &\leqslant (1+(1+\omega)^2)\|\nabla^2 u\|^2_{L^2(\Omega_\omega)},\end{aligned} \tag{2.8}$$

for all $u \in H^2(\Omega_\omega) \cap H^1_{0,\Gamma^-_\omega}(\Omega_\omega)$, which gives inequality (3.3).

Finally, assume $u \in H^2(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ and $\omega \in (0, \pi]$. Then the Laplacian inequality (3.4) follows immediately, thanks to the fundamental convexity estimate for the Laplacian (operator) $\|\nabla^2 u\|_{L^2(\Omega_\omega)} \leqslant \|\Delta u\|_{L^2(\Omega_\omega)}$, cf. Chapter 2.3 of [16], where the author's proof was based on convexity, integration by parts in $H^3(\Omega_\omega) \cap H^1_0(\Omega_\omega)$ and a density argument. The proof of the lemma is finished. $\square$

Remark 2.1. Estimate (2.5) is the classical Poincaré's inequality obtained using only the fact that $u \in H^1(\Omega_\omega)$ with the boundary condition $u = 0$ on $\Gamma^+$. The higher order inequality (2.7) needs more regularity and a Dirichlet boundary condition in another direction as $C_\omega$.

**Lemma 2.2.** *Let $h_\omega := \mathrm{Tr}(z_\pi)$ on $\Gamma^-_\omega$ where $z_\pi \in H^{m+1}(\Omega_\pi) \cap H^1_0(\Omega_\pi)$, $m \geqslant 0$, then for any integer $k$, $0 \leqslant k \leqslant m$,*

$$\int_0^1 r\Big|\frac{\mathrm{d}^k}{\mathrm{d}r^k}\frac{h_\omega(r\delta_\omega)}{r}\Big|^2\,\mathrm{d}r + \int_0^1 \frac{1}{r}\Big|\frac{\mathrm{d}^k}{\mathrm{d}r^k}h_\omega(r\delta_\omega)\Big|^2\,\mathrm{d}r \leqslant (\pi-\omega)\|z_\pi\|^2_{H^{m+1}(\Omega_\pi)}. \tag{2.9}$$

*In particular, one has*

$$\lim_{\omega\to\pi}\|h_\omega\|_{H^m(\Gamma^-_\omega)} = 0. \tag{2.10}$$

*Moreover, if $m \geqslant 1$, then $h_\omega \in \mathcal{C}^{m-1}(\overline{\Gamma^-_\omega})$ and $h_\omega(0) = h_\omega(\delta_\omega) = 0$.*

Proof of Lemma 2.2. Taking tangential derivatives up to order $m$ along $\Gamma^-_\omega$, we obtain in polar coordinates $(r, \theta)$, for any $k$, $0 \leqslant k \leqslant m$,

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}h_\omega(r\delta_\omega) = \frac{\partial^k}{\partial r^k}\overline{z}_\pi(r,\omega) = -\int_\omega^\pi \frac{\partial^{k+1}}{\partial r^k\partial\theta}\overline{z}_\pi(r,\theta)\,\mathrm{d}\theta,$$

since $z_\pi \in H^{m+1}(\Omega_\pi) \cap H^1_0(\Omega_\pi)$ implies that all radial derivatives of $\overline{z}_\pi(r,\theta)$ are zero at $\theta = \pi$. Thus, thanks to the Cauchy-Schwarz inequality and the fact that $z_\pi \in H^{m+1}(\Omega_\pi)$, for all $k \leqslant m$ we obtain the first estimate

$$\int_0^1 \frac{1}{r}\Big|\frac{\partial^k}{\partial r^k}\overline{z}_\pi(r,\omega)\Big|^2 \,\mathrm{d}r \leqslant (\pi-\omega)\int_0^1\int_0^\pi \Big|\frac{1}{r}\frac{\partial^{k+1}}{\partial r^k \partial\theta}\overline{z}_\pi(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.11}$$

Similarly, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}\frac{h_\omega(r\delta_\omega)}{r} = \frac{\partial^k}{\partial r^k}\frac{\overline{z}_\pi(r,\omega)}{r} = -\int_\omega^\pi \frac{\partial^k}{\partial r^k}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\overline{z}_\pi(r,\theta)\,\mathrm{d}\theta,$$

and by the same arguments above one obtains for all $k \leqslant m$ the second estimate

$$\int_0^1 r\Big|\frac{\partial^k}{\partial r^k}\frac{\overline{z}_\pi}{r}(r,\omega)\Big|^2 \,\mathrm{d}r \leqslant (\pi-\omega)\int_0^1\int_0^\pi \Big|\frac{\partial^k}{\partial r^k}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\overline{z}_\pi(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.12}$$

