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On tangent cones to Schubert varieties in type E

Mikhail V. Ignatyev, Aleksandr A. Shevchenko

Abstract. We consider tangent cones to Schubert subvarieties of the flag variety G/B, where B is a Borel subgroup of a reductive complex algebraic group G of type E_6, E_7 or E_8. We prove that if w_1 and w_2 form a good pair of involutions in the Weyl group W of G then the tangent cones C_{w_1} and C_{w_2} to the corresponding Schubert subvarieties of G/B do not coincide as subschemes of the tangent space to G/B at the neutral point.

1 Introduction and the main result

Let G be a complex reductive algebraic group, T a maximal torus in G, B a Borel subgroup in G containing T, and U the unipotent radical of B. Let Φ be the root system of G with respect to T, Φ^+ the set of positive roots with respect to B, Δ the set of simple roots, and W the Weyl group of Φ (see [4], [10] and [11] for basic facts about algebraic groups and root systems).

Denote by $\mathcal{F} = G/B$ the flag variety and by $X_w \subseteq \mathcal{F}$ the Schubert subvariety corresponding to an element w of the Weyl group W. Denote by $\mathcal{O} = \mathcal{O}_{p,X_w}$ the local ring at the point $p = eB \in X_w$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. The decreasing sequence of ideals

$$\mathcal{O} \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \dots$$

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is a filtration on $\mathcal{O}$. We define R to be the graded algebra

$$R = \operatorname{gr} \mathcal{O} = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

By definition, the *tangent cone* C_w to the Schubert variety X_w at the point p is the spectrum of R: $C_w = \operatorname{Spec} R$. Obviously, C_w is a subscheme of the tangent space $T_pX_w \subseteq T_p\mathcal{F}$. A hard problem in studying geometry of X_w is to describe C_w [5, Chapter 7].

In 2011, D.Yu. Eliseev and A.N. Panov computed tangent cones C_w for all $w \in W$ in the case $G = \mathrm{SL}_n(\mathbb{C})$, $n \leq 5$ [8]. Using their computations, A.N. Panov formulated the following Conjecture.

Conjecture 1 (A.N. Panov, 2011). *Let w_1, w_2 be involutions, i.e., $w_1^2 = w_2^2 = \mathrm{id}$. If $w_1 \neq w_2$, then $C_{w_1} \neq C_{w_2}$ as subschemes of $T_p\mathcal{F}$.*

One can easily check that it is enough to prove the Conjecture for irreducible root systems (see Remark 2 below). In 2013, D.Yu. Eliseev and the first author proved this Conjecture in types A_n, F_4 and G_2 [9]. In [3], M.A. Bochkarev and the authors proved the Conjecture in types B_n and C_n. In [12], we proved that the Conjecture is true if Φ is of type D_n and w_1, w_2 are so-called basic involutions. In this paper, we prove that the Conjecture is true for so-called good pairs of involutions (see Definition 4) for $\Phi = E_6$, E_7 and E_8. Precisely, our main result is as follows.

Theorem 1. *Assume that every irreducible component of Φ is of type E_6, E_7 or E_8. Let w_1, w_2 be a good pair of involutions in the Weyl group of Φ. Then the tangent cones C_{w_1} and C_{w_2} do not coincide as subschemes of $T_p\mathcal{F}$.*

Remark 1. One can also consider reduced tangent cones. Let $\mathcal{A}$ be the symmetric algebra of the vector space $\mathfrak{m}/\mathfrak{m}^2$, or, equivalently, the algebra of regular functions on the tangent space T_pX_w. Since R is generated as $\mathbb{C}$-algebra by $\mathfrak{m}/\mathfrak{m}^2$, it is a quotient ring $R = \mathcal{A}/I$. By definition, the *reduced tangent cone* C_w^{red} to X_w at the point p is the common zero locus in T_pX_w of the polynomials $f \in I \subseteq \mathcal{A}$. Clearly, if $C_{w_1}^{\mathrm{red}} \neq C_{w_2}^{\mathrm{red}}$, then $C_{w_1} \neq C_{w_2}$. It was proved in [3] that if Φ is of type B_n or C_n and w_1 and w_2 are distinct involutions in W, then $C_{w_1}^{\mathrm{red}}$ and $C_{w_2}^{\mathrm{red}}$ do not coincide as subvarieties of $T_p\mathcal{F}$. In [12], the similar result was obtained for basic involutions in type D_n. For type E, this question still remains open even for good pairs of involutions.

The paper is organized as follows. In Section 2, we introduce the main technical tool used in the proof of Theorem 1. Namely, to each element $w \in W$ one can assign a polynomial d_w in the algebra of regular functions on the Lie algebra of the maximal torus T. These polynomials are called Kostant-Kumar polynomials [1], [13], [14], [15]. In [15] S. Kumar showed that if w_1 and w_2 are arbitrary elements of W and $d_{w_1} \neq d_{w_2}$, then $C_{w_1} \neq C_{w_2}$. We give three equivalent definitions of Kostant-Kumar polynomials and formulate their properties needed for the sequel. In Section 3, we recall basic definitions and facts about root systems of type E and prove the main technical fact about divisibility of Kostant-Kumar polynomials, see Proposition 1. Finally, Section 4 contains the notion of a good pair of involutions and the proof of our main result, Theorem 1, based on Proposition 1 and detailed consideration of configurations of roots, see Proposition 2.

2 Kostant-Kumar polynomials

Let w be an element of the Weyl group W. Here we recall the precise definition of the Kostant-Kumar polynomial d_w, explain how to compute it in combinatorial terms, and show that it depends only on the scheme structure of C_w, see [15] for the details.

