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Mathematica Bohemica, Vol. 145 (2020), No. 4, 401–414

Persistent URL: http://dml.cz/dmlcz/148432

SOME MONOUNARY ALGEBRAS WITH EKP

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Received September 27, 2018. Published online November 27, 2019.
Communicated by Simion Breaz

Abstract. An algebra $\mathcal{A}$ is said to have the endomorphism kernel property (EKP) if every congruence on $\mathcal{A}$ is the kernel of some endomorphism of $\mathcal{A}$. Three classes of monounary algebras are dealt with. For these classes, all monounary algebras with EKP are described.

Keywords: monounary algebra; endomorphism; congruence; kernel

MSC 2010: 08A60, 08A30, 08A35

1. Introduction

The notions of homomorphism and congruence in universal algebra are of cardinal importance. The well-known fundamental homomorphism theorem says that there is a correspondence between congruences on an algebra $\mathcal{A}$ and kernels of homomorphisms of the same algebra.

We deal with algebras possessing the endomorphism kernel property (EKP). An algebra is defined to have EKP if every congruence on $\mathcal{A}$ is the kernel of an endomorphism of $\mathcal{A}$. This notion was introduced in [2] for distributive lattices. EKP was studied in finite distributive lattices and de Morgan algebras (see [2]), Stone algebras and modular p-algebras (see [8]–[10]). Further, the strong EKP (i.e. each congruence of $\mathcal{A}$ is a kernel of a strong endomorphism of $\mathcal{A}$) was investigated in [3], [4], [6], [7], and [11]–[13].

We focus on EKP in monounary algebras. The importance of theory of unary and monounary algebras is pointed out for example in the monographs [24], [15], [19], [25]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. Several authors concentrate

The research has been supported by the grant VEGA 2/0044/16.

DOI: 10.21136/MB.2019.0128-18

on endomorphisms of monounary algebras, see e.g. [1], [5], [16], [17], [18], [21], [26], [27] and of injective monounary algebras (see [20], [22], [23]).

The main result of the paper is a characterization of a monounary algebra $\mathcal{A}$ with EKP if

(i) $\mathcal{A}$ consists of finitely many connected components (Theorem 3.1),
(ii) $\mathcal{A}$ is injective (Theorem 4.1),
(iii) each cyclic element of $\mathcal{A}$ has only finitely many ancestors (Theorem 5.1 and 5.2).

2. Preliminaries

The set of all positive integers is denoted by $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of a set A is denoted by $\|A\|$. If ψ is a mapping from a set A into a set B, then $\ker(\psi)$ denotes the kernel of ψ.

We deal with monounary algebras. The fundamental operation will be mostly denoted by f. The identity operation is denoted by id.

Let $\mathcal{A} = (A, f)$ be a monounary algebra. We denote by $\mathbf{S}(\mathcal{A})$ the class of all algebras that are isomorphic to a subalgebra of $\mathcal{A}$. The algebra $\mathcal{A}$ is *connected* if for every $a, b \in A$ there exist $m, n \in \mathbb{N}$ such that $f^n(a) = f^m(b)$. We say that a set B is a *component* of the algebra $\mathcal{A}$ if B has the following properties:

(1) $B \subseteq A$,
(2) $f(B) \subseteq B$,
(3) (B, f) is connected,
(4) if $a \in A$ is such that $f(a) \in B$, then $a \in B$.

If $\|A\| = 1$, then the algebra $\mathcal{A}$ is called *trivial.* A component B of an algebra $\mathcal{A}$ is called trivial if the algebra (B, f) is trivial.

We say that a set C is a *cycle* of the algebra $\mathcal{A}$ if C has the following properties:

(1) C is a finite subset of A,
(2) $f(C) = C$,
(3) (C, f) is connected.

If C is a cycle, then $\|C\|$ is called the length of the cycle C. Algebra $\mathcal{A}$ is called a cycle if A is a cycle of $\mathcal{A}$.