Then relation (2.9) follows by summing inequalities (2.11) and (2.12). Henceforth, for all $k$, $0 \leqslant k \leqslant m$,

$$\lim_{\omega\to\pi}\Big\|\frac{\mathrm{d}^k}{\mathrm{d}r^k}h_\omega(r\delta_\omega)\Big\|^2_{L^2(\Gamma^-_\omega)} = 0,$$

i.e. (2.10) holds.

The last assertion is a direct consequence of Sobolev's embedding theorem $H^m(\Gamma^-_\omega) \hookrightarrow \mathcal{C}^{m-1}(\overline{\Gamma^-_\omega})$ with $m \geqslant 1$ and the fact that $z_\pi \in H^{m+1}(\Omega_\pi) \cap H^1_0(\Omega_\pi)$ which yields $z_\pi \in \mathcal{C}^0(\overline{\Omega_\pi})$, in particular $h_\omega(0) = \overline{z}_\pi(0,0) = 0$ and $h_\omega(\delta_\omega) = \overline{z}_\pi(1,\omega) = 0$. The proof of the lemma is finished. $\square$

The following lemma gives a weaker version of the existence of extension operators for traces on $\Gamma^-_\omega$ of functions at least $H^2(\Omega_\pi) \cap H^1_0(\Omega_\pi)$. The interest in such a lemma lies in the fact that the solution $u_\pi$ of our problem in $\Omega_\pi$ has an extra regularity of $H^4$, keeping it useful and powerful in the proof of the main result.

**Lemma 2.3.** *Let $\omega \in (0,\pi]$, if $g_\omega = \mathrm{Tr}(z_\pi)$ on $\Gamma^-_\omega$ and $z_\pi \in H^{m+1}(\Omega_\pi) \cap H^1_0(\Omega_\pi)$, $m \geqslant 1$, then the function $G_\omega$ (extension of $g_\omega$ to $\Omega_\omega$) defined in polar coordinates by*

$$\overline{G}_\omega(r,\theta) := \overline{g}_\omega(r)\frac{\theta}{\omega}$$

*has the following properties:*

(i) $\mathrm{Tr}(G_\omega) = g_\omega$ *on* $\Gamma^-_\omega$ *and* $G_\omega \in H^m(\Omega_\omega) \cap H^1_{0,\Gamma^-_\omega}(\Omega_\omega)$.

(ii) *There exists a constant $C(m) > 0$ dependent only on $m$, such that*

$$\|G_\omega\|^2_{H^m(\Omega_\omega)} \leqslant C(m)\Big(\omega\|g_\omega\|^2_{H^m(\Gamma^-_\omega)} + \frac{\pi-\omega}{\omega}\|z_\pi\|^2_{H^{m+1}(\Omega_\pi)}\Big), \tag{2.13}$$

$$\lim_{\substack{\omega\to\pi \\ <}}\|G_\omega\|_{H^m(\Omega_\omega)} = 0. \tag{2.14}$$

P r o o f of Lemma 2.3. By definition of $G_\omega$ and using Lemma 2.2 with $m \geqslant 1$, we obtain $g_\omega \in H^1(\Gamma_\omega^-) \hookrightarrow \mathcal{C}^0(\overline{\Gamma_\omega^-})$ and evidently $\overline{G}_\omega(r,\omega) = \overline{g}_\omega(r)$, $\overline{G}_\omega(r,0) = 0$ for all $r \in (0,1]$ and $\overline{G}_\omega(1,\theta) = 0$ for all $\theta \in (0,\omega)$; hence $G_\omega \in H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$ which finishes the proof of (i).