The torus T acts on the Schubert variety X_w by left multiplications (or, equivalently, by conjugations). The point p is invariant under this action, hence there is the structure of a T-module on the local ring $\mathcal{O}$. The action of T on $\mathcal{O}$ preserves the filtration by powers of the ideal $\mathfrak{m}$, so we obtain the structure of a T-module on the algebra $R = \operatorname{gr}\mathcal{O}$. By [15, Theorem 2.2], R can be decomposed into a direct sum of its finite-dimensional weight subspaces:

$$R = \bigoplus_{\lambda \in \mathfrak{X}(T)} R_\lambda\,.$$

Here $\mathfrak{h}$ is the Lie algebra of the torus T, $\mathfrak{X}(T) \subseteq \mathfrak{h}^*$ is the character lattice of T and $R_\lambda = \{f \in R \mid t \cdot f = \lambda(t)f\}$ is the weight subspace of weight λ. Let Λ be the $\mathbb{Z}$-module consisting of all (possibly infinite) $\mathbb{Z}$-linear combinations of linearly independent elements e^λ, $\lambda \in \mathfrak{X}(T)$. The *formal character* of R is an element of Λ of the form

$$\operatorname{ch} R = \sum_{\lambda \in \mathfrak{X}(T)} m_\lambda e^\lambda\,,$$

where $m_\lambda = \dim R_\lambda$.

Now, pick an element $a = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda e^\lambda \in \Lambda$. Assume that there are finitely many $\lambda \in \mathfrak{X}(T)$ such that $n_\lambda \neq 0$. Given $k \geq 0$, one can define the polynomial

$$[a]_k = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda \cdot \frac{\lambda^k}{k!} \in S = \mathbb{C}[\mathfrak{h}]\,.$$

Denote $[a] = [a]_{k_0}$, where k_0 is minimal among all non-negative numbers k such that $[a]_k \neq 0$. For instance, if $a = 1 - e^\lambda$, then $[a]_0 = 0$ and $[a] = [a]_1 = -\lambda$ (here we denote $1 = e^0$).

Let A be the submodule of Λ consisting of all finite linear combinations. It is a commutative ring with respect to the multiplication $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$. In fact, it is just the group ring of $\mathfrak{X}(T)$. Denote the field of fractions of the ring A by Q. To each element of Q of the form $q = a/b$, a, $b \in A$, one can assign the element

$$[q] = \frac{[a]}{[b]} \in \mathbb{C}(\mathfrak{h})$$

of the field of rational functions on $\mathfrak{h}$. Note that this element is well-defined [15].

There exists an involution $q \mapsto q^*$ on Q defined by

$$e^{\lambda} \mapsto (e^{\lambda})^* = e^{-\lambda}.$$

It turns out [15, Theorem 2.2] that the character $\operatorname{ch} R$ belongs to Q, hence $(\operatorname{ch} R)^* \in Q$, too. (One can consider the field Q of rational functions as a subring of the ring Λ.) Finally, we put

$$c_w = [(\operatorname{ch} R)^*], \quad d_w = (-1)^{l(w)} \cdot c_w \cdot \prod_{\alpha \in \Phi^+} \alpha.$$

Here $l(w)$ is the length of w in the Weyl group W with respect to the set of simple roots Δ. Evidently, c_w and d_w belong to $\mathbb{C}(\mathfrak{h})$; in fact, d_w is a polynomial, i.e., it belongs to the algebra $S = \mathbb{C}[\mathfrak{h}]$ of regular functions on $\mathfrak{h}$, see [14] and [5, Theorem 7.2.6].

Definition 1. Let w be an element of the Weyl group W. The polynomial $d_w \in S$ is called the *Kostant-Kumar polynomial* associated with w.

It follows from the definition that c_w and d_w depend only on the canonical structure of a T-module on the algebra R of regular functions on the tangent cone C_w. Thus, to prove that the tangent cones corresponding to elements w_1, w_2 of the Weyl group are distinct, it is enough to check that $c_{w_1} \neq c_{w_2}$, or, equivalently, $d_{w_1} \neq d_{w_2}$.

On the other hand, there is a purely combinatorial description of Kostant-Kumar polynomials. To give this description, we need some more notation. Let w, v be elements of W. Fix a reduced decomposition of the element $w = s_{i_1} \ldots s_{i_l}$. (Here $\alpha_1, \ldots, \alpha_n \in \Delta$ are simple roots and $s_i = s_{\alpha_i}$ is the simple reflection corresponding to α_i.) Put

$$c_{w,v} = (-1)^{l(w)} \cdot \sum \frac{1}{s_{i_1}^{\epsilon_1}\alpha_{i_1}} \cdot \frac{1}{s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2}\alpha_{i_2}} \cdot \ldots \cdot \frac{1}{s_{i_1}^{\epsilon_1} \ldots s_{i_l}^{\epsilon_l}\alpha_{i_l}},$$

where the sum is taken over all sequences $(\epsilon_1, \ldots, \epsilon_l)$ of zeroes and units such that $s_{i_1}^{\epsilon_1} \ldots s_{i_l}^{\epsilon_l} = v$. Actually, the element $c_{w,v} \in \mathbb{C}(\mathfrak{h})$ depends only on w and v, not on the choice of a reduced decomposition of w [15, Section 3].

Example 1. Let $\Phi = A_n$. Put $w = s_1 s_2 s_1$. To compute $c_{w,\mathrm{id}}$, we should take the sum over two sequences, $(0, 0, 0)$ and $(1, 0, 1)$. Hence

$$c_{w,\mathrm{id}} = (-1)^3 \cdot \left(\frac{1}{\alpha_1\alpha_2\alpha_1} + \frac{1}{-\alpha_1(\alpha_1 + \alpha_2)\alpha_1} \right) = -\frac{1}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)}.$$

A remarkable fact is that $c_{w,\mathrm{id}} = c_w$, hence to prove that the tangent cones to Schubert varieties do not coincide as subschemes, we need only combinatorics of the Weyl group. Note also that for classical Weyl groups, elements $c_{w,v}$ are closely related to Schubert polynomials [1].