A subset B of A is termed as a *chain of* $\mathcal{A}$ if for every $a, b \in B$ there is $n \in \mathbb{N}_0$ such that either $f^n(a) = b$ or $f^n(b) = a$. If A is a chain of $\mathcal{A}$, then we will say that $\mathcal{A}$ is a *basic algebra.* Basic monounary algebras were introduced in [14].

A connected monounary algebra with a one-element cycle is called a *root monounary algebra* or simply a *root*, cf. [15]. By a c-root we mean the root with the cycle $\{c\}$.

Let $b \in A$. We denote

$$\begin{aligned} f^{-1}(b) &= \{a \in A\colon\ f(a) = b\},\\ \downarrow b &= \{a \in A\colon\ f^k(a) = b \text{ for some } k \in \mathbb{N}_0\},\\ \uparrow b &= \{f^k(b), k \in \mathbb{N}\},\\ C_{\mathcal{A}} &= \{a \in A\colon\ a \text{ is an element of some cycle of } \mathcal{A}\},\\ C^*_{\mathcal{A}} &= \{a \in A\colon\ f(a) = a\}. \end{aligned}$$

Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$. Let κ be a cardinal number. The following condition is denoted by (γ).

(γ) If $\mathcal{A}$ contains κ cycles of length k, then $\mathcal{A}$ contains $\kappa \cdot \aleph_0$ cycles of length l.

Let $\mathcal{B} = (B, f)$ and $A \cap B = \emptyset$. The algebra $(A \cup B, f)$ will be denoted by $\mathcal{A} + \mathcal{B}$. Let $a \in A$, $b \in B$. Let $\mathcal{A}$ be an a-root and $\mathcal{B}$ be a b-root. Then

$$\mathcal{A} \oplus \mathcal{B} = ((A \cup B) \setminus \{b\}, g),$$

where

$$g(x) = \begin{cases} f(x) & \text{if } x \in (A \cup B) \setminus f^{-1}(b),\\ a & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (A, f)$. Now we present three lemmas without proofs; they are easy to show directly from definitions.

Lemma 2.1. *Let $\mathcal{B} = (B, f)$ be a subalgebra of $\mathcal{A}$ and $\theta \in \mathrm{Con}(\mathcal{B})$. If $\theta' = \theta \cup \{(a, a), a \in A \setminus B\}$, then $\theta' \in \mathrm{Con}(\mathcal{A})$.*

Lemma 2.2. *Let $\mathcal{B} = (B, f) \in \mathbf{S}(\mathcal{A})$. Then the following statements are valid:*

(1) *If $k \in \mathbb{N}$, κ is a cardinal number and $\mathcal{B}$ contains κ cycles of length k, then $\mathcal{A}$ contains κ cycles of length k.*

(2) *If $c \in C^*_{\mathcal{B}}$, then there exists $c' \in C^*_{\mathcal{A}}$ such that*

$$\|f^{-1}(c)\| \leqslant \|f^{-1}(c')\|.$$

(3) *If f is injective on the set $A \setminus C_{\mathcal{A}}$, then f is injective on the set $B \setminus C_{\mathcal{B}}$.*

(4) *Let $\mathcal{D} = (D, \mathrm{id})$ and $D \cap B = \emptyset$. If $\|D\| + \|C^*_{\mathcal{B}}\| \leqslant \|C^*_{\mathcal{A}}\|$, then $\mathcal{B} + \mathcal{D} \in \mathbf{S}(\mathcal{A})$.*

(5) *Let $\mathcal{B}$ be connected and its operation be not injective.*
If $\mathcal{A} = \mathcal{D}_1 + \mathcal{D}_2$ and the operation of $\mathcal{D}_1$ is injective, then $\mathcal{B}$ is a subalgebra of $\mathcal{D}_2$.