We also have, for any $k$, $0 \leqslant k \leqslant m$, where $\theta^2/\omega^2$ is estimated by 1,

$$\int_0^\omega \int_0^1 \Big|\frac{\partial^k}{\partial r^k}\overline{G}_\omega(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta \leqslant \omega \int_0^1 \Big|\frac{\mathrm{d}^k\overline{g}_\omega(r)}{\mathrm{d}r^k}\Big|^2 r\,\mathrm{d}r \leqslant \omega\|g_\omega\|^2_{H^m(\Gamma_\omega^-)}, \tag{2.15}$$

$$\int_0^\omega \int_0^1 \Big|\frac{1}{r^k}\frac{\partial^k}{\partial \theta^k}\overline{G}_\omega(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \begin{cases} \dfrac{1}{\omega}\displaystyle\int_0^1 \frac{1}{r}|\overline{g}_\omega(r)|^2\,\mathrm{d}r & \text{if } k=1,\\ 0 & \text{if } k \geqslant 2. \end{cases} \tag{2.16}$$

For the cross derivatives, we need to consider only those which are first order with respect to $\theta$, since all second-order $\theta$-derivatives vanish identically. So, for any integer $k$, $0 \leqslant k \leqslant m-1$,

$$\int_0^\omega \int_0^1 \Big|\frac{1}{r}\frac{\partial^{k+1}}{\partial r^k\partial\theta}\overline{G}_\omega(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{\omega}\int_0^1 \frac{1}{r}\Big|\frac{\mathrm{d}^k\overline{g}_\omega(r)}{\mathrm{d}r^k}\Big|^2\,\mathrm{d}r, \tag{2.17}$$

$$\int_0^\omega \int_0^1 \Big|\frac{\partial^k}{\partial r^k}\Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big)\overline{G}_\omega(r,\theta)\Big|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{\omega}\int_0^1 r\Big|\frac{\mathrm{d}^k}{\mathrm{d}r^k}\frac{\overline{g}_\omega(r)}{r}\Big|^2\,\mathrm{d}r. \tag{2.18}$$

Henceforth, since $g_\omega = \mathrm{Tr}(z_\pi)$ on $\Gamma_\omega^-$ and $z_\pi \in H^{m+1}(\Omega_\pi) \cap H_0^1(\Omega_\pi)$, then by applying Lemma 2.2 to the right-hand sides of (2.16), (2.17) and (2.18) and summing the resulting inequality with (2.15) for all values of $k = 0,1,\ldots,m$, one obtains (2.13). Finally, (2.14) holds thanks to Lemma 2.2 using (2.10) by passing to the limit in (2.13) as $\omega \to \pi$ and the proof is finished. □

## 3. The main result

We consider the family of problems defined in the previous section $(P_\omega)_{\omega\in(2\pi/3,\pi]}$, and we assume that $f_\omega$ depends continuously on $\omega$ according to Assumption 1.1. Then, the following theorem is optimal with respect to the Sobolev exponent. However, it should be pointed that $u_\omega$ and $u_\pi$ have different domains of definition which justifies the notion of convergence below with respect to norms defined on different Sobolev spaces $H^2(\Omega_\omega)$ as $\omega$ tends to $\pi$.

**Theorem 3.1.** *Let $u_\omega$ the family of solutions to problems $(P_\omega)$, $\omega \in (2\pi/3,\pi]$, then*

$$\lim_{\omega\to\pi} \|u_\omega - u_\pi\|_{H^2(\Omega_\omega)} = 0.$$

Notice that the restriction of $u_\omega - u_\pi$ is actually in $H^2(\Omega_\omega)$ and not necessarily in $H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ since one does not necessarily have $u_\pi = 0$ on $\Gamma_\omega^-$. We will make use of the following lemma which first gives a (weaker) convergence result of solutions $u_\omega$ of problems $(P_\omega)$ as $\omega \to \pi$ with respect to the energy norm.

**Lemma 3.1.** *Let $u_\omega$ the family of solutions to problems $(P_\omega)$, $\omega \in (2\pi/3, \pi]$. Then*

$$\lim_{\omega \to \pi} \|\Delta u_\omega - \Delta u_\pi\|_{H^1(\Omega_\omega)} = 0, \tag{3.1}$$

$$\lim_{\omega \to \pi} \|u_\omega - u_\pi\|_{H^1(\Omega_\omega)} = 0. \tag{3.2}$$

P r o o f of Lemma 3.1. We set $v_\omega := \Delta u_\omega$ and $v_\pi := \Delta u_\pi$. These are solutions of the two following problems:

$$\begin{cases} \Delta v_\omega = f_\omega \in L^2(\Omega_\omega) & \text{in } \Omega_\omega, \\ v_\omega = 0 & \text{on } \partial\Omega_\omega, \end{cases} \tag{3.3}$$

$$\begin{cases} \Delta v_\pi = f_\pi \in L^2(\Omega_\pi) & \text{in } \Omega_\pi, \\ v_\pi = 0 & \text{on } \partial\Omega_\pi. \end{cases} \tag{3.4}$$