Finally, we will present an original definition of elements $c_{w,v}$ using so-called nil-Hecke ring (see [15] and [5, Section 7.1]). The group W naturally acts on $\mathbb{C}(\mathfrak{h})$ by automorphisms. Denote by Q_W the vector space over $\mathbb{C}(\mathfrak{h})$ with basis $\{\delta_w,\ w \in W\}$. It is a ring with respect to the multiplication

$$f\delta_v \cdot g\delta_w = fv(g)\delta_{vw}\,.$$

This ring is called the *nil-Hecke ring*. To each i from 1 to n put

$$x_i = \alpha_i^{-1}(\delta_{s_i} - \delta_{\mathrm{id}})\,.$$

Let $w \in W$ and $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition of w. Then the element

$$x_w = x_{i_1} \dots x_{i_l}$$

does not depend on the choice of a reduced decomposition of w [13, Proposition 2.1].

Moreover, it turns out that $\{x_w,\ w \in W\}$ is a $\mathbb{C}(\mathfrak{h})$-basis of Q_W [13, Proposition 2.2], and

$$x_w = \sum\nolimits_{v \in W} c_{w,v}\delta_v\,.$$

Actually, if $w, v \in W$, then

$$\begin{aligned}
&\text{a) } x_v \cdot x_w = \begin{cases} x_{vw}\,, & \text{if } l(vw) = l(v) + l(w),\\ 0\,, & \text{otherwise,}\end{cases}\\
&\text{b) } c_{w,v} = -v(\alpha_i)^{-1}(c_{ws_i,v} + c_{ws_i,vs_i}),\ \text{if } l(ws_i) = l(w) - 1,\\
&\text{c) } c_{w,v} = \alpha_i^{-1}(s_i(c_{s_iw,s_iv}) - c_{s_iw,v}),\ \text{if } l(s_iw) = l(w) - 1.
\end{aligned} \tag{1}$$

The first property is proved in [13, Proposition 2.2]. The second and the third properties follow immediately from the first one and the definitions (see also the proof of [15, Corollary 3.2]).

Remark 2. Suppose Φ is a union of its subsystems Φ_1 and Φ_2 contained in mutually orthogonal subspaces. Let W_1, W_2 be the Weyl groups of Φ_1, Φ_2 respectively, so $W = W_1 \times W_2$. Denote $\Delta_1 = \Delta \cap \Phi_1 = \{\alpha_1, \dots, \alpha_r\}$ and $\Delta_2 = \Delta \cap \Phi_2 = \{\beta_1, \dots, \beta_s\}$, then

$$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s]\,.$$

Given $v \in W_i$, $i = 1, 2$, denote by d_v^i its Kostant-Kumar polynomial. We can consider d_v^1 (respectively, d_v^2) as an element of $\mathbb{C}[\mathfrak{h}]$ depending only on $\alpha_1, \dots, \alpha_r$ (respectively, on $\beta_1, \dots, \beta_s$). We define $c_v^i \in \mathbb{C}(\mathfrak{h})$, $i = 1,\ 2$, by a similar way. Let $w \in W$, $w_1 \in W_1$, $w_2 \in W_2$ and $w = w_1w_2$. Repeating literally the proof of [9, Proposition 1.6], we obtain the following:

$$d_w = d^1_{w_1} d^2_{w_2}\,, \quad c_w = c^1_{w_1} c^2_{w_2}\,.$$

Thus, to prove Theorem 1 it is enough to prove this theorem for irreducible root systems of type E, because $\mathbb{C}[\mathfrak{h}]$ is a unique factorization domain.

3 Divisibility in $\mathbb{C}[\mathfrak{h}]$

Throughout this section, Φ denotes an irreducible root system of type E_6, E_7 or E_8. Below we briefly recall some facts about Φ. (We follow the notation from [4].) Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of the Euclidean space $\mathbb{R}^n$. As usual, we identify the set Φ^+ of positive roots with the following subset of $\mathbb{R}^n$:

$$\begin{aligned}
E_6^+ &= \{(\pm\epsilon_i + \epsilon_j),\ 1 \le i < j \le 5\} \\
&\cup \left\{ \frac{1}{2}\left(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}\epsilon_i\right),\ \sum_{i=1}^{5}\nu(i) \text{ is even} \right\}, \\
E_7^+ &= \{(\pm\epsilon_i + \epsilon_j) 1 \le i < j \le 6\} \cup \{(\epsilon_7 - \epsilon_8)\} \\
&\cup \left\{ \frac{1}{2}\left(\epsilon_7 - \epsilon_8 + \sum_{i=1}^{6}(-1)^{\nu(i)}\epsilon_i\right),\ \sum_{i=1}^{6}\nu(i) \text{ is even} \right\}, \\
E_8^+ &= \{\pm\epsilon_i + \epsilon_j 1 \le i < j \le 8\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{8}(-1)^{\nu(i)}\epsilon_i,\ \sum_{i=1}^{8}\nu(i) \text{ is even} \right\},
\end{aligned}$$

so W can be considered as a subgroup of the orthogonal group $O(\mathbb{R}^n)$.

The simple roots have the following form.

$$\begin{aligned}
\Phi = E_6\colon\ & \alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \\
& \alpha_2 = \epsilon_1 + \epsilon_2, \qquad \alpha_3 = \epsilon_2 - \epsilon_1, \\
& \alpha_4 = \epsilon_3 - \epsilon_2, \qquad \alpha_5 = \epsilon_4 - \epsilon_3, \\
& \alpha_6 = \epsilon_5 - \epsilon_4; \\
\Phi = E_7\colon\ & \alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \\
& \alpha_2 = \epsilon_1 + \epsilon_2, \qquad \alpha_3 = \epsilon_2 - \epsilon_1, \\
& \alpha_4 = \epsilon_3 - \epsilon_2, \qquad \alpha_5 = \epsilon_4 - \epsilon_3, \\
& \alpha_6 = \epsilon_5 - \epsilon_4, \qquad \alpha_7 = \epsilon_6 - \epsilon_5. \\
\Phi = E_8\colon\ & \alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \\
& \alpha_2 = \epsilon_1 + \epsilon_2, \qquad \alpha_3 = \epsilon_2 - \epsilon_1, \\
& \alpha_4 = \epsilon_3 - \epsilon_2, \qquad \alpha_5 = \epsilon_4 - \epsilon_3, \\
& \alpha_6 = \epsilon_5 - \epsilon_4, \qquad \alpha_7 = \epsilon_6 - \epsilon_5, \\
& \alpha_8 = \epsilon_7 - \epsilon_6.
\end{aligned} \tag{2}$$