Lemma 2.3. *The following statements are valid:*

(1) *If $\mathcal{A}$ is connected, then any homomorphic image of $\mathcal{A}$ is connected.*
(2) *Let $k \in \mathbb{N}$. If $\mathcal{A}$ is a cycle of length k and $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$, then $\mathcal{B}$ is a cycle of length l, where l divides k.*
(3) *Let κ be the number of components of $\mathcal{A}$. Then the algebra (B, id) such that $\|B\| \leqslant \kappa$ is a homomorphic image of $\mathcal{A}$.*
(4) *Let $k \in \mathbb{N}$. If $\mathcal{A}$ is connected without a cycle, then there are at least two homomorphic images of $\mathcal{A}$ such that they are basic algebras with a cycle of length k.*
(5) *If $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ such that $C^*_{\mathcal{B}} \neq \emptyset$, then $\mathcal{B}$ is a homomorphic image of $\mathcal{A} + \mathcal{D}$ for every algebra $\mathcal{D}$.*

Lemma 2.4. *Let $\{A_i\colon\ i \in I\}$ be the set of all components of $\mathcal{A}$ without a cycle. Let $a_i \in A_i$ be such that $f^{-1}(a_i) \neq \emptyset$ for every $i \in I$.*

Then there exist $c \notin A$ and a c-root $\mathcal{D} = (D, g)$ and a homomorphism φ from $\mathcal{A}$ onto $\mathcal{D}$ such that

(1) *φ is injective on the set $A \setminus \Big(C_{\mathcal{A}} \cup \bigcup\limits_{i\in I} \uparrow a_i\Big)$,*
(2) $\|g^{-1}(c)\| = \sum\limits_{x\in C_{\mathcal{A}}} (\|f^{-1}(x)\| - 1) + \sum\limits_{i\in I} \|f^{-1}(a_i)\| + 1.$

P r o o f. Put $D = \{c\} \cup \Big[A \setminus \Big(C_{\mathcal{A}} \cup \bigcup\limits_{i\in I} \uparrow a_i\Big)\Big]$. For $z \in D$ put

$$g(z) = \begin{cases} c & \text{if } z = c \text{ or } f(z) \in C_{\mathcal{A}} \text{ or } f(z) = a_i \text{ for some } i \in I, \\ f(z) & \text{otherwise.} \end{cases}$$

Then the algebra (D, g) is the c-root such that

$$\|g^{-1}(c)\| = \sum_{x\in C_{\mathcal{A}}} (\|f^{-1}(x)\| - 1) + \sum_{i\in I} \|f^{-1}(a_i)\| + 1.$$

It is a homomorphic image of $\mathcal{A}$ since

$$\varphi(x) = \begin{cases} x & \text{if } x \in D, \\ c & \text{otherwise,} \end{cases}$$

is a homomorphism from $\mathcal{A}$ onto (D, g). $\square$

3. Several properties of algebras with EKP

We say that an algebra $\mathcal{A}$ has an *endomorphism kernel property* if every congruence relation on $\mathcal{A}$ is a kernel of some endomorphism of $\mathcal{A}$, i.e.

$$\mathrm{Con}(\mathcal{A}) = \{\ker(\varphi)\colon\ \varphi \text{ is an endomorphism of } \mathcal{A}\}.$$

Shortly, we will write that $\mathcal{A}$ *has EKP*.

The next lemma is a very useful tool for manipulation with EKP in monounary algebras. It will be often used in this paper.

Lemma 3.1. *The algebra $\mathcal{A}$ has EKP if and only if $\mathcal{B} \in \mathbf{S}(\mathcal{A})$ for every homomorphic image $\mathcal{B}$ of $\mathcal{A}$.*

Proof. It follows immediately from Lemma 2.1 of [13]. □

Example 3.1.

(1) An n-element cycle, $n > 1$, has not EKP.
(2) Let $A \neq \emptyset$. Algebras (A, id), (A, const), where const is a constant operation, have EKP.
(3) Let κ be an infinite cardinal. Let $\mathcal{A} = (A, f)$ be such that
   (a) $f^2(x) = f(x)$ for every $x \in A$,
   (b) $\mathcal{A}$ consists of at most κ components,
   (c) every component has the cardinality κ.
   Then $\mathcal{A}$ has EKP.

