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ON p-ADIC EULER CONSTANTS

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Abstract. The goal of this article is to associate a p-adic analytic function to the Euler constants $\gamma_p(a, F)$, study the properties of these functions in the neighborhood of $s = 1$ and introduce a p-adic analogue of the infinite sum $\sum_{n \geqslant 1} f(n)/n$ for an algebraic valued, periodic function f. After this, we prove the theorem of Baker, Birch and Wirsing in this setup and discuss irrationality results associated to p-adic Euler constants generalising the earlier known results in this direction. Finally, we define and prove certain properties of p-adic Euler-Briggs constants analogous to the ones proved by Gun, Saha and Sinha.

Keywords: p-adic Euler-Lehmer constant; linear forms in logarithms

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1. INTRODUCTION

The Euler's constant γ is defined as:

$$\gamma = \lim_{x \to \infty} \left(\sum_{n \leqslant x} \frac{1}{n} - \log x \right).$$

This constant is associated to well known number theoretic functions. For instance, if we denote $\zeta(s)$ as the Reimann Zeta function, then γ is the constant term of the Laurent series expansion of $\zeta(s)$ around $s = 1$.

Focussing on the p-adic setup, in 1964, Kubota and Leopoldt in [11] introduced the p-adic analogue of the Zeta function $\zeta_p(s)$ and the Hurwitz zeta function $H_p(s, a, F)$ whenever $p \mid F$. Since γ could also be realised as the derivative of log gamma function, Morita in [14] introduced a p-adic analogue of the gamma function, and defined p-adic Euler's constant γ_p.

Around the same time, Diamond in [7] defined the p-adic analogue of the log gamma function different from Morita's log gamma function, and also defined the

p-adic analogue of Lehmer constants $\gamma_p(a, F)$, building upon the work of Lehmer (see [13]) in this context. He however noted that Euler's constant defined by him is the same as Morita's upto a rational factor. Soon, certain constants γ_p and $\gamma_p(a, p^m)$ were realised as integrals by Koblitz, see [10].

In this article, we restrict ourselves to odd primes p and study the p-adic Euler constants by associating an analytic function analogous to p-adic Hurwitz zeta function $H_p(s, a, F)$. To elaborate further, let $\mathbb{N}$ denote the set of natural numbers and $1_{a,F} \colon \mathbb{N} \to \{0, 1\}$ be the indicator function of the natural numbers n congruent to $a \mod F$. Given our procedure in the classical setup, we can recover the constant $\gamma(a, F)$ via a Dirichlet series associated to $1_{a,F}$. We would like to imitate this process in the p-adic setup. More precisely:

Question 1. *Can we realise $\gamma_p(a, F)$ as the constant term of the Laurent expansion of a p-adic "Dirichlet series" associated to the arithmetic function $1_{a,F}$ around $s = 1$?*

We now elaborate on the issue pertaining to the above question. We begin with the construction of the p-adic function $H_p(s, a, F)$, see [21]. Given the Hurwitz zeta function

$$
(1) \qquad H(s, a, F) = \sum_{n \geqslant 0} \frac{1}{(a + nF)^s}, \quad \operatorname{Re} s > 1,
$$

one can interpolate it continuously to a p-adic analytic function $H_p(s, a, F)$ by considering the meromorphic continuation of $H(s, a, F)$ to the whole complex plane. When $p \mid F$, $p \nmid a$, we interpolate at certain negative integers and we have $H_p(1 - n, a, F) = H(1 - n, a, F)$ whenever $n \equiv 0 \mod(p - 1)$. In the region $\{s \in \mathbb{C}_p \colon |s|_p < p^{1 - 1/(p-1)}\}$, we obtain the following expansion for $H_p(s, a, F)$:

$$
H_p(s, a, F) = \frac{\langle a \rangle^{1-s}}{F(s - 1)} \sum_{j=0}^{\infty} \binom{1 - s}{j} B_j \left(\frac{F}{a} \right)^j,
$$

where B_j denotes the jth Bernoulli number, $\binom{t}{j} = (t)(t - 1) \ldots (t - j + 1)/j!$, and $\langle a \rangle^t := \lim_{n \to t} \langle a \rangle^n$. (See Section 2 for the definition of $\langle a \rangle$.) As mentioned in Proposition 3, we have

$$
(2) \qquad \frac{\mathrm{d}}{\mathrm{d}s}(s - 1)H_p(s, a, F)\big|_{s=1} = \gamma_p(a, F).
$$

We can extend the above definition (see [19]), to periodic Dirichlet series in the p-adic setup as follows: If f is an even periodic function of period F, the p-adic *periodic*

Dirichlet series $L_p(s, f)$ is defined by

$$(3) \qquad L_p(s, f) = \sum_{\substack{a=0, \\ (a,p)=1}}^{\mathcal{K}_f} f(a) H_p(s, a, \mathcal{K}_f),$$

where $\mathcal{K}_f$ denotes the least common multiple of F and p. We shall refer to the above series as the *Washington series*. This series does not provide a satisfactory answer to Question 1. The analogy mentioned by Morita in [15] also does not give the required analytic function. The issue in both places is that the function f is twisted by the principal character $\chi_0 \bmod p$. This does not affect the properties of the Dirichlet characters or more generally completely multiplicative functions, but as an "additive" analogue, we note that we are missing the contribution from the p 'parts'. In Section 3, to function $1_{a,F}$, we associate a p-adic L series $\overline{H_p}(s, a, F)$ when $p \nmid F$. This (see Definition 5) is primarily motivated by the definition of $\gamma_p(a, F)$ given by Diamond. Naturally, we will be working with the definiton of γ_p given by Diamond. We prove the distribution formula (see Proposition 2) for $\overline{H_p}(s, a, F)$, and with this, we can unambiguously define an analogue of periodic Dirichlet series $\overline{L_p}(s, f)$ in this context. It should be noted here that for any odd periodic function f, we have $\overline{L_p}(s, f) \equiv 0$ and at the same time, there are nonzero periodic even functions f for which $\overline{L_p}(s, f) \equiv 0$. While we are not able to classify all such periodic functions, we rule out this scenario in the case when $p \nmid F$. More precisely, we prove the following theorem in the end of Section 4.

Theorem 1. *Let F be a natural number greater than one, co-prime to p and let f be a nonzero even arithmetic function of period F taking algebraic values. Then we have $\overline{L_p}(s, f) \not\equiv 0$.*

In Section 5, we give an affirmative answer to Question 1. With the definition of $\overline{L_p}(s, f)$, we can talk about its value at $s = 1$ whenever it exists. The theory now follows just as in the classical setup owing to the Gauss formula given by Diamond in the p-adic setup. By appealing to the theorem of Brumer, which is the p-adic analogue of Baker's theorem in linear forms of logarithms of algebraic numbers, we can discuss certain linear independence and irrationality results. We present the theorem of Baker, Birch and Wirsing (see [2], Theorem 1) in this context.

Theorem 2. *Let f be an even periodic arithmetic function of period F taking algebraic values and satisfying $f(a) = 0$ whenever $1 < (a, F) < F$. If $\overline{L_p}(s, f) \not\equiv 0$, then $\overline{L_p}(1, f) \neq 0$.*

The above theorem is different from the one mentioned in [18] as we are not working with (3). With the above theorem, we prove the following:

Theorem 3. *Let F be a positive integer which is not a power of p. Then at most one of the elements of the set*

$$\{\gamma_p, \gamma_p(a, F)\colon 1 \leqslant a \leqslant F, \, (a, F) = 1\}$$

is algebraic.

This theorem was proved by Murty and Saradha in [18] for F being a prime. Our setting also extends the results of Chatterjee and Gun, see [5].

Finally, we introduce a p-adic analogue of generalised Euler-Briggs constant $\gamma_p(\Omega, a, F)$ for a finite set of primes Ω (not containing p) in Section 6, and prove the same results mentioned in [8], [9], in the p-adic context. We prove the following theorem.