We say that v is less or equal to w with respect to the *Bruhat order*, written $v \le w$, if some reduced decomposition for v is a subword of some reduced decomposition for w. It is well-known that this order plays the crucial role in many geometric aspects of theory of algebraic groups. For instance, the Bruhat order encodes the incidences among Schubert varieties, i.e., X_v is contained in X_w if and only if $v \le w$. It turns out that $c_{w,v}$ is non-zero if and only if $v \le w$ [15, Corollary 3.2]. For example, $c_w = c_{w,\mathrm{id}}$ is non-zero for *any* w, because id is the smallest element of W with respect to the Bruhat order. Note that given $v, w \in W$, there exists $g_{w,v} \in S = \mathbb{C}[\mathfrak{h}]$ such that

$$c_{w,v} = g_{w,v} \cdot \prod_{\substack{\alpha>0,\\ s_\alpha v \leq w}} \alpha^{-1}\,, \tag{3}$$

see [7] and [5, Theorem 7.1.11]

Since we fixed the order on the set of simple roots, one can consider the lexicographic total order on the set of positive roots: given $\alpha = \sum a_i\alpha_i$ and $\beta = \sum b_i\alpha_i$, we write $\alpha \prec \beta$ if there exists j such that $a_i = b_i$ for all $i < j$ and $a_i < b_i$. Let w be an involution in the Weyl group W of Φ. Denote by s_α the reflection in W corresponding to a root α. Denote by β_1 the maximal (with respect to the order $\preceq$) root among all roots $\beta \in \Phi^+$ for which $w(\beta) = -\beta$. Next, for $i \geq 1$, denote by β_{i+1} the maximal root among all roots $\beta \in \Phi^+$ such that $w_i(\beta) = -\beta$, where

$$w_i = s_{\beta_i} \circ s_{\beta_{i-1}} \circ \ldots \circ s_{\beta_1} \circ w\,.$$

One can easily check that w_k coincides with the identity element of W for certain k.

Definition 2. The set $\mathrm{Supp}(w) = \{\beta_1, \ldots, \beta_k\}$ is called the *support* $\mathrm{Supp}(\sigma)$ of w. It turns out that $\mathrm{Supp}(w)$ is an orthogonal subset of Φ^+ [16, Theorem 5.4]. Note that

$$w = \prod_{\beta \in \mathrm{Supp}(w)} s_\beta\,,$$

where the product is taken in any fixed order.

Lemma 1. *Let w_1, w_2 be involutions in W. If $\mathrm{Supp}(w_1) \subset \mathrm{Supp}(w_2)$ then $w_1 \leq w_2$.*

Proof. The well-known Strong Exchange Condition (see, e.g., [6, Proposition 3.1 (ii)]) implies that, given $w \in W$ and $\alpha \in \Phi$, one has $l(ws_\alpha) > l(w)$ if and only if $w\alpha \in \Phi^+$. On the other hand (see, e.g., [2, Definition 2.1.1]), $l(ws_\alpha) > l(w)$ if and only if $ws_\alpha > w$. Hence, $w(\alpha) \in \Phi^+$ if and only if $ws_\alpha > w$. Let

$$\mathrm{Supp}(w_2) \setminus \mathrm{Supp}(w_1) = \{\beta_1,\ \ldots,\ \beta_k\}\,,$$

then

$$w_2 = w_1 \cdot \prod_{\beta \in \mathrm{Supp}(w_2)\setminus\mathrm{Supp}(w_1)} s_\beta = w_1 s_{\beta_1} \ldots s_{\beta_k}\,.$$

Next, denote $v_i = w_1 s_{\beta_1} \ldots s_{\beta_i - 1}$ for $1 \leq i \leq k+1$, so that $v_1 = w_1$ and $v_{k+1} = w_2$. Then, clearly, $v_i(\beta_i) = \beta_i \in \Phi^+$, thus, $w_1 = v_1 < v_2 < \ldots < v_{k+1} = w_2$, as required. □

Definition 3. The subset $\mathcal{C}_1 = \{\beta \in \Phi^+ \mid \alpha_1 \preceq \beta\}$ is called the *first column of* Φ^+.

We will essentially use the following standard fact about parabolic subgroups of the Weil group W.

Theorem 2. *[11, Proposition 1.10 (c)] Let I be a subset of the set Δ of simple roots. Denote by W_I the parabolic subgroup of W generated by the simple reflections s_α, $\alpha \in I$. Put also*

$$W^I = \{w \in W \mid l(ws_\alpha) > l(w) \text{ for all } \alpha \in I\}\,.$$

Given $w \in W$, there exist unique $u \in W^I$ and $v \in W_I$ such that $w = uv$. Their lengths satisfy $l(w) = l(u) + l(v)$.

The following proposition plays the crucial role in the proof of the main result (cf. [9, Lemmas 2.4, 2.5], [3, Lemma 2.6] and [12, Lemma 2.7]).

Proposition 1. *Let $w \in W$ be an involution. Assume that* $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \{\beta\}$ *and the reflection s_β has a reduced decomposition of the form $s_\beta = u_\beta v_\beta$ for a certain element v_β from the subgroup $\widetilde{W}$ of W generated by the reflections s_i, $i \neq 1$, so that $u_\beta = v_\beta^{-1} s_1$ and $l(u_\beta s_i) = l(u) + 1$ for all $i \neq 1$. Then β does not divide d_w in $\mathbb{C}[\mathfrak{h}]$.*

Proof. Denote

$$\begin{aligned}\widetilde{W}^1 &= \{w \in W \mid l(ws_i) = l(w) + 1 \text{ for all } i \neq 1\}\\ &= \{w \in W \mid w(\alpha_i) \in \Phi^+ \text{ for all } i \neq 1\}\,.\end{aligned}$$

Applying Theorem 2 to the subset $I = \Delta \setminus \{\alpha_1\}$, we see that there exist unique $u \in \widetilde{W}^1 = W^I$ and $v \in \widetilde{W} = W_I$ such that $w = uv$. We claim that in fact $u = u_\beta$. Indeed, denote $w' = \prod_{\alpha \in \mathrm{Supp}(w),\ \alpha \neq \beta} s_\alpha$, then one can write

$$w = \prod_{\alpha \in \mathrm{Supp}(w)} s_\alpha = s_\beta w' = u_\beta v_\beta w'.$$

But $v_\beta w' \in \widetilde{W}$, while $u_\beta \in \widetilde{W}^1$, which means that $u = u_\beta$ and $v = v_\beta w'$.