Lemma 3.2. *Let $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, \mathrm{id})$ and $A \cap B = \emptyset$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ has EKP,*
(ii) *$\mathcal{A} + \mathcal{B}$ has EKP.*

Lemma 3.3. *Let $\mathcal{A} = (A, f)$ have EKP. If B is a component of $\mathcal{A}$, then the algebra $(A \setminus B, f)$ has EKP.*

Proof. Denote $D = A \setminus B$ and $\mathcal{D} = (D, f)$. Let $\theta \in \mathrm{Con}\mathcal{D}$. Consider $\theta' = \theta \cup \{(b, b), b \in B\}$. Then $\theta' \in \mathrm{Con}\mathcal{A}$ according to Lemma 2.1. The assumption $\mathcal{A}$ has EKP implies that there exists an endomorphism φ of $\mathcal{A}$ such that $\ker(\varphi) = \theta'$. We have that $\ker(\varphi|D) = \theta$ and φ is injective on B. If $\varphi(D) \subseteq D$, then $\varphi|D$ is an endomorphism of $\mathcal{D}$.

Suppose that $\varphi(D) \cap B \neq \emptyset$. Denote $E = \varphi^{-1}(B)$. Then E consists of some components of $\mathcal{A}$. Let $\mathcal{B}' = (B', f)$ be a component of $\mathcal{A}$ such that $\varphi(B) \subseteq B'$.

Assume that there exists $a \in D$ such that $\varphi(a) \in B'$. Take $b \in B$. Then there exist $m, n \in \mathbb{N}$ such that $f^n(\varphi(a)) = f^m(\varphi(b))$ since $\mathcal{B}'$ is connected. Therefore $\varphi(f^n(a)) = \varphi(f^m(b))$. We have

$$f^n(a) \notin B,\ f^m(b) \in B \text{ and } (f^n(a), f^m(b)) \in \ker(\varphi),$$

a contradiction. Hence $\varphi(D) \cap B' = \emptyset$.

Take $\mathcal{B}'' = (\varphi(B), f)$. We have $\mathcal{B}'' \cong \mathcal{B}$ because φ is injective on B. Let us define ε, the mapping from A into A, such that

$$\varepsilon(x) = \begin{cases} x & \text{if } x \in B, \\ \varphi^2(x) & \text{if } x \in E, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

We obtain that ε is an endomorphism of $\mathcal{A}$, $\ker(\varepsilon) = \theta'$ and $\varepsilon|D$ is an endomorphism of $\mathcal{D}$, $\ker(\varepsilon|D) = \theta$. □

Lemma 3.4. *Let $\mathcal{A} = (A, f)$ have EKP. Then the algebra $\mathcal{A}$ satisfies condition (γ).*

Proof. Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$. Let κ be a cardinal number.

Suppose that $\mathcal{A}$ contains κ cycles of length k. Then these κ cycles can be homomorphically mapped onto κ cycles of length l. That means $\mathcal{A}$ contains κ cycles of length k and κ cycles of length l. These 2κ cycles can be mapped by a homomorphism onto 2κ cycles of length l. Therefore $\mathcal{A}$ contains κ cycles of length k and 2κ cycles of length l, etc. It yields that $\mathcal{A}$ contains $\aleph_0 \cdot \kappa$ cycles of length l. □

Let $\{A_i\colon\ i \in I\}$ be the component partition of $\mathcal{A}$.

Lemma 3.5. *Let $\mathcal{A} = (A, f)$ have EKP. Then $\|C_{\mathcal{A}}\| = \|C^*_{\mathcal{A}}\| = \|I\|$.*

Proof. Every connected monounary algebra contains at most one cycle. This yields $\|C^*_{\mathcal{A}}\| \leqslant \|I\|$.

The algebra (I, id) is a homomorphic image of $\mathcal{A}$ and $C^*_{\mathcal{A}} \subseteq C_{\mathcal{A}}$. It implies that $\|C_{\mathcal{A}}\| \geqslant \|C^*_{\mathcal{A}}\| \geqslant \|I\|$ in view of Lemma 3.1.