Theorem 4. *Let Ω be a finite set of primes and let*

$$V_{\overline{Q}, N} := \overline{Q}\langle \gamma_p(\Omega, a, m)\colon 1 \leqslant a < N, \, 1 \leqslant m \leqslant N, \, (a, m) = (m, P_\Omega) = 1 \rangle.$$

Then we have $\dim_{\overline{Q}} V_{\overline{Q}, N} \gg_\Omega N^2/\log N$ as $N \to \infty$.

2. Preliminaries

We state the theorems and remarks required to prove the results. Let $\mathbb{N}$ denote the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{Q}$ the rationals, and $\overline{\mathbb{Q}}$ the field of algebraic numbers. For a rational number a whose denominator is co-prime to F, we denote $\overline{a}_F$ as the representative of $a \bmod F$ in the set $\{0, \ldots, F - 1\}$. Throughout, we consider periodic arithmetic functions $f\colon \mathbb{N} \to \overline{\mathbb{Q}}$ of period F, i.e. $f(n + F) = f(n)$ for all natural numbers n, and we extend the domain to $\mathbb{Z}$ by setting $f(n + F) = f(n)$ for all $n \in \mathbb{Z}$. We say that an *arithmetic function* f of period F is *even* if $f(-n) = f(n)$, and *odd* if $f(-n) = -f(n)$ for all integers n. Given any arithmetic function f of period F, we can decompose it as the odd and even part. We define the *odd* and *even parts* of f, denoted by f_o and f_e, to be

$$f_\mathrm{o}(n) := \frac{f(n) - f(-n)}{2} \quad \text{and} \quad f_\mathrm{e}(n) := \frac{f(n) + f(-n)}{2}.$$

Since f is of period F, it follows that f_o and f_e are arithmetic functions of period F. Note that f_o is an odd arithmetic function whereas f_e is an even arithmetic function. Also, an *arithmetic function* f of period F is said to be of *Dirichlet type* if it is supported at the co-prime residue classes $(\mathbb{Z}/F\mathbb{Z})^*$. Such functions can be written as $\overline{\mathbb{Q}}$ linear combinations of Dirichlet characters of period F. We now mention certain results in the classical setup.

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2.1. Classical setup. If we denote $L(s, f) = \sum\limits_{n=1}^{\infty} f(n)/n^s$ for $\operatorname{Re} s > 1$, the condition $\sum\limits_{a=1}^{F} f(a) = 0$ ensures that $L(1, f)$ exists. In fact, we have the following expression for $L(1, f)$.

$$(4) \qquad L(1, f) = \sum_{a=1}^{F} f(a)\gamma(a, F) = -\sum_{a=1}^{F-1} \hat{f}(a) \log(1 - \mathrm{e}^{2\pi \mathrm{i} a/F}),$$

where $\hat{f}(a) = F^{-1} \sum\limits_{b=0}^{F-1} f(b)\mathrm{e}^{-2\pi \mathrm{i} ba/F}$.

Finally, we define the generalised Euler-Briggs constants following Gun, Saha and Sinha, see [9].

Definition 1. Let Ω be a finite set of primes, F a natural number co-prime to elements of Ω and a such that $1 \leqslant a \leqslant F$. We define the generalised Euler-Briggs constants $\gamma(\Omega, a, F)$ by the following limit:

$$\gamma(\Omega, a, F) := \lim_{x \to \infty} \left(\sum_{\substack{n \leqslant x \\ n \equiv a \bmod F, \\ (n, P_\Omega) = 1}} \frac{1}{n} - \delta_\Omega \frac{\log x}{F} \right).$$

Here they denote

$$P_\Omega := \prod_{p \in \Omega} p \quad \text{and} \quad \delta_\Omega := \prod_{p \in \Omega} \left(1 - \frac{1}{p} \right).$$

By convention, $P_\Omega = 1$, $\delta_\Omega = 1$ when $\Omega = \emptyset$.

2.2. The p-adic setup. Let p be an *odd* prime number, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ be the completion of $\mathbb{Z}$ and $\mathbb{Q}$, respectively, under the p-adic metric. Let $v_p \colon \overline{\mathbb{Q}}_p^* \to \mathbb{Q}$ be the valuation map with $v_p(p) = 1$ and $|p|_p = 1/p$. Moreover, let $\mathbb{C}_p$ denote the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ under this metric, and U_p denote the units of the integral closure of $\mathbb{Z}_p$ in $\mathbb{C}_p$. Let us denote

$$\mathbb{Q}_p[[X]] = \left\{ \sum_{i \geqslant 0}^{\infty} a_i X^i \colon a_i \in \mathbb{Q}_p \right\}, \quad \mathbb{Q}_p\{X\} = \left\{ \sum_{i \geqslant 0}^{\infty} a_i X^i \colon a_i \to 0 \right\}.$$

Following Cohen in [6], we define the operator $\langle . \rangle \colon \mathbb{Q}_p^* \to U_1$, where $U_1 = \{x \in \mathbb{Z}_p \colon |x - 1|_p \leqslant 1/p\}$. If we denote μ_∞ as the roots of unity in $\mathbb{C}_p$, then we have the Teichmuller character $\omega \colon \mathbb{Z}_p^* \to \mu_\infty$, i.e. $\omega(x)$ is the unique $(p-1)$st root of unity such that $\omega(x) \equiv x \bmod p$.

Definition 2. The *map* $\langle . \rangle\colon \mathbb{Z}_p^* \to U_1$ is defined as

$$\langle x \rangle := \frac{x}{\omega(x)}.$$

This map is extended to $\mathbb{Q}_p^*$ by setting $\langle x \rangle := \langle x/p^{v_p(x)} \rangle$.

2.2.1. The Iwasawa logarithm. With the analogy in the classical case, we write

$$\log_p(1+X) = \sum_{k \geqslant 1} (-1)^{k-1} \frac{X^k}{k}.$$

We note that $\log_p(1 + X) \in \mathbb{Q}_p[[X]]$, and moreover, it has a radius of convergence $|p|_p^{1/(p-1)}$. The $\log_p$ map can be analytically extended to $\mathbb{C}_p^*$ satisfying $\log_p p = 0$ and

$$\log_p(ab) = \log_p(a) + \log_p(b) \quad \forall\, a, b \in \mathbb{C}_p^*.$$

Moreover, we have the following proposition, see [20], page 257.

Proposition 1. *Let* $t \in p\mathbb{Z}_p$. *Then the derivative of*

$$x \to (1+t)^x\colon \mathbb{Z}_p \to \mathbb{C}_p$$

at the origin is $\log_p(1+t)$.

In particular, $(\mathrm{d}/\mathrm{d}s)\langle d \rangle^s \big|_{s=0} = \log_p \langle d \rangle = \log_p d$.

Remark 1. Note that the derivative mentioned above is the strict differentiation as defined in [20], page 218, but as mentioned on page 238, for restricted power series $f \in \mathbb{Q}_p\{X\}$ one may also take the derivative with respect to X and evaluate it at a point $a \in \mathbb{Z}_p$.

2.2.2. The Volkenborn integral.

Definition 3. The *Volkenborn integral* of a function $f \in \mathbb{Q}_p\{X\}$ is by definition:

$$(5) \qquad \int_{\mathbb{Z}_p} f(t)\,\mathrm{d}t = \lim_{r \to \infty} \frac{1}{p^r} \sum_{0 \leqslant n < p^r} f(n).$$

Moreover, if we define

$$F(x) = \int_{\mathbb{Z}_p} f(x+t)\,\mathrm{d}t,$$

then we have

$$\frac{\mathrm{d}}{\mathrm{dx}} F(x) = \int_{\mathbb{Z}_p} \frac{\partial}{\partial x} f(x+t) \, \mathrm{d}t.$$

We require the Volkenborn Integral mainly for the above property.

2.3. Some transcendental prerequisites in p-adic domain. We mention the remarkable theorem of Brumer (see [4]) which is the p-adic analogue of Baker's theorem of linear forms in logarithms of algebraic numbers.

Theorem 5 (Brumer)**.** *Let $\alpha_1, \ldots, \alpha_n$ be the elements of U_p which are algebraic over $\mathbb{Q}$ and whose p-adic logarithms are linearly independent over $\mathbb{Q}$. These logarithms are then linearly independent over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}_p$.*

In fact, the non vanishing of $L_p(1, \chi)$ is a consequence of the above theorem. We require the following corollary by Murty and Saradha (see [18]) which asserts the transcendence of $\overline{\mathbb{Q}}$-linear forms of p-adic logarithms of algebraic numbers under some conditions.