We claim that

$$\begin{aligned}c_w &= -\frac{c_{us_1,g_0} g_0(c_{v,g_0^{-1}})}{\beta} - \sum_{\substack{g \leq u,\, g^{-1} \leq v,\\ g \neq g_0}} \frac{c_{us_1,g} g(c_{v,g^{-1}})}{g(\alpha_1)}\\ &= \beta^{-1} \cdot g_0(c_{v,g_0^{-1}}) \cdot \frac{K}{L} + \frac{M}{N}\end{aligned} \tag{4}$$

Here $g_0 = us_1$ and K, L and $M, N \in \mathbb{C}[\mathfrak{h}]$ are pairs of coprime polynomials such that the root β (considered as an element of $\mathbb{C}[\mathfrak{h}]$) divides neither K nor N.

Indeed, one can prove (4) using (1) and arguing as in the proof of [9, Lemma 2.5]. Namely, since $l(w) = l(u) + l(v)$, formula (1)a shows that

$$\begin{aligned}x_w &= \sum_{s \in W} c_{w,s}\delta_s = x_u x_v = \sum_{g,h \in W} c_{u,g}\delta_g \cdot c_{v,h}\delta_h\\ &= \sum_{g,h \in W} c_{u,g} g(c_{v,h})\delta_{gh} = \sum_{s \in W}\left(\sum_{g \in W} c_{u,g} g(c_{v,g^{-1}s})\right)\delta_s\,.\end{aligned}$$

Thus, for any $s \in W$, the coefficient of δ_s is equal to

$$c_{w,s} = \sum_{g \in W} c_{u,g} g(c_{v,g^{-1}s}),$$

in particular,

$$c_w = c_{w,\mathrm{id}} = \sum_{g \in W} c_{u,g} g(c_{v,g^{-1}}).$$

Moreover, since $c_{p,q} \neq 0$ if and only if $p \geq q$, the sum in the right hand side is taken over permutations g such that $u \geq g$ and $v \geq g^{-1}$. Denote the set of such permutations by U. Note that $g \in U$ implies that g is obtained from $u = v_\beta^{-1} s_1$ by deleting s_1 and, possibly, some other simple reflections. (If s_1 is not deleted, then the condition $v \geq g^{-1}$ does not hold.) Hence

$$c_w = c_{w,\mathrm{id}} = \sum_{g \in U} c_{u,g} g(c_{v,g^{-1}}).$$

Using (1)b) and the fact that $l(us_1) = l(u) - 1$, we obtain

$$c_{u,g} = -g(\alpha_1)^{-1}(c_{us_1,g} + c_{us_1,gs_1}) = -g(\alpha_1)^{-1} c_{us_1,g},$$

because $us_1 \not\geq gs_1$ and so $c_{us_1,gs_1} = 0$. Thus,

$$c_w = -\sum_{g \in U} \frac{c_{us_1,g} g(c_{v,g^{-1}})}{g(\alpha_1)}.$$

It is easy to check that there is at most one element g such that $g(\alpha_1) = \beta$ and $g \in U$, namely, the element $g_0 = us_1 = v_\beta^{-1}$. Indeed,

$$s_\beta = v_\beta^{-1} s_1 v_\beta = s_{v_\beta^{-1}(\alpha_1)},$$

hence $v_\beta^{-1}(\alpha_1) = \pm\beta$. But v_β^{-1} belongs to $\widetilde{W}$, consequently,

$$v_\beta^{-1}(\alpha_1) = \alpha_1 + \ldots \in \Phi^+.$$

We conclude that $g_0 = v_\beta^{-1}$ sends α_1 to β. On the other hand, if $g(\alpha_1) = \beta$ for some $g \neq v_\beta^{-1}$ from U, then $s_\beta = v_\beta^{-1} s_1 v_\beta$ is not a reduced decomposition of s_β, a contradiction.

Assume for a moment that g_0 belongs to U, i.e., $v \geq g_0^{-1}$. Then

$$c_w = -\frac{c_{us_1,g_0} g_0(c_{v,g_0^{-1}})}{\beta} - \sum_{\substack{g \in U,\\ g \neq g_0}} \frac{c_{us_1,g} g(c_{v,g^{-1}})}{g(\alpha_1)}. \tag{5}$$

By S' (resp. Q') denote the subalgebra of $S = \mathbb{C}[\mathfrak{h}]$ (resp. the subfield of $\mathbb{C}(\mathfrak{h})$) generated by α_i, $i \neq 1$, then $c_{v,g_0^{-1}} \in Q'$ because v, $g_0^{-1} \in \widetilde{W}$. Since $g \in \widetilde{W}$, $g(c_{v,g_0^{-1}}) \in Q'$, too. In particular, if $g(c_{v,g_0^{-1}}) = G_1/G_2$ and G_1, $G_2 \in S'$ are coprime, then β does not divide G_1. On the other hand, $c_{us_1,g_0} \in Q'$, because both $us_1 = g_0$ belongs to $\widetilde{W}$. We conclude that the first summand in (5) has the form $g_0(c_{v,g_0^{-1}}) \cdot K/\beta L$ for some coprime $K, L \in \mathbb{C}[\mathfrak{h}]$. Finally, if $g \in U$ and $g \neq g_0$, then $g(c_{v,g^{-1}}) \in Q'$. Again, since us_1 and g belong to $\widetilde{W}$, one has $c_{us_1,g} \in Q'$. We see that if the latter sum in (5) is equal to M/N, where $M, N \in \mathbb{C}[\mathfrak{h}]$ are coprime, then β does not divide N.