Let I be finite. Then every component of $\mathcal{A}$ contains a 1-element cycle according to $(I, \mathrm{id}) \in \mathbf{S}(\mathcal{A})$. Thus $\|C_{\mathcal{A}}\| = \|C^*_{\mathcal{A}}\|$.

Let I be infinite. Then

$$\|C_{\mathcal{A}}\| \leqslant \|I\| \sum_{k=1}^{\infty} k = \|I\| \cdot \aleph_0 = \|I\|.$$

□

Corollary 3.1. *Let $\mathcal{A}$ be a basic algebra. The algebra $\mathcal{A}$ has EKP if and only if $\mathcal{A}$ has a 1-element cycle.*

P r o o f. If $\mathcal{A}$ contains a 1-element cycle, then it has EKP by Lemma 3.1. If $\mathcal{A}$ has EKP, then the previous assertion gives $C^*_{\mathcal{A}} \neq \emptyset$. $\square$

Theorem 3.1. *Let I be a finite set and $\{A_i\colon\ i \in I\}$ be the component partition of the algebra $\mathcal{A}$. The algebra $\mathcal{A}$ has EKP if and only if*

(1) *the algebra (A_i, f) has EKP for every $i \in I$,*
(2) *if $J \subseteq I$, $J = \{j_1, \ldots, j_m\}$, then there exists $j \in J$ such that*

$$\mathcal{A}_{j_1} \oplus \ldots \oplus \mathcal{A}_{j_m} \in \mathbf{S}(\mathcal{A}_j).$$

P r o o f. Suppose that $\mathcal{A}$ has EKP. In view of Lemma 3.2 we suppose that $\mathcal{A}$ has no trivial component. The first assertion follows from Lemma 3.3.

Denote $\mathcal{D} = (D, f) = \mathcal{A}_{j_1} \oplus \ldots \oplus \mathcal{A}_{j_m}$. Consider the algebra $\mathcal{A}'$ that has a component partition $\{D\} \cup \{A_i\colon\ i \in I \setminus J\}$. Then $\mathcal{A}'$ is a homomorphic image of $\mathcal{A}$. In view of Lemma 3.1 we have $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. Therefore there exists $k \in I$ such that $\mathcal{D} \in \mathbf{S}(\mathcal{A}_k)$. If $k \in J$, then condition (2) is satisfied. Assume that $k \in I \setminus J$. Then there exists $k_1 \in I \setminus \{k\}$ such that $\mathcal{A}_k \in \mathbf{S}(\mathcal{A}_{k_1})$ according to $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. We have $\mathcal{D} \in \mathbf{S}(\mathcal{A}_k) \subseteq \mathbf{S}(\mathcal{A}_{k_1})$. Therefore if $k_1 \in J$, then condition (2) is satisfied. If $k_1 \in I \setminus J$, then we will continue to take $k_2 \in I \setminus \{k, k_1\}$ such that $\mathcal{A}_k \in \mathbf{S}(\mathcal{A}_{k_1})$ according to $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. After at most $\|I\| - m$ steps we obtain an element from J.

Suppose that (1), (2) are valid. Let $\mathcal{B} = (B, f)$ be a homomorphic image of $\mathcal{A}$, the mapping φ be a corresponding homomorphism and $\{B_k, k \in K\}$ be a component partition of $\mathcal{B}$. Define $\psi\colon I \to K$ such that if $\varphi(A_i) \subseteq B_k$, then $\psi(i) = k$. Take $k \in K$. Denote $L = \psi^{-1}(k)$. If $\|L\| = 1$, then $\varphi(A_{\psi^{-1}(k)}) = B_k$ and $\mathcal{B}_k \in \mathbf{S}(\mathcal{A}_{\psi^{-1}(k)})$ according to (1). Let $\|L\| > 1$. Then $\bigcup\limits_{i \in L} \varphi(A_i) = B_k$. Take $j \in L$ such that $\sum\limits_{i \in L} \mathcal{A}_i \in \mathbf{S}(\mathcal{A}_j)$ according to (2). We obtain $\varphi(A_i) \in \mathbf{S}(A_i) \subseteq \mathbf{S}(\mathcal{A}_j)$ for every $i \in L$. Therefore $B_k = \bigcup\limits_{i \in L} \varphi(A_i) \in \mathbf{S}(\mathcal{A}_j)$. $\square$