Corollary 1. *Let K be a number field. There exists a constant $c > 0$ depending on K such that the following holds: Suppose $\alpha_1, \ldots, \alpha_m$ are multiplicatively independent algebraic numbers in K satisfying $|\alpha_i - 1|_p \leqslant p^{-c}$ for $1 \leqslant i \leqslant m$ and $\beta_1, \ldots, \beta_m \in K$. The linear form*

$$\beta_1 \log_p \alpha_1 + \ldots + \beta_m \log_p \alpha_m$$

is either zero or transcendental.

Remark 2. Taking the Iwasawa logarithm, we can remove the condition on c. Indeed, this is true as we can write

$$\sum_{i=1}^{n} \beta_i \log_p \alpha_i = \sum_{i=1}^{n} \frac{\beta_i}{n} \log_p \alpha_i^n.$$

Therefore, choosing n such that $|\alpha_i^n - 1|_p < p^{-c}$, we can say that $\sum_{i=1}^{n} \beta_i \log_p \alpha_i$ is either zero or transcendental. The existence of n is guaranteed by the following fact: Let $K/\mathbb{Q}_p$ be a finite field extension with ring of integers O_K and prime ideal $\mathfrak{p}$. Then the quotient group $(1 + \mathfrak{p}O_K)/(1 + \mathfrak{p}^m O_K)$ is finite for all natural numbers m, see [12], page 47. This was also worked out in [5].

2.4. On p-adic Euler constants. We end the preliminaries by defining the p-adic Euler constants as done by Diamond and stating a few properties.

Definition 4. Let a, F be integers with $v_p(a/F) < 0$. We then define

$$\gamma_p(a, F) = -\lim_{k \to \infty} \frac{1}{Fp^k} \sum_{\substack{m=0, \\ m \equiv a \bmod F}}^{Fp^k - 1} \log_p m.$$

When $v_p(a) \geqslant v_p(F)$, we write $F = p^k F^*$ with $(p, F^*) = 1$ and set $\phi = \varphi(F^*)$, where φ denotes the Euler Phi function. We define

$$\gamma_p(a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0, \\ v_p(a+nF) < \phi + k}}^{p^\phi - 1} \gamma_p(a + nF, p^\phi F).$$

We state the following results of Diamond which are required for certain computations in Section 4.

Theorem 6 (Diamond)**.** *Let F be a natural number greater than one and r be a positive integer less than F. We have the following properties.*
(1) *If $d \mid (r, F)$, then $F\gamma_p(r, F) = F/d\,\gamma_p(r/d, F/d) - \log_p d$.*
(2) $\gamma_p(r, F) = \gamma_p(F - r, F)$.
(3) *If $b \in \mathbb{Z}^+$, then $\gamma_p(r, F) = \sum_{n=0}^{b-1} \gamma_p(r + nF, bF)$.*

Theorem 7 (Diamond)**.** *Let $F > 1$ and ζ_F be a primitive Fth root of unity. Then*

$$F\gamma_p(a, F) = \gamma_p - \sum_{r=1}^{F-1} \zeta_F^{-ar} \log_p(1 - \zeta_F^r).$$

3. On a p-adic analogue of Hurwitz zeta series

For natural integers a, F such that $1 \leqslant a \leqslant F - 1$, $p \mid F$, $p \nmid a$, we rewrite the p-adic Hurwitz zeta function $H_p(s, a, F)$ as follows:

$$H_p(s, a, F) = \frac{\langle a \rangle^{1-s}}{F(s-1)} \int_{\mathbb{Z}_p} \left\langle 1 + \frac{F}{a}t \right\rangle^{1-s} \mathrm{d}t.$$

The above integral is the same as the infinite series mentioned in the introduction by [20], pages 173 and 270. We note that $H_p(s, a, F) = H_p(s, F - a, F)$ by [20], Proposition 4, page 268. We also have a well known distribution formula for $H_p(s, a, F)$, see [6], page 286:

$$\text{(6)} \qquad\qquad H_p(s, a, F) = \sum_{n=0}^{r-1} H_p(s, a + nF, Fr),$$

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where r is a natural number greater than one. When $p \nmid F$, we observe that the Volkenborn integral does not exist. In order to define a p-adic analogue for the Hurwitz zeta function when $p \nmid F$, we first construct a suitable series in the classical setup. Let $H(s, a, F)$ denote the Dirichlet series:

$$(7) \qquad H(s, a, F) = \sum_{n \geqslant 0} \frac{1}{(a + nF)^s}, \quad \text{where } \operatorname{Re} s > 1.$$

The distribution relation mentioned in (6) is also valid for this Dirichlet series and therefore we obtain the following equality when $r = p$:

$$(8) \qquad H(s, a, F) - \frac{1}{p^s} H(s, \overline{ap^{-1}}_F, F) = \sum_{\substack{n=0, \\ n \neq aF^{-1}_p}}^{p-1} H(s, a + nF, pF),$$

where $\overline{a}_F$ denotes the representative of $a \bmod F$ in the set $\{0, \dots, F-1\}$. As we vary a over the set $\{\overline{ap^{-i}}_F\}_{i=0}^{\phi-1}$ (ϕ denotes the order of p in $(\mathbb{Z}/FZ)^*$), we obtain the following identity:

$$(9) \qquad \left(1 - \frac{1}{p^{\phi s}}\right) H(s, a, F) = \sum_{i=0}^{\phi-1} \frac{1}{p^{is}} \sum_{\substack{n=0, \\ \overline{ap^{-i}}_F + nF \neq 0 \bmod p}}^{p-1} H(s, \overline{ap^{-i}}_F + nF, pF).$$

Since we have the p-adic analogues for the Dirichlet series on the right-side, it is enough to give the corresponding analogue for the Dirichlet polynomial $1/p^{is}$. To do so, we revert back to the p-adic setup. When $p \mid F_1$ and $p \nmid ad$, we have

$$H_p(s, da, dF_1) = \frac{\langle d \rangle^{1-s}}{d} H_p(s, a, F_1).$$

This shows that $\langle d \rangle^{1-s}/d$ is the required analogue of the Dirichlet polynomial d^{-s}, and when $d = p$, we will have a factor $1/p$ as $\langle p \rangle^{1-s} = 1$. This shall allow us to 'extend' the definition of $H_p(s, a, F)$ when $p \nmid F$. The same principle was followed by Diamond (see [7]), while defining the p-adic analogue of Euler constants in Definition 4. We now define $\overline{H_p}(s, a, F)$ for non negative integers a and F with $a < F$.

Definition 5. We define $\overline{H_p}(s, 0, 1)$ as

$$\overline{H_p}(s, 0, 1) = \frac{p}{p-1} \sum_{a=1}^{p-1} H_p(s, a, p).$$

If $F > 1$, $0 \leqslant a < F$ with $v_p(a) < v_p(F)$. We set

$$\overline{H_p}(s, a, F) = \frac{1}{p^k} H_p\left(s, \frac{a}{p^k}, \frac{F}{p^k}\right) \quad \text{if } p^k \| (a, F).$$

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If we further assume that $p \nmid F$ and $(a, F) = 1$, we set

$$\overline{H_p}(s, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{n \in N(a, F)} \overline{H_p}(s, a + nF, p^\phi F),$$

where ϕ is the order of $p \bmod F$ and

$$N(a, F) = \{0 \leqslant n < p^\phi \colon v_p(a + nF) < \phi\}.$$

Finally, if $d = (a, F)$ and $d > 1$, we define

$$\overline{H_p}(s, a, F) = \frac{\langle d \rangle^{1-s}}{d} \overline{H_p}\left(s, \frac{a}{d}, \frac{F}{d}\right).$$

Remark 3. We have the following equivalent expressions for $\overline{H_p}(s, a, F)$ which we use frequently for computational reasons.