To prove that β does not divide d_w, it is enough to show that $c_{v,g_0^{-1}} \neq 0$, i.e., $v \geq g_0^{-1}$ (or, equivalently, $v^{-1} \geq g_0$). By Lemma 1, $s_\beta \leq w$, According to [2, Chapter 2, Exercise 21], this is equivalent to $v_\beta \leq v$. Hence, $g_0 = v_\beta^{-1} \leq v^{-1}$, which concludes the proof. □

4 Good pairs of involutions

In this section, we formulate and prove the main result of the paper, Theorem 1. To do this, we need to introduce the notion of a good pair of involutions. Recall the set of simple roots from (2). We will order the simple roots as follows.

Type of Φ	Order of simple roots
E_6	$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6$ or $\alpha_2,\ \alpha_6,\ \alpha_3,\ \alpha_5,\ \alpha_4,\ \alpha_1$
E_7	$\alpha_3,\ \alpha_7,\ \alpha_4,\ \alpha_6,\ \alpha_5,\ \alpha_2,\ \alpha_1$
E_8	$\alpha_4,\ \alpha_8,\ \alpha_5,\ \alpha_7,\ \alpha_6,\ \alpha_3,\ \alpha_2,\ \alpha_1$

Since the set of simple roots is ordered, the support of an involution and the first column are well-defined.

Definition 4. Let w_1, w_2 be involutions in W. We say that they form a *good pair of involutions* if $\mathrm{Supp}(w_i) \cap \mathcal{C}_1 = \{\beta_i\}$ for $i = 1,\ 2$ such that $\beta_1 \neq \beta_2$, both β_1 and β_2 are not maximal in C_1 for $\Phi = E_8$, and $s_{\beta_1} \not\leq w_2$ or $s_{\beta_2} \not\leq w_1$.

Now we are ready to prove our main result, Theorem 1, which claims that if w_1, w_2 is a good pair of involutions then the corresponding tangent cones C_{w_1} and C_{w_2} do not coincide. This follows immediately from the following proposition.

Proposition 2. *Let w_1, w_2 be a good pair of involutions in W. Then $d_{w_1} \neq d_{w_2}$.*

Proof. Let $\beta = \beta_1$ or β_2, and $s_\beta = uv_\beta$ be as in Proposition 1. In the tables below we list the elements u for all possible β. The first (respectively, the second) column of the table contains the sequence $(c_1, \ldots, c_8)$ (respectively, $(b_1, \ldots, b_n)$) if

$$\beta = \sum_{i=1}^{8} c_i \epsilon_i = \sum_{i=1}^{n} \alpha_i,\ n = \mathrm{rk}\,\Phi\,.$$

The third column contains a reduced decomposition of u.

Case $\Phi = E_6$ with the order $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6$		
$(c_1,\ \ldots,\ c_8)$	$(b_1,\ \ldots,\ b_6)$	Reduced decomposition of u
$(\frac12,-\frac12,-\frac12,-\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	100000	s_1
$(-\frac12,\frac12,-\frac12,-\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	101000	s_3s_1
$(-\frac12,-\frac12,\frac12,-\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	101100	$s_4s_3s_1$
$(\frac12,\frac12,\frac12,-\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	111100	$s_2s_4s_3s_1$
$(-\frac12,-\frac12,-\frac12,\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	101110	$s_5s_4s_3s_1$
$(-\frac12,-\frac12,-\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	101111	$s_6s_5s_4s_3s_1$
$(\frac12,\frac12,-\frac12,\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	111110	$s_5s_2s_4s_3s_1$
$(\frac12,-\frac12,\frac12,\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	111210	$s_4s_5s_2s_4s_3s_1$
$(\frac12,\frac12,-\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	111111	$s_6s_5s_2s_4s_3s_1$
$(-\frac12,\frac12,\frac12,\frac12,-\frac12,-\frac12,-\frac12,\frac12)$	112210	$s_3s_4s_5s_2s_4s_3s_1$
$(\frac12,-\frac12,\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	111211	$s_6s_4s_5s_2s_4s_3s_1$
$(-\frac12,\frac12,\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	112211	$s_3s_6s_4s_5s_2s_4s_3s_1$
$(\frac12,-\frac12,-\frac12,\frac12,\frac12,-\frac12,-\frac12,\frac12)$	111221	$s_5s_6s_4s_5s_2s_4s_3s_1$
$(-\frac12,\frac12,-\frac12,\frac12,\frac12,-\frac12,-\frac12,\frac12)$	112221	$s_3s_5s_6s_4s_5s_2s_4s_3s_1$
$(-\frac12,-\frac12,\frac12,\frac12,\frac12,-\frac12,-\frac12,\frac12)$	112321	$s_4s_3s_5s_6s_4s_5s_2s_4s_3s_1$
$(\frac12,\frac12,\frac12,\frac12,\frac12,-\frac12,-\frac12,\frac12)$	122321	$s_2s_4s_3s_5s_6s_4s_5s_2s_4s_3s_1$

Case $\Phi = E_6$ with the order $\alpha_2,\ \alpha_6,\ \alpha_3,\ \alpha_5,\ \alpha_4,\ \alpha_1$		
$(c_1,\ \ldots,\ c_8)$	$(b_1,\ \ldots,\ b_6)$	Reduced decomposition of u
$(0,0,0,-1,1,0,0,0)$	000001	s_6
$(0,0,-1,0,1,0,0,0)$	000011	s_5s_6
$(0,-1,0,0,1,0,0,0)$	000111	$s_4s_5s_6$
$(1,0,0,0,1,0,0,0)$	010111	$s_2s_4s_5s_6$
$(-1,0,0,0,1,0,0,0)$	001111	$s_3s_4s_5s_6$
$(-\frac12,-\frac12,-\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	101111	$s_1s_3s_4s_5s_6$
$(0,1,0,0,1,0,0,0)$	011111	$s_2s_3s_4s_5s_6$
$(0,0,1,0,1,0,0,0)$	011211	$s_4s_2s_3s_4s_5s_6$
$(\frac12,\frac12,-\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	111111	$s_1s_2s_3s_4s_5s_6$
$(0,0,0,1,1,0,0,0)$	011221	$s_5s_4s_2s_3s_4s_5s_6$
$(\frac12,-\frac12,\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	111211	$s_4s_1s_2s_3s_4s_5s_6$
$(-\frac12,\frac12,\frac12,-\frac12,\frac12,-\frac12,-\frac12,\frac12)$	112211	$s_3s_4s_1s_2s_3s_4s_5s_6$