If we use the same notation of a function defined on $\Omega_\pi$ and its restriction to $\Omega_\omega$, then one has on $\Omega_\omega$:

$$\begin{cases} \Delta v_\pi = f_\pi \in L^2(\Omega_\omega), & \\ v_\pi = 0 & \text{on } \partial\Omega_\omega \setminus \overline{\Gamma_\omega^-}, \\ v_\pi = g_\omega & \text{on } \Gamma_\omega^-, \end{cases} \tag{3.5}$$

where $g_\omega = \operatorname{Tr}(v_\pi)$ on $\Gamma_\omega^-$. Since $u_\pi \in H^4(\Omega_\pi)$, then $v_\pi \in H^2(\Omega_\pi) \cap H_0^1(\Omega_\pi)$ and Lemma 2.3 with $m = 1$ gives an extension of $g_\omega$ to $\Omega_\omega$, in polar coordinates $\overline{G}_\omega(r,\theta) = \overline{g}_\omega(r)\theta/\omega$, such that $G_\omega \in H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$ and

$$\lim_{\substack{\omega \to \pi \\ <}} \|G_\omega\|_{H^1(\Omega_\omega)} = 0. \tag{3.6}$$

We deduce by comparison of trace operators that $v_\pi - v_\omega - G_\omega \in H_0^1(\Omega_\omega)$. On the other hand, using (3.3) and (3.5), $v_\pi - v_\omega - G_\omega$ is a weak solution of the following Dirichlet problem with homogeneous boundary conditions:

$$\begin{cases} \Delta \Phi_\omega = f_\pi - f_\omega - \Delta G_\omega & \text{in } \mathcal{D}'(\Omega_\omega), \\ \Phi_\omega = 0 & \text{on } \partial\Omega_\omega. \end{cases}$$

Hence, for any test function $\varphi \in \mathcal{D}(\Omega_\omega)$,

$$\int_{\Omega_\omega} \nabla(v_\pi - v_\omega - G_\omega) \cdot \nabla\varphi \,\mathrm{d}x = \int_{\Omega_\omega} (f_\pi - f_\omega)\varphi \,\mathrm{d}x - \int_{\Omega_\omega} \nabla G_\omega \cdot \nabla\varphi \,\mathrm{d}x.$$

By density this last equality holds for $\varphi$ in $H_0^1(\Omega_\omega)$, since $f_\pi - f_\omega$ and $\nabla G_\omega$ are in $L^2(\Omega_\omega)$. Thus, choosing $\varphi = v_\pi - v_\omega - G_\omega$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(3.7)\qquad \|\nabla(v_\pi - v_\omega - G_\omega)\|^2_{L^2(\Omega_\omega)} &\leqslant \|f_\pi - f_\omega\|_{L^2(\Omega_\omega)}\|v_\pi - v_\omega - G_\omega\|_{L^2(\Omega_\omega)} \\ &\quad + \|\nabla G_\omega\|_{L^2(\Omega_\omega)}\|\nabla(v_\pi - v_\omega - G_\omega)\|_{L^2(\Omega_\omega)}.\end{aligned}$$

Following the lines of the proof of Lemma 2.1 and Remark 2.1, we have by the Poincaré's inequality (2.5),

$$(3.8)\qquad \|v_\omega - v_\pi - G_\omega\|_{L^2(\Omega_\omega)} \leqslant \sqrt{\omega}\|\nabla(v_\omega - v_\pi - G_\omega)\|_{L^2(\Omega_\omega)}.$$

Thus, (3.7) implies

$$\|\nabla(v_\pi - v_\omega - G_\omega)\|_{L^2(\Omega_\omega)} \leqslant \sqrt{\omega}\|f_\pi - f_\omega\|_{L^2(\Omega_\omega)} + \|\nabla G_\omega\|_{L^2(\Omega_\omega)}.$$

It follows thanks to (3.6) and Assumption 1.1 on the sequence $(f_\omega)_\omega$ that

$$(3.9)\qquad \lim_{\omega\to\pi} \|\nabla(v_\omega - v_\pi - G_\omega)\|_{L^2(\Omega_\omega)} = 0.$$

Therefore, (3.8) and (3.9) yield

$$(3.10)\qquad \lim_{\omega\to\pi} \|v_\omega - v_\pi - G_\omega\|_{H^1(\Omega_\omega)} = 0$$

and by the triangular inequality

$$\|v_\pi - v_\omega\|_{H^1(\Omega_\omega)} \leqslant \|v_\pi - v_\omega - G_\omega\|_{H^1(\Omega_\omega)} + \|G_\omega\|_{H^1(\Omega_\omega)},$$

the proof of (3.1) follows thanks to (3.10) and (3.6).