(1) When $p \nmid F$, following the same lines as (8) and (9), we have

$$(10) \qquad \overline{H_p}(s, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_i^F(a, n) H_p(s, \overline{ap^{-i}}_F + nF, pF).$$

(2) For a fixed F co-prime to p, let ϕ_1 be such that $p^{\phi_1} \equiv 1 \bmod F$. Then we have

$$\overline{H_p}(s, a, F) = \frac{p^{\phi_1}}{p^{\phi_1} - 1} \sum_{n \in N_1(a, F)} \overline{H_p}(s, a + nF, p^{\phi_1} F),$$

where $N_1(a, F) = \{0 \leqslant n < p^{\phi_1} \colon v_p(a + nF) < \phi_1\}$. The above is true as (9) holds when we replace ϕ by ϕ_1.

We start with the proof of the distribution relation for $\overline{H_p}(s, a, F)$ providing all the details. We remark here that any proof of [7], Theorem 14, statement (iv) will follow through here. The proof purely relies on the distribution formula (6) and appropriate rearrangements. When p is co-prime to F, we also set

$$\delta_i^F(a, n) := \begin{cases} 1 & \text{if } \overline{ap^{-i}}_F + nF \not\equiv 0 \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2. *Let N be a natural number greater than 1, F be a positive integer and a be an integer such that $0 \leqslant a \leqslant F - 1$. We have the following distribution relation:*

$$(11) \qquad \overline{H_p}(s, a, F) = \sum_{n=0}^{N-1} \overline{H_p}(s, a + nF, NF).$$

P r o o f. Without loss of generality for $F > 1$ we can assume that $(a, F) = 1$ and moreover by the surjective group homomorphism $\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ whenever $N \mid M$, it suffices to do the same when N is a prime. Indeed, if it is true for primes q and r, then it is true for $F = qr$ as shown below:

$$\overline{H_p}(s, a, F) = \sum_{n=0}^{q-1} \overline{H_p}(s, a + nF, pF) = \sum_{n=0}^{q-1} \sum_{m=0}^{r-1} \overline{H_p}(s, a + nF + mqF, qrF)$$

$$= \sum_{n=0}^{qr-1} \overline{H_p}(s, a + nF, qrF).$$

If $p \mid F$, we have the distribution relation as mentioned in (6) for $H_p(s, a, F)$, so we need to consider the case when $p \nmid F$. We have two cases, namely $N = p$ and $N \neq p$. Let us denote

$$\mathcal{F}(s) := \sum_{n=0}^{p-1} \overline{H_p}(s, a + nF, pF).$$

On evaluation of $\mathcal{F}(s)$, we obtain

$$\mathcal{F}(s) = \sum_{\substack{n=0, \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} \overline{H_p}(s, a + nF, pF) + \overline{H_p}(s, a + \overline{-aF^{-1}}_p F, pF)$$

$$= \sum_{\substack{n=0, \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} \overline{H_p}(s, a + nF, pF) + \frac{1}{p} \overline{H_p}(s, \overline{ap^{-1}}_F, F).$$

From (10) we have

$$\overline{H_p}(s, \overline{ap^{-1}}_F, F) = \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_{i+1}^F(a, n) H_p(s, \overline{ap^{-i-1}}_F + nF, pF).$$

Note that when $i = \phi - 1$, we have $\overline{ap^{-i-1}}_F + nF = a + nF$. Therefore, substituting this in the above, we get

$$\mathcal{F}(s) = \sum_{\substack{n=0, \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a + nF, pF) + \frac{1}{p^\phi - 1} \sum_{\substack{n=0, \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a + nF, pF)$$

$$+ \frac{1}{p} \left(\frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-2} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_{i+1}^F(a, n) H_p(s, \overline{ap^{-i-1}}_F + nF, pF) \right)$$

$$= \frac{p^\phi}{p^\phi - 1} \left(\sum_{\substack{n=0, \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a + nF, pF) \right.$$

$$\left. + \sum_{i=0}^{\phi-2} \frac{1}{p^{i+1}} \sum_{n=0}^{p-1} \delta_{i+1}^F(a, n) H_p(s, \overline{ap^{-i-1}}_F + nF, pF) \right)$$

$$= \frac{p^\phi}{p^\phi - 1} \left(\sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_i^F(a, n) H_p(s, \overline{ap^{-i-1}}_F + nF, pF) \right) = \overline{H_p}(s, a, F).$$

We still need to consider the case when $N \neq p$ and N is a prime. Consider the sum

$$\mathcal{F}(s) = \sum_{m=0}^{N-1} \overline{H_p}(s, a + mF, NF).$$

If we set ϕ as the order of the element $p \bmod NF$, expanding each term as mentioned in Remark 3, statement (2), we get

(12) $$\mathcal{F}(s) = \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{m=0}^{N-1} \sum_{n=0}^{p-1} \delta_i^{NF}(a+mF, n) H_p(\overline{((a + mF)p^{-i}}_{NF} + nNF, pNF).$$

Consider the set of elements

$$\mathcal{S}_i := \{ \overline{(a + mF)p^{-i}}_{NF} + nNF \colon 0 \leqslant m \leqslant N - 1, 0 \leqslant n \leqslant p - 1,$$
$$\overline{(a + mF)p^{-i}}_{NF} + nNF \not\equiv 0 \bmod p \}.$$

For every fixed i such that $0 \leqslant i \leqslant \phi - 1$, we note that the terms in (12) vary over the elements of $\mathcal{S}_i$. We note that $\mathcal{S}_i$ has $N(p - 1)$ elements, and if $y \in \mathcal{S}_i$, then $y \equiv \overline{ap^{-i}}_F \bmod F$. Therefore all the elements are the representatives of preimages of $\overline{ap^{-i}}_F \bmod F$ in $(\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/NF\mathbb{Z})$. For every j such that $1 \leqslant j \leqslant p - 1$, we denote by $b_{j,i}$ the unique representative in $\{0, \ldots, pF - 1\}$ such that $b_{j,i} \equiv j \bmod p$, $b_{j,i} \equiv ap^{-i} \bmod F$. Let us also denote

$$T_{j,i} := \{ b_{j,i} + kpF \colon 0 \leqslant k \leqslant N - 1 \}.$$

Moreover, by the distribution relation (6) for the set $T_{j,i}$, we observe that

$$\sum_{c \in T_{j,i}} H_p(s, c, pNF) = H_p(s, b_{j,i}, pF).$$

Now we have $\mathcal{S}_i = \bigcup_{j=1}^{p-1} T_{j,i}$. Naturally, the sets $T_{j,i}$ are disjoint. We also note that

$$\{b_{j,i}\}_{j=1}^{p-1} = \{ \alpha_n \colon \alpha_n = \overline{ap^{-i}}_F + nF \text{ with } 0 \leqslant n \leqslant p - 1 \text{ and } p \nmid \alpha_n \}.$$

We can therefore write (12) as follows:

$$\mathcal{F}(s) = \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_i^F(a,n) \overline{H_p}(s, \overline{ap^{-i}}_F + nF, pF) = \overline{H_p}(s, a, F).$$

We get the last equality from Remark 3, statement (2). The above proof works verbatim when $F = 1$, $a = 0$, thereby proving the proposition. $\qquad\square$

4. On the p-adic series $\overline{L_p}(s, f)$

Definition 6. Let $f \colon \mathbb{N} \to \mathbb{C}_p$ be a periodic function of period F. We define

$$\overline{L_p}(s, f) := \sum_{a=0}^{F-1} f(a) \overline{H_p}(s, a, F).$$

Remark 4. We make the following remarks.

(1) If f and g are two periodic functions, then $\overline{L_p}(s, f) + \overline{L_p}(s, g) = \overline{L_p}(s, f + g)$. This is a consequence of Proposition 6.