$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	111221	$s_5s_4s_1s_2s_3s_4s_5s_6$
$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	112221	$s_3s_5s_4s_1s_2s_3s_4s_5s_6$
$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	112321	$s_4s_3s_5s_4s_1s_2s_3s_4s_5s_6$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	122321	$s_2s_4s_3s_5s_4s_1s_2s_3s_4s_5s_6$

Case $\Phi = E_7$ with the order $\alpha_3,\ \alpha_7,\ \alpha_4,\ \alpha_6,\ \alpha_5,\ \alpha_2,\ \alpha_1$		
$(c_1,\ \ldots,\ c_8)$	$(b_1,\ \ldots,\ b_7)$	Reduced decomposition of u
$(0,0,0,0,-1,1,0,0)$	0000001	s_7
$(0,0,0,-1,0,1,0,0)$	0000011	s_6s_7
$(0,0,-1,0,0,1,0,0)$	0000111	$s_5s_6s_7$
$(0,-1,0,0,0,1,0,0)$	0001111	$s_4s_5s_6s_7$
$(1,0,0,0,0,1,0,0)$	0101111	$s_2s_4s_5s_6s_7$
$(-1,0,0,0,0,1,0,0)$	0011111	$s_3s_4s_5s_6s_7$
$(0,1,0,0,0,1,0,0)$	0111111	$s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1011111	$s_1s_3s_4s_5s_6s_7$
$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1111111	$s_2s_1s_3s_4s_5s_6s_7$
$(0,0,1,0,0,1,0,0)$	0112111	$s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1112111	$s_1s_4s_2s_3s_4s_5s_6s_7$
$(0,0,0,1,0,1,0,0)$	0112211	$s_5s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1112211	$s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(0,0,0,0,1,1,0,0)$	0112221	$s_6s_5s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1122111	$s_3s_1s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1112221	$s_6s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1122211	$s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1123211	$s_4s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1122221	$s_6s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1123221	$s_4s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1223211	$s_2s_4s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1123321	$s_5s_4s_3s_5s_1s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1223221	$s_6s_2s_4s_3s_5s_1$ $s_4s_2s_3s_4s_5s_6s_7$
$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	1223321	$s_5s_6s_2s_4s_3s_5s_1$ $s_4s_2s_3s_4s_5s_6s_7$

$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	1224321	$s_4s_5s_6s_2s_4s_3s_5s_1$ $s_4s_2s_3s_4s_5s_6s_7$
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	1234321	$s_3s_4s_5s_6s_2s_4s_3s_5s_1$ $s_4s_2s_3s_4s_5s_6s_7$
$(0,0,0,0,0,0,-1,1)$	2234321	$s_1s_3s_4s_5s_6s_2s_4s_3s_5s_1$ $s_4s_2s_3s_4s_5s_6s_7$

Case $\Phi = E_8$ with the order $\alpha_4,\ \alpha_8,\ \alpha_5,\ \alpha_7,\ \alpha_6,\ \alpha_3,\ \alpha_1,\ \alpha_1$		
$(c_1,\ \ldots,\ c_8)$	$(b_1,\ \ldots,\ b_7)$	Reduced decomposition of u
$(0,0,0,0,0,-1,1,0)$	00000001	s_8
$(0,0,0,0,-1,0,1,0)$	00000011	s_7s_8
$(0,0,0,-1,0,0,1,0)$	00000111	$s_6s_7s_8$
$(0,0,-1,0,0,0,1,0)$	00001111	$s_5s_6s_7s_8$
$(0,-1,0,0,0,0,1,0)$	00011111	$s_4s_5s_6s_7s_8$
$(-1,0,0,0,0,0,1,0)$	00111111	$s_3s_4s_5s_6s_7s_8$
$(1,0,0,0,0,0,1,0)$	01011111	$s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	10111111	$s_1s_3s_4s_5s_6s_7s_8$
$(0,1,0,0,0,0,1,0)$	01111111	$s_3s_2s_4s_5s_6s_7s_8$
$(0,0,1,0,0,0,1,0)$	01121111	$s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11111111	$s_1s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,1,0,0,1,0)$	01122111	$s_5s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11121111	$s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11122111	$s_5s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11221111	$s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,0,1,0,1,0)$	01122211	$s_6s_5s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11222111	$s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,0,0,1,1,0)$	01122221	$s_7s_6s_5s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11122211	$s_1s_6s_5s_1s_4s_3s_2$ $s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11122221	$s_7s_1s_6s_5s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11222211	$s_3s_1s_6s_5s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$

$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11232111	$s_4s_5s_3s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11232211	$s_6s_4s_5s_3s_1s_4$ $s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11222221	$s_7s_3s_1s_6s_5s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12232111	$s_2s_4s_5s_3s_1s_4s_3s_2$ $s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11233211	$s_5s_6s_4s_5s_3s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12232211	$s_6s_2s_4s_5s_3s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11232221	$s_7s_6s_4s_5s_3s_1s_4s_3$ $s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12233211	$s_5s_6s_2s_4s_5s_3s_1s_4$ $s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12232221	$s_2s_7s_6s_4s_5s_3s_1s_4$ $s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11233221	$s_7s_5s_6s_4s_5s_3s_1s_4$ $s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	11233321	$s_6s_7s_5s_6s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12233221	$s_5s_2s_7s_6s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12243211	$s_4s_5s_6s_2s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12343211	$s_3s_4s_5s_6s_2s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12243221	$s_7s_4s_5s_6s_2s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12233321	$s_6s_5s_2s_7s_6s_4s_5s_3s_1$ $s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12243321	$s_4s_6s_5s_2s_7s_6s_4s_5s_3$ $s_1s_4s_3s_2s_4s_5s_6s_7s_8$