Similarly, the second assertion (3.2) can be proved by applying the same arguments to $u_\omega$ and $u_\pi$ which are solutions of the second-order elliptic homogeneous problems

$$\begin{cases} \Delta u_\omega = v_\omega \in L^2(\Omega_\omega) & \text{in } \Omega_\omega, \\ u_\omega = 0 & \text{on } \partial\Omega_\omega, \end{cases}$$
$$\begin{cases} \Delta u_\pi = v_\pi \in L^2(\Omega_\pi) & \text{in } \Omega_\pi, \\ u_\pi = 0 & \text{on } \partial\Omega_\pi, \end{cases}$$

which completes the proof. □

P r o o f of Theorem 3.1. Let $h_\omega$ be the trace of $u_\pi$ on $\Gamma_\omega^-$. Since $u_\pi \in H^4(\Omega_\pi) \cap H_0^1(\Omega_\pi)$, then Lemma 2.3 with $m = 3$ provides us an extension $H_\omega$ of $h_\omega$ to $\Omega_\omega$, in polar coordinates $\overline{H}_\omega(r,\theta) = \overline{h}_\omega(r)\theta/\omega$, such that $H_\omega \in H^3(\Omega_\omega) \cap H^1_{0,\Gamma_\omega^-}(\Omega_\omega)$ and

$$\lim_{\substack{\omega \to \pi \\ <}} \|H_\omega\|_{H^3(\Omega_\omega)} = 0. \tag{3.11}$$

In particular $u_\omega - (u_\pi - H_\omega) \in H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$ and since $\omega \leqslant \pi$, thus Lemma 2.1 gives us the elliptic inequality for the Laplacian

$$\begin{aligned} \|u_\omega - u_\pi - H_\omega)\|_{H^2(\Omega_\omega)} &\leqslant C(\omega)\|\Delta u_\omega - \Delta u_\pi + \Delta H_\omega\|_{L^2(\Omega_\omega)} \\ &\leqslant C(\omega)\|\Delta u_\omega - \Delta u_\pi\|_{L^2(\Omega_\omega)} + C(\omega)\|\Delta H_\omega\|_{L^2(\Omega_\omega)} \\ &\leqslant C(\omega)\|\Delta u_\omega - \Delta u_\pi\|_{L^2(\Omega_\omega)} + C(\omega)\|H_\omega\|_{H^3(\Omega_\omega)}, \end{aligned} \tag{3.12}$$

with $C(\omega) \to \sqrt{1 + (1+\pi)^2}$ as $\omega \to \pi$ and the proof is finished thanks to Lemma 2.3, equation (3.1) and relation (3.11). □

R e m a r k 3.1. In the proof of the main result in Theorem 3.1 we did not need the extra regularity of $H^4$ of the solution $u_\pi$. In fact, $H^3$ is enough to have the convergence result of the extension $H_\omega$ in $H^2$-norm which is more than enough to continue the proof with the estimate (3.12).

## Conclusion

The main result given throughout this paper allows us to describe the local behaviour of solutions of a biharmonic problem posed in a polygonal convex open set near approximately flat boundaries. The convergence result is given with respect to the best expected topology, in our case the $H^2$-norm which gives actually a justification of the Taylor expansion as obtained in the former work [18]. Even though it can also be obtained from [17], Chapter 4 with different and more involved explicit computation *via* Fourier methods, the techniques presented here are new and may possibly apply to other operators as well.

Following Remark 3.1, it will be also interesting to consider the case of a more general right-hand side $f \in H^{-1}$ provided that one gives the right sense to the Navier boundary conditions. Nevertheless, the authors claim that the optimal topology for the convergence result becomes $H^1$ since for $\omega < \pi$ the solution $u_\omega$ of the biharmonic problem $(P_\omega)$ will have at most a regularity $H^\sigma$ for $\sigma < \pi/\omega$.

The case of non convex domains ($\omega > \pi$) needs other techniques. Beyond Fourier methods as in [17], the authors claim that techniques such as singular perturbation methods and asymptotic analysis with respect to the angle parameter $\omega$ close to $\pi$ may be powerful and more interesting to explore in forthcoming works.

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