(2) Since $H_p(s, a, F) = H_p(s, F - a, F)$, whenever $p \mid F$, $p \nmid a$ for arbitrary positive numbers b and F, we have $\overline{H_p}(s, b, F) = \overline{H_p}(s, F - b, F)$. Hence, if f is odd of period F, then $\overline{L_p}(s, f) \equiv 0$. But the map $f \to \overline{L_p}(s, f)$ need not be injective even when the function f is even. Consider the example $f \colon \mathbb{N} \to \mathbb{Q}$ of period p such that $f(n) = 1$ whenever $(n, p) = 1$ and $f(n) = -(p-1)$ otherwise. Therefore

$$\overline{L_p}(s, f) = \frac{f(0)}{p} \overline{H_p}(s, 0, 1) + \sum_{a=1}^{p-1} \overline{H_p}(s, a, p).$$

From the definition, we have $\overline{L_p}(s, f) \equiv 0$. More generally, the same is true whenever f is a periodic function of period p^k such that $f(n) = 1$ whenever $(n, p) = 1$, $f(np^k) = -\varphi(p^k)$ and $f(n) = 0$ otherwise.

(3) $\overline{L_p}(s, \chi)$ is slightly different from the Kubota p-adic L function $L_p(s, \chi)$ for a primitive Dirichlet character χ of conductor k. We have

$$L_p(s, \chi) = \sum_{\substack{a=0, \\ (a,p)=1}}^{kp} \chi(a) H_p(s, a, kp),$$

where the sum runs over $0 \leqslant a < kp$ with $(a, p) = 1$. On the contrary, we have

$$\overline{L_p}(s, \chi) = \sum_{a=0}^{k-1} \chi(a) \overline{H_p}(s, a, k) = \sum_{a=0}^{kp-1} \chi(a) \overline{H_p}(s, a, kp)$$

$$= \sum_{\substack{a=0, \\ (a,p)=1}}^{kp} \chi(a)\overline{H_p}(s,a,kp) + \sum_{0 \leqslant a < k} \chi(pa)\overline{H_p}(s,pa,kp)$$

$$\Rightarrow \left(1 - \frac{\chi(p)}{p}\right)\overline{L}_p(s,\chi) = L_p(s,\chi).$$

So upto an "Euler" factor, both the p-adic L functions are equal.

(4) Let f be an even periodic function of period F. The Washington series $L_p(s,f)$ is the same as $\overline{L}_p(s, f\chi_0)$, where χ_0 denotes the principal character $\mod p$.

We now prove Theorem 1. The strategy of the proof is to first show that the functions $\{H_p(s,a,pF) \colon 1 \leqslant a \leqslant \frac{1}{2}pF, (a,p) = 1\}$ are linearly independent over $\overline{\mathbb{Q}}$. To prove this, we transfer the expression to the real setup using analytic continuation. We then proceed to prove that if $f \colon \mathbb{N} \to \overline{\mathbb{Q}}$ is an even arithmetic function of period F co-prime to p such that

$$\sum_{a=0}^{F-1} f(a)\overline{H_p}(s,a,F) \equiv 0,$$

then $f \equiv 0$. The proof involves a valuation argument.

P r o o f of Theorem 1. Let F be greater than one such that $p \mid F$. Suppose there exists $\alpha_a \in \overline{\mathbb{Q}}$ such that

$$\sum_{\substack{a=1, \\ (a,p)=1}}^{F/2} \alpha_a H_p(s,a,F) = 0.$$

Then substituting $s = 1 - k$, where k runs over the positive integers satisfying $k \equiv 0 \mod(p-1)$, and noting that $H_p(1-k,a,F) = H(1-k,a,F)$ for such integers k (recall here that $H(s,a,F)$ is the Dirichlet series (7)), we have

(13)
$$\sum_{\substack{a=1, \\ (a,p)=1}}^{F/2} \alpha_a H(1-k,a,F) = 0.$$

We now construct an even arithmetic function g of period F as follows:

$$g(n) = \begin{cases} \alpha_a & \text{if } (n,p) = 1 \text{ and } n \equiv \pm a \mod F, \\ 0 & \text{otherwise.} \end{cases}$$

Note that from (13) and using $H(1-k,a,F) = H(1-k,F-a,F)$ we obtain

$$\sum_{a=1}^{F} g(a)H(1-k,a,F) = 0 \Rightarrow L(1-k,g) = 0 \text{ whenever } k \equiv 0 \mod(p-1).$$

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Now, from [17], Lemma 2.1 we have $L(k, \widehat{g}) = 0$ for positive integers $k \equiv 0 \mod(p-1)$ and consequently $\widehat{g} \equiv 0$ (see [1], Theorem 11.3). This implies $g \equiv 0$. Therefore we have the following:

$$(14) \qquad \sum_{\substack{a=1, \\ (a,p)=1}}^{F/2} \alpha_a H_p(s, a, F) = 0 \Rightarrow \alpha_a = 0 \text{ for all } a \text{ such that } 1 \leqslant a \leqslant F/2, \ (a, p) = 1.$$

Now let f be an even algebraic arithmetic function of period F co-prime to p such that

$$\sum_{a=0}^{F-1} f(a) \overline{H_p}(s, a, F) = 0.$$

We therefore have

$$f(0) \frac{\langle F \rangle^{1-s}}{F} \overline{H_p}(s, 0, 1)$$

$$+ \frac{p^\phi}{p^\phi - 1} \sum_{a=0}^{F-1} f(a) \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_i(a, n) \overline{H_p}(s, \overline{ap^{-i}}_F + nF, pF) = 0,$$

where ϕ denotes the order of p in $(\mathbb{Z}/F\mathbb{Z})^*$. Rewriting the equation, we get

$$\frac{p}{p-1} f(0) \sum_{n=1}^{p-1} H_p(s, nF, pF) + \frac{p^\phi}{p^\phi - 1} \sum_{\substack{a=1, \\ (a,p)=1, \\ F \nmid a}}^{pF} \sum_{i=0}^{\phi-1} \frac{f(ap^i)}{p^i} H_p(s, a, pF) = 0.$$

Since f is even, as mentioned in (14) we have $f(0) = 0$, and for all a such that $(a, p) = 1$,

$$(15) \qquad \sum_{i=0}^{\phi-1} \frac{f(ap^i)}{p^i} = 0.$$

Note that the above equality is also valid when $p \mid a$ as we can replace a by $a + F$. Suppose $f(b) \neq 0$ for some b. Let $M_{[b]} = \min\{v_p(f(bp^i)) : 0 \leqslant i \leqslant \phi - 1\}$. We note that there exists j such that $M_{[b]} = v_p(f(bp^j))$. In (15), by replacing a with bp^{j+1}, we obtain

$$\sum_{i=0}^{\phi-1} \frac{f(bp^{i+j+1})}{p^i} = 0.$$

But this is not possible, as

$$v_p\left(\sum_{i=0}^{\phi-1} \frac{f(bp^{i+j+1})}{p^i}\right) = v_p\left(\frac{f(bp^j)}{p^{\phi-1}}\right) = M_{[b]} - \phi + 1,$$

a contradiction to (15). Hence $f(b) = 0$ for all b. $\qquad \square$

5. On the value of $\overline{L_p}(1, f)$ for periodic functions f

5.1. p-adic Euler constants. We begin by associating the constants $\gamma_p(a, F)$ to the p-adic Hurwitz zeta function.

Proposition 3. *Let F be a natural number greater than one with $p \mid F$ and $a < F$ such that $p \nmid a$. We have*

$$\gamma_p(a, F) = \frac{\mathrm{d}}{\mathrm{d}s}(s-1)H_p(s, a, F)\big|_{s=1}.$$

P r o o f. Since $\langle Ft/a + 1 \rangle^s \in \mathbb{Q}_p\{t\}$, we can evaluate

$$\frac{\mathrm{d}}{\mathrm{d}s}(s-1)H_p(s, a, F) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\langle a \rangle^{1-s}}{F}\int_{\mathbb{Z}_p}\left\langle 1 + \frac{F}{a}t\right\rangle^{1-s}\mathrm{d}t$$

$$= \frac{1}{F}\left(-\log_p\langle a \rangle \langle a \rangle^{1-s}\int_{\mathbb{Z}_p}\left\langle 1 + \frac{F}{a}t\right\rangle^{1-s}\mathrm{d}t + \langle a \rangle^{1-s}\int_{\mathbb{Z}_p}\frac{\partial}{\partial s}\left\langle 1 + \frac{F}{a}t\right\rangle^{1-s}\mathrm{d}t\right).$$

Evaluating the derivative at $s = 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(s-1)H_p(s, a, F)\big|_{s=1} = -\frac{1}{F}\left(\log_p\langle a \rangle + \int_{\mathbb{Z}_p}\log_p\left\langle 1 + \frac{F}{a}t\right\rangle\right)\mathrm{d}t = \gamma_p(a, F),$$

the last equality from Definitions 4 and 5. $\qquad\square$

From [10] we recall that we can write

$$(16) \qquad\qquad \gamma_p = \frac{p}{p-1}\sum_{a=1}^{p-1}\gamma_p(a, p).$$

In particular, the above shows that γ_p is the derivative of $(s-1)\overline{H_p}(s, 0, 1)$ evaluated at $s = 1$. We set

$$\gamma_p(0, F) := \frac{1}{F}\gamma_p - \frac{\log_p F}{F}$$

so that we can realise it as the derivative of $(s-1)\overline{H_p}(s, 0, F)$ evaluated at $s = 1$.