$(0,0,0,0,0,-1,0,1)$	22343211	$s_1s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12343221	$s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,0,-1,0,0,1)$	22343221	$s_1s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12244321	$s_5s_4s_6s_5s_2s_7s_6s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12343321	$s_6s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,-1,0,0,0,1)$	22343321	$s_1s_6s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12344321	$s_5s_6s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,-1,0,0,0,0,1)$	22344321	$s_1s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	12354321	$s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	13354321	$s_2s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,-1,0,0,0,0,0,1)$	22354321	$s_1s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(1,0,0,0,0,0,0,1)$	23354321	$s_2s_1s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(-1,0,0,0,0,0,0,1)$	22454321	$s_3s_1s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,1,0,0,0,0,0,1)$	23454321	$s_2s_3s_1s_4s_5s_6s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,1,0,0,0,0,1)$	23464321	$s_4s_2s_3s_1s_4s_5s_6$ $s_7s_3s_4s_5s_6s_2s_4s_5$ $s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,1,0,0,0,1)$	23465321	$s_5s_4s_2s_3s_1s_4s_5s_6$ $s_7s_3s_4s_5s_6s_2s_4$ $s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$

$(0,0,0,0,1,0,0,1)$	23465421	$s_6s_5s_4s_2s_3s_1s_4s_5$ $s_6s_7s_3s_4s_5s_6$ $s_2s_4s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$
$(0,0,0,0,0,1,0,1)$	23465431	$s_7s_6s_5s_4s_2s_3s_1s_4$ $s_5s_6s_7s_3s_4s_5s_6$ $s_2s_4s_5s_3s_1s_4s_3s_2s_4s_5s_6s_7s_8$

All these tables were generated using computer algebra system SAGE [17]; the listing of the code can be found in the Appendix.

One can immediately check that s_β and u satisfy the conditions of Proposition 1. Hence, according to this proposition, β_i does not divide d_{w_i} in $\mathbb{C}[\mathfrak{h}]$ for $i = 1, 2$. On the other hand, formula (3) implies that, given $w \in W$, there exists $g \in \mathbb{C}[\mathfrak{h}]$ such that

$$d_w = c_w \cdot \prod_{\alpha\in\Phi^+} \alpha = g \cdot \prod_{\substack{\alpha\in\Phi^+,\\ s_\alpha\nleq w}} \alpha\,,$$

hence if $s_\alpha \neq w$ then α divides d_w. But if, for example, $s_{\beta_1} \nleq w_2$ then β_1 divides d_{w_2}. At the same time, β_1 does not divide d_{w_1}, thus, $d_{w_1} \neq d_{w_2}$. The proof is complete. □

Appendix

Below we present the listing of the code generating tables from the proof of Proposition 2 using computer algebra system SAGE.

```
rank=8 # the rank of the root system
column_number=8 # the number of the first column
W=WeylGroup(['E',rank],prefix='s', implementation='permutation')
ref=W.reflections()
s=W.simple_reflections()
R=RootSystem(['E',rank]).ambient_space();
simple_roots=R.simple_roots()
phi_plus=W.positive_roots()
C1=[]
C1el=[]
for i in range(0,len(phi_plus)):
  if phi_plus[i][column_number-1]!=0:
    C1.append(phi_plus[i])
    C1el.append(ref[i+1])
U=[s[column_number]]
Ulistver=[[column_number]]
for i in range(1,len(C1)):
  u=copy(Ulistver[i-1])
  index=-1
```

```
  difference=C1[i]-C1[i-1]
  difference_abs=[abs(ele) for ele in difference]
  if sum(difference_abs)!=1:
    b=0
    for j in range(i-2,1,-1):
      difference1=C1[i]-C1[j]
      difference_abs1=[abs(ele) for ele in difference1]
      if sum(difference_abs1)==1 and b==0:
        b=1
        difference=C1[i]-C1[j]
        difference_abs=[abs(ele) for ele in difference]
        u=copy(Ulistver[j])
  if sum(difference)==1:
    for j in range(0,len(difference)):
      if difference[j]==1:
        index=j
  u1=[]
  u2=s[column_number]*s[column_number]
  if index!=-1:
    u1=[index+1]
    u2=s[index+1]
    for j in range(0,len(u)):
      u1.append(u[j])
      u2=u2*s[u[j]]
  U.append(u2)
  Ulistver.append(u1)
list_of_indexes=[]
for i in range(1,rank+1):
  if i!=column_number:
    list_of_indexes.append(i)
for u in U:
  b=1
  for i in list_of_indexes:
    u1=u*s[i]
    if u1.length()<u.length():
      b=0
    if b==0:
      print('false')
      print(u)
V=[s[column_number]*s[column_number]]
Vlistver=[[]]
for i in range(1,len(Ulistver)):
  u=copy(Ulistver[i])
  v1=[]
  v2=s[column_number]*s[column_number]
  for j in range(len(u)-2,-1,-1):
```

```
    v1.append(u[j])
    v2=v2*s[u[j]]
  V.append(v2)
  Vlistver.append(v1)
C1roots=[R.simple_root(column_number)]
for i in range(1,len(Vlistver)):
  root=R.simple_root(column_number)
  for j in range(0,len(Vlistver[i])):
    root=root+R.simple_root(Vlistver[i][j])
  C1roots.append(root)
for i in range(0,len(U)):
  u=U[i]
  v=V[i]
  r=C1el[i]
  if (u*v!=r) or (u.length()+v.length()!=r.length()):
    print(false)
for i in range(0,len(U)):
  print(C1roots[i],C1[i],U[i])
```

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