Corollary 2. *Let F be a natural number greater than or equal to 1, and let $0 \leqslant a < F$. The residue of $\overline{H_p}(s, a, F)$ at $s = 1$ is $1/F$. Moreover, we have*

$$\frac{\mathrm{d}}{\mathrm{d}s}(s-1)\overline{H_p}(s, a, F)\big|_{s=1} = \gamma_p(a, F).$$

P r o o f. We need to consider the case when $a \geqslant 1$ as the case $a = 0$ is already dealt with. When $p \mid F$ and $p \nmid a$, we have the residue of $H_p(s, a, F)$ at $s = 1$ to be $1/F$, and by Proposition 3, the constant term is given by $\gamma_p(a, F)$. So, we need to consider the case when $p \nmid F$. We further restrict to the case when $(a, F) = 1$. From Remark 3, statement (1) we have

$$\lim_{s \to 1}(s - 1)\overline{H_p}(s, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1} \delta_i^F(a, n) \lim_{s \to 1}(s - 1)H_p(s, \overline{ap^{-i}}_F + nF, pF).$$

$$= \frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \frac{p - 1}{pF} = \frac{1}{F},$$

thereby concluding the first part of the corollary. The second part simply follows from the definition by setting $F^* = F$, and noting that

$$\gamma_p(a_1, F_1) = \frac{1}{p}\gamma_p\left(\frac{a_1}{p}, \frac{F_1}{p}\right)$$

whenever $p \mid (a_1, F_1)$.

If we assume $p \nmid F$ and $(a, F) = d$ for some $d > 1$, we have

$$\lim_{s \to 1} \overline{H_p}(s, a, F) = \frac{1}{d} \lim_{s \to 1} \overline{H_p}\left(s, \frac{a}{d}, \frac{F}{d}\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(s - 1)\overline{H_p}(s, a, F)\Big|_{s=1} = -\frac{\log_p d}{d} \lim_{s \to 1} \overline{H_p}\left(s, \frac{a}{d}, \frac{F}{d}\right) + \frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}s}(s - 1)\overline{H_p}\left(s, \frac{a}{d}, \frac{F}{d}\right)\Big|_{s=i}$$

Since we have the corollary for the case when $(a, F) = 1$, we apply Theorem 6 to complete the proof. $\square$

We also have the following corollary which was proved by Diamond [7] only for the case $v_p(F) > 0$. We remove this condition here.

Corollary 3. *Let*

$$\Phi_p(F) = \sum_{\substack{r=1, \\ (r,F)=1}}^{F} \gamma_p(r, F).$$

We then have

$$\Phi_p(F) = \frac{\varphi(F)}{F}\gamma_p + \frac{\varphi(F)}{F} \sum_{d|F} \frac{\log_p d}{d - 1},$$

where d runs over all the primes dividing F.

P r o o f. Let us denote

$$\Phi_p(s, F) := \sum_{\substack{a=1, \\ (a,F)=1}}^{F} \overline{H_p}(s, a, F),$$

$$\Phi_p(s, F) = \sum_{a=0}^{F-1} \sum_{d|(a,F)} \mu(d)\overline{H_p}(s, a, F) = \sum_{d|F} \mu(d) \sum_{\substack{a=0, \\ d|a}}^{F-1} \overline{H_p}(s, a, F)$$

$$= \sum_{d|F} \mu(d) \frac{\langle d \rangle^{1-s}}{d} \sum_{a=0}^{F/d-1} \overline{H_p}\left(s, a, \frac{F}{d}\right) = \prod_{r|F} \left(1 - \frac{\langle r \rangle^{1-s}}{r}\right)\overline{H_p}(s, 0, 1),$$

where r runs over all the primes dividing F. The last equality is obtained by Proposition 2, as

$$\overline{H_p}(s, 0, 1) = \sum_{a=0}^{F/d-1} \overline{H_p}\left(s, a, \frac{F}{d}\right).$$

Noting that $\overline{H_p}(s, 0, 1)$ has a pole at $s = 1$ with residue 1, we arrive at the corollary by evaluating $(\mathrm{d}/\mathrm{d}s)(s-1)\Phi_p(s, F)\big|_{s=1}$, by Proposition 1 and Corollary 2. $\qquad\square$

By appealing to Theorem 7, we can compute the value of $\overline{L_p}(1, f)$ whenever it exists. More precisely:

Corollary 4. *The residue of the function $\overline{L_p}(s, f)$ at $s = 1$ is $1/F \sum_{a=0}^{F-1} f(a)$. In particular, $\overline{L_p}(1, f)$ exists if and only if $\sum_{a=0}^{F-1} f(a) = 0$. Under these assumptions, the explicit value is given by*

$$\overline{L_p}(1, f) = \sum_{a=0}^{F-1} f(a)\gamma_p(a, F) = -\sum_{r=1}^{F-1} \hat{f}(r) \log_p(1 - \zeta_F^r),$$

where $\hat{f}(a) = 1/F \sum_{r=0}^{F-1} f(a)\zeta_F^{-ar}$.

P r o o f. From Corollary 2, along with

$$\overline{H_p}(s, 0, F) = \frac{\langle F \rangle^{1-s}}{F} H_p(s, 0, 1),$$

we have

$$\lim_{s \to 1}(s - 1)\overline{L_p}(s, f) = \sum_{a=0}^{F-1} \lim_{s \to 1}(s - 1)\overline{H_p}(s, a, F) = \sum_{a=0}^{F-1} \frac{f(a)}{F}.$$

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This proves the first part, and therefore, from now on, we assume that $\sum_{a=0}^{F-1} f(a) = 0$. For $0 \leqslant a < F$, again from Corollary 2 we have

$$\overline{H_p}(s, a, F) = \frac{1}{F(s-1)} + \gamma_p(a, F) + O(s-1).$$

Substituting this expression into Definition 6, we get the first equality. The second equality is an immediate consequence of the first equality, Theorem 7 and the identity $F = \prod_{r=1}^{F-1} (1 - \zeta_F^r)$. $\qquad\square$

One should observe the similarity between the expression of $\overline{L_p}(1, f)$ and the value of $L(1, f)$ as mentioned in (4).

5.2. Proofs of Theorems 2 and 3.

P r o o f of Theorem 2.   By Corollaries 3 and 4 we know that

$$\overline{L_p}(1, f) = f(0)\gamma_p(0, F) + \sum_{(a,F)=1} f(a)\gamma_p(a, F)$$

$$= \sum_{(a,F)=1} \left(f(a) + \frac{f(0)}{\varphi(F)} \right) \gamma_p(a, F) - \frac{f(0)}{F} \sum_{r|F} \frac{\log_p r}{r - 1} - \frac{f(0)}{F} \log_p F,$$

where r runs over the primes dividing F. We can write the above equation as

(17) $$\overline{L_p}(1, f) = \overline{L_p}(1, g) - \frac{f(0)}{F} \sum_{r|F} \left(\frac{1}{r - 1} + v_r(F) \right) \log_p r,$$

where $g \colon \mathbb{N} \to \overline{\mathbb{Q}}$ is the arithmetic function

$$g(n) = \begin{cases} f(n) + \dfrac{f(0)}{\varphi(F)} & \text{if } (n, F) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since g is a Dirichlet type even function, we note that $g \equiv \sum_{\substack{\chi \bmod F, \\ \chi(-1)=1}} a_\chi \chi$ with $a_\chi \in \overline{\mathbb{Q}}$ and therefore

$$\overline{L_p}(1, g) = \sum_{\substack{\chi \bmod F, \\ \chi(-1)=1}} a_\chi \overline{L_p}(1, \chi).$$

In the above, note that $a_{\chi_0} = 0$ as $\sum_{a=0}^{F-1} g(a) = 0$. From Remark 4, statement (3) and [16], Corollary 8, we note that $\overline{L_p}(1, g) \neq 0$ unless $g \equiv 0$. Thus, if we assume $\overline{L_p}(1, f) = 0$, then we only have the following cases by Theorem 5.

(1) If $F \neq p^k$, then $f(0) = 0$ and therefore $f(a) = 0$ (as $g \equiv 0$) whenever $(a, F) = 1$. Therefore $f \equiv 0$.

(2) $F = p^k$, then we have for all $(a, F) = 1$

$$\frac{f(0)}{\varphi(F)} + f(a) = 0 \Rightarrow f(a) = -\frac{f(0)}{(p-1)p^{k-1}}.$$

By the example mentioned in Remark 4, statement (2), we have $\overline{L_p}(1, f) \equiv 0$. In fact, this proof also shows that f is the only nonzero even function of period p^k satisfying the conditions of the theorem and such that $\overline{L_p}(s, f) \equiv 0$. $\qquad \square$

P r o o f of Theorem 3. We first claim that at most one of the elements of the set

$$S_f := \{\gamma_p(a, F): 1 \leqslant a \leqslant F, \ (a, F) = 1\}$$

is algebraic. To prove this, for any two equivalence classes $[a]$, $[b]$ with $[b] \neq [-a]$, we can construct a periodic function f of period F as follows:

$$f(n) = \begin{cases} 1 & \text{if } n \equiv \pm a \bmod F, \\ -1 & \text{if } n \equiv \pm b \bmod F, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\overline{L_p}(s, f) \not\equiv 0$. Now we can apply Theorem 2 to conclude that $\overline{L_p}(1, f)$ is nonzero and hence by Corollary 1, $\overline{L_p}(1, f)$ is transcendental. However, we note that $\overline{L_p}(1, f) = 2(\gamma_p(a, F) - \gamma_p(b, F))$. This proves the claim.

It remains to consider $S_f \cup \{\gamma_p\}$. Suppose γ_p and $\gamma_p(a, F)$ is algebraic. Then by Theorem 7 and Corollary 1 we have

(18) $$F\gamma_p(a, F) = \gamma_p.$$

We first construct a periodic function f of period F as follows:

$$f(n) = \begin{cases} -2 & \text{if } n \equiv 0 \bmod F, \\ 1 & \text{if } n \equiv \pm a \bmod F, \\ 0 & \text{otherwise.} \end{cases}$$

From (18) and Corollary 4 we have $\overline{L_p}(1, f) = +2 \log_p F / F$. Now construct another periodic function g as follows:

$$g(n) = \begin{cases} 1 & \text{if } (n, F) = 1, \\ -\varphi(F) & \text{if } n \equiv 0 \bmod F, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollaries 3 and 4 we have

$$\overline{L_p}(1,g) = \frac{\varphi(F)}{F} \sum_{r|F} \frac{\log_p r}{r-1} + \frac{\varphi(F)}{F} \log_p F.$$

Now consider the arithmetic function $h(n) := \varphi(F)f(n) - 2g(n)$. Note that

$$\overline{L_p}(1,h) = -2\frac{\varphi(F)}{F} \sum_{r|F} \frac{\log_p r}{r-1}.$$

But we also note that h is of Dirichlet type and therefore $\overline{L_p}(1,h)$ is a $\overline{\mathbb{Q}}$-linear combination of p-adic logarithm of units (since $\overline{L_p}(1,\chi)$ also satisfies the same property). Hence by Theorem 5, we obtain a contradiction unless $\overline{L_p}(1,h) \equiv 0$, which is not possible as $F \neq p^k$. This proves the theorem. $\qquad\square$

6. The p-adic analogue of Euler-Briggs constant

Now we define the p-adic analogue of the Generalised Euler-Briggs constant. In the p-adic context, we need the condition $(pF, P_\Omega) = 1$. We denote $1_\Omega(n) \colon \mathbb{N} \to \{0,1\}$ as

$$1_\Omega(n) = \begin{cases} 1 & \text{if } (n, P_\Omega) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We begin by recalling another theorem of Diamond from [7].

Theorem 8 (Diamond). *Suppose we have non negative rational integers a, b, M with $M \geqslant 1$. Let R be an open set in $\mathbb{C}_p$ with $a + M\mathbb{Z}_p \subset R$ and $f \colon R \to \mathbb{C}_p$ be locally holomorphic. If we define the sum*

$$S(k,b) := \frac{1}{bp^k} \sum_{\substack{n=0, \\ n \equiv a \bmod M}}^{Mbp^k - 1} f(n),$$

then $L = \lim_{k\to\infty} S(k,b)$ exists and is independent of the choice of b.

The above theorem ensures the existence of the limit in the following definition:

Definition 7. Let Ω be a finite set of primes not containing non negative integers p, a, F such that F is co-prime to P_Ω and $0 \leqslant a < F - 1$. If $v_p(a) < v_p(F)$, we set

$$\gamma_p(\Omega, a, F) = -\lim_{k\to\infty} \frac{1}{P_\Omega F p^k} \sum_{\substack{n=0, \\ n \equiv a \bmod F, \\ (n, P_\Omega) = 1}}^{FP_\Omega p^k - 1} \log_p n.$$

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When $v_p(a) \geqslant v_p(F)$, we write $F = p^k F^*$ with $(p, F^*) = 1$ and set $\phi = \varphi(F^*)$, where φ denotes the Euler Phi function. We define

$$\gamma_p(\Omega, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0, \\ v_p(a+nF) < \phi+k}}^{p^\phi - 1} \gamma_p(\Omega, a + nF, p^\phi F).$$

We immediately have the following proposition:

Proposition 4. *Assuming the same notations and conditions as mentioned above, we obtain*

$$\gamma_p(\Omega, a, F) = \frac{\mathrm{d}}{\mathrm{d}s}(s-1)\overline{L_p}(s, f)\big|_{s=1},$$

where $f \colon \mathbb{N} \to \{0, 1\}$ is the arithmetic function $f(n) = 1_{a,F}(n)1_\Omega(n)$.

P r o o f. Assume Ω to be a finite set of primes not containing the prime divisors of pF. We first evaluate $(\mathrm{d}/\mathrm{d}s)(s-1)\overline{L_p}(s, f)\big|_{s=1}$. We have

$$1_{a,F}(n)1_\Omega(n) = \sum_{d|P_\Omega} \mu(d)1_{0,d}(n)1_{a,F}(n) = \sum_{d|P_\Omega} \mu(d)1_{a_d, Fd}(n),$$

where a_d is the representative of the congruences $a \bmod F$, $0 \bmod d$ in $(\mathbb{Z}/Fd\mathbb{Z})^*$ in the set $\{0, \ldots, Fd - 1\}$. Therefore we have

$$\overline{L_p}(s, f) = \sum_{d|P_\Omega} \mu(d)\overline{L_p}(s, 1_{a_d, Fd})$$

$$= \sum_{d|P_\Omega} \mu(d)\frac{\langle d \rangle^{1-s}}{d}\overline{L_p}(s, 1_{\overline{ad^{-1}}_F, F}) = \sum_{d|P_\Omega} \mu(d)\frac{\langle d \rangle^{1-s}}{d}\overline{H_p}(s, \overline{ad^{-1}}_F, F).$$

We obtain the following expression (19) with the help of Proposition 1 and Corollary 2:

$$(19) \qquad \frac{\mathrm{d}}{\mathrm{d}s}(s-1)\overline{L_p}(s, f)\big|_{s=1} = -\frac{1}{F}\sum_{d|P_\Omega}\frac{\mu(d)}{d}\log_p d + \sum_{d|P_\Omega}\frac{\mu(d)}{d}\gamma_p(\overline{ad^{-1}}_F, F).$$

It remains to show that the above value is the same as $\gamma(\Omega, a, F)$. We have 3 cases.

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(1) When $p \mid F$, $v_p(a) < v_p(F)$. From Definition 7 we note that

$$
(20) \qquad \gamma_p(\Omega, a, F) = -\lim_{k \to \infty} \frac{1}{P_\Omega F p^k} \sum_{\substack{n=0, \\ n \equiv a \bmod F}}^{F P_\Omega p^k - 1} \sum_{d \mid (n, P_\Omega)} \mu(d) \log_p n
$$

$$
= -\sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \lim_{k \to \infty} \frac{d}{P_\Omega F p^k} \sum_{\substack{n=0, \\ n \equiv a \bmod F, \\ n \equiv 0 \bmod d}}^{F P_\Omega p^k - 1} \log_p n
$$

$$
= -\frac{1}{F} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \log_p d + \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \gamma_p(\overline{ad^{-1}}_F, F).
$$

(2) When $p \nmid F$, substituting (20) to the terms in Definition 7, we have

$$
(21) \quad \gamma_p(\Omega, a, F) = \frac{p^\phi}{p^\phi - 1}
$$

$$
\times \sum_{\substack{n=0, \\ v_p(a+nF) < \phi}}^{p^\phi - 1} \left(-\frac{1}{p^\phi F} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \log_p d + \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \gamma_p(\overline{((a+nF)d^{-1}}_{p^\phi F}, p^\phi F) \right).
$$

Note that there is exactly one n such that $v_p(a + nF) \geqslant \phi$. We can prove that there is at most one such n by noting that if $0 \leqslant n_1 < n_2 \leqslant p^\phi - 1$, then

$$
\phi > v_p((n_2 - n_1)F) = v_p(a + n_2 F - a - n_1 F)
$$
$$
\geqslant \min\{v_p(a + n_2 F), v_p(a + n_1 F)\}.
$$

The existence of n is also guaranteed by the same reason, as no two terms $a + n_1 F$ and $a + n_2 F$ can lie in the same equivalence class. For each $d \mid P_\Omega$ we note that the elements of the set $\{\overline{(a + nF)d^{-1}}_{p^\phi F}: 0 \leqslant n \leqslant p^\phi - 1, v_p(a + nF) < \phi\}$ satisfy the congruence $y \equiv ad^{-1} \bmod F$, $y \not\equiv 0 \bmod p^\phi$. With these observations, we can rewrite (21) as follows:

$$
\gamma_p(\Omega, a, F) = -\frac{1}{F} \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \log_p d
$$

$$
+ \sum_{d \mid P_\Omega} \frac{\mu(d)}{d} \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0, \\ v_p(\overline{ad^{-1}}_F + nF) < \phi}}^{p^\phi - 1} \gamma_p(\overline{ad^{-1}}_F + nF, p^\phi F).
$$

For each $d \mid P_\Omega$, note that the second summand is $\gamma_p(\overline{ad^{-1}}_F, F)$ from Definition 4 and the above expression coincides with (19).

(3) When $p^k||(a, F)$, then we reduce the question to the case when $p \nmid (a, F)$ (one of the above cases) as

$$\gamma_p(\Omega, a, F) = \frac{1}{p^k}\gamma_p(\Omega, a', F') \quad \text{and} \quad \overline{L_p}(s, f) = \frac{1}{p^k}\overline{L_p}(s, f'),$$

where $f'(n) = 1_{a', F'}(n)1_\Omega(n)$ with $a' = p^{-k}a$, $F' = p^{-k}F$. $\qquad\square$

With the above definition, we now write $\gamma_p(\Omega, a, F)$ in terms of γ_p and linear forms of p-adic logarithms of algebraic numbers. This statement is the p-adic analogue of [9], Lemma 8.

Corollary 5. *With the same conditions on Ω and F, we have*

$$\gamma_p(\Omega, a, F) = \frac{1}{\varphi(F)} \sum_{\substack{\chi \bmod F, \\ \chi \neq \chi_0, \chi \text{ even}}} (\chi(a))^{-1}\overline{L_p}(1, \chi) \prod_{p_1 \in \Omega} \left(1 - \frac{\chi(p_1)}{p_1}\right)$$

$$+ \frac{\delta_\Omega}{F}\left(\gamma_p + \sum_{p_1 | F} \frac{\log p_1}{p_1 - 1} + \sum_{p_1 \in \Omega} \frac{\log p_1}{p_1 - 1}\right).$$

P r o o f. Let f be the arithmetic function defined in Proposition 4. From the orthogonality relations of the characters $\bmod F$, we note that

$$1_{a,F}(n) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\chi(n).$$

Now since the sum runs over the co-prime residue classes of P_Ω, we have

$$f(n) = 1_{a,F}(n)1_\Omega(n) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\chi(n)1_\Omega(n).$$

Evaluating $\overline{L_p}(s, f)$,

$$\overline{L_p}(s, f) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\overline{L_p}(s, \chi 1_\Omega)$$

as $\chi 1_\Omega$ is a periodic function of period $P_\Omega F$. Now, by the same argument mentioned in Corollary 3, we have

$$\overline{L_p}(s, \chi 1_\Omega) = \prod_{p_1 \in \Omega} \left(1 - \frac{\chi(p_1)\langle p_1 \rangle^{1-s}}{p_1}\right)\overline{L_p}(s, \chi),$$

$$\overline{L_p}(s, f) = \frac{1}{\varphi(F)}$$

$$\times \left(\overline{L_p}(s, \chi_0) \prod_{r \in \Omega} \left(1 - \frac{\langle r \rangle^{1-s}}{r}\right) + \sum_{\substack{\chi \bmod F, \\ \chi \neq \chi_0}} (\chi(a))^{-1}\overline{L_p}(s, \chi) \prod_{r \in \Omega} \left(1 - \frac{\chi(r)\langle r \rangle^{1-s}}{r}\right)\right),$$

where r runs over the primes in Ω. Again by the same reasoning we also have

$$\overline{L_p}(s, \chi_0) = \prod_{r \mid F} \left(1 - \frac{\langle r \rangle^{1-s}}{r}\right) H(s, 0, 1).$$

Noting that $\overline{L_p}(s, \chi)$ exists at $s = 1$ for $\chi \neq \chi_0$ and applying Corollary 3 for the first sum and $\overline{L_p}(s, \chi) \equiv 0$ when χ is odd by Remark 4, statement (2), we have the result. $\qquad \square$

We end the section with the proof of Theorem 4.

P r o o f of Theorem 4.   We first note that for a fixed period $F > 1$ co-prime to P_Ω, and a non-principal even Dirichlet character $\chi \bmod F$, we have

$$\overline{L_p}(1, \chi) \in \overline{\mathbb{Q}} \langle \gamma_p(\Omega, a, F) \colon 1 \leqslant a \leqslant F, \ (a, F) = 1 \rangle.$$

This is indeed true from Corollary 5 and the orthogonality relations of Dirichlet characters, as

$$\overline{L_p}(1, \chi) \prod_{p_1 \in \Omega} \left(1 - \frac{\chi(p_1)}{p_1}\right) = \sum_{\substack{a=1, \\ (a,F)=1}}^{F} \chi(a) \gamma_p(\Omega, a, F).$$

The remaining part of the proof can be carried out along the same lines as [3], Theorem 3. Indeed, we note that as r varies over the primes $\mathcal{P}$, the elements

$$\{\overline{L_p}(1, \chi) \colon r \in \mathcal{P}, \ \chi \bmod r, \ \chi \text{ is non principal, even}\}$$

are linearly independent over $\overline{\mathbb{Q}}$. Therefore

$$\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}}, N} \geqslant \sum_{\substack{r \leqslant N, \\ r \in \mathcal{P}}} \left(\frac{r-1}{2} - 1\right) = \frac{N^2}{4 \log N} + O\left(\frac{N}{\log N}\right).$$

This proves the theorem. $\qquad \square$

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