Corollary 3.2. *Let $\mathcal{A}$ consist of finitely many components. If $\mathcal{A}$ has EKP, then there exists at most one $c \in C^*_{\mathcal{A}}$ such that*

$$1 < \|f^{-1}(c)\| < \aleph_0.$$

Corollary 3.3. *Let $\mathcal{A}$ be such that*

(1) *$\mathcal{A}$ consists of finitely many components,*
(2) *$f^2(a) = f(a)$ for every $a \in A$,*
(3) *at most one component of $\mathcal{A}$ is finite.*

Then $\mathcal{A}$ has EKP.

4. Injective algebras

A monounary algebra is called *injective* if its fundamental operation is injective. Finite components of injective algebras are cycles and infinite components contain no cycle. If $C_{\mathcal{A}} \neq \emptyset$, then the algebra $(C_{\mathcal{A}}, f)$ is injective.

In this section, we describe all injective monounary algebras with EKP. Then a method how to obtain a new algebra with EKP from an injective one will be derived. Namely, we will see that an algebra with EKP that has finitely many components can be added.

Denote by $\mathcal{I}$ the class of all injective monounary algebras.

Theorem 4.1. *Let $\mathcal{A} \in \mathcal{I}$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ has EKP.*
(ii) *Every component of $\mathcal{A}$ is a cycle and $\mathcal{A}$ satisfies (γ).*

Proof. Let (i) be valid. Assume that $\mathcal{A}$ contains a component without a cycle. Then this component can be mapped by a homomorphism onto a connected monounary algebra with a cycle and a non-injective operation according to Lemma 2.3 (4). That means $\mathcal{A}$ has not EKP by Lemma 3.1. Condition (γ) is valid by Lemma 3.4.

Suppose that (ii) is valid. Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$. Then every component of $\mathcal{B}$ is a cycle. For every $i \in \mathbb{N}$ we denote by ν_i the number of cycles of length i of $\mathcal{A}$ and by μ_i the number of cycles of length i of $\mathcal{B}$.

Assume that $i \in \mathbb{N}$. We need to prove that $\mu_i \leqslant \nu_i$. Then (i) is satisfied according to Lemma 3.1. In view of Lemma 2.3 (2) we have

$$\mu_i \leqslant \sum_{j \in \mathbb{N}} \nu_{i \cdot j}.$$

Suppose that there exists $j \neq 1$ such that $\nu_{i \cdot j} \neq 0$. Then condition (γ) implies that $\nu_i \geqslant \aleph_0$ and $\nu_{i \cdot k} \leqslant \nu_i$ for every $k \in \mathbb{N}$. Therefore

$$\sum_{j \in \mathbb{N}} \nu_{i \cdot j} = \nu_i.$$

□

Corollary 4.1. *If $\mathcal{A}$ has EKP, then the algebra $(C_{\mathcal{A}}, f)$ has EKP.*

Proof. We have $(C_{\mathcal{A}}, f) \in \mathcal{I}$. The assertion follows from Lemma 3.4, Lemma 3.5 and the previous theorem. $\square$

Theorem 4.2. *Let algebras $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, f)$ be such that*

(a) $\mathcal{A} \in \mathcal{I}$,

(b) $A \cap B = \emptyset$,

(c) *$\mathcal{B}$ consists of finitely many components,*

(d) *$(D, f) \notin \mathcal{I}$ for every component D of $\mathcal{B}$.*

Then the following statements are equivalent:

(1) *The algebra $\mathcal{A} + \mathcal{B}$ has EKP.*

(2) *Algebras $\mathcal{A}$, $\mathcal{B}$ have EKP.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ have EKP. Thus, $\mathcal{A}$ consists of cycles according to Theorem 4.1 and every component of $\mathcal{B}$ has a cycle according to Lemma 3.5. Assume that ε is a homomorphism from $\mathcal{A}+\mathcal{B}$ onto $\mathcal{D}' = (D', f)$. Then $\varepsilon(B)$ is a union of some components of $\mathcal{D}'$ according to (a). Therefore $(\varepsilon(B), f) \in \mathbf{S}(\mathcal{B})$ since $\mathcal{B}$ has EKP. Further, the set $\varepsilon(A) \setminus \varepsilon(B)$ is a union of some components of $\mathcal{D}'$, too, and it determines a subalgebra of the algebra $(\varepsilon(A), f)$. Thus, the algebra $(\varepsilon(A) \setminus \varepsilon(B), f) \in \mathbf{S}(\mathcal{A})$ since $\mathcal{A}$ has EKP. We conclude that $\mathcal{D}' \in \mathbf{S}(\mathcal{A} + \mathcal{B})$.

Suppose that $\mathcal{A}+\mathcal{B}$ has EKP. Then $\mathcal{A}$ has EKP according to (c) and Lemma 3.3. Let n be the number of components of $\mathcal{B}$. Then $n \in \mathbb{N}$ according to (c) and the number of nontrivial roots of $\mathcal{A}+\mathcal{B}$ is at most n according to (a). Suppose that B' is a component of $\mathcal{B}$ that does not determine a root. In view of (d) take $b \in B'$ such that $\|f^{-1}(b)\| > 1$. Let $d \notin A \cup B$. For $x \in A \cup B$ we define

$$\zeta(x) = \begin{cases} d & \text{if } x \in \uparrow b, \\ x & \text{otherwise.} \end{cases}$$

Then the algebra $(\zeta(A \cup B), f)$, where $f(d) = d$, is a homomorphic image of $\mathcal{A}+\mathcal{B}$ that contains $n+1$ nontrivial roots. That means that this algebra is not a subalgebra of $\mathcal{A}+\mathcal{B}$, a contradiction. We obtain that every component of $\mathcal{B}$ is a nontrivial root. Let $C_{\mathcal{B}} = \{c_1, \ldots, c_n\}$.

Assume that φ is a homomorphism from $\mathcal{B}$ onto $\mathcal{D} = (D, f)$, $A \cap D = \emptyset$. Then every component of $\mathcal{D}$ is a root and $\mathcal{D}$ consists of at most n components. If every component of $\mathcal{D}$ is trivial, then $\mathcal{D} \in \mathbf{S}(\mathcal{B})$. Suppose that $\mathcal{D}$ contains nontrivial components. Let $d_1, \ldots, d_m$ be all cyclic points of $\mathcal{D}$ such that $\|f^{-1}(d_i)\| > 1$, $i = 1, \ldots, m$. Then $m \leqslant n$. Put

$$\psi(x) = \begin{cases} x & \text{if } x \in A, \\ \varphi(x) & \text{if } x \in B. \end{cases}$$

Then ψ is a homomorphism from $\mathcal{A}+\mathcal{B}$ onto $\mathcal{A}+\mathcal{D}$. Thus $\mathcal{A}+\mathcal{D} \in \mathbf{S}(\mathcal{A}+\mathcal{B})$. Assume that ξ is an embedding of $\mathcal{A}+\mathcal{D}$ into $\mathcal{A}+\mathcal{B}$. Then $\xi(d_i) \in B$ for every $i = 1, \ldots, m$ according to (a). Therefore $\mathcal{D} \in \mathbf{S}(\mathcal{B})$ according to $\left\| D \setminus \left(\bigcup\limits_{i=1}^{m} {\downarrow} d_i \right) \right\| + m \leqslant n$ and Lemma 2.2 (6). $\square$

The sum of two algebras with EKP need not have EKP:

Example 4.1. Let $A = \{a, a'\}$ and $f(a) = f(a') = a$. The algebra $\mathcal{A} = (A, f)$ has EKP. Let $B \cap A = \emptyset$ and $\mathcal{B} = (B, f)$ be isomorphic to $\mathcal{A}$. The algebra $\mathcal{A} + \mathcal{B}$ does not have EKP.

5. Class $\mathcal{F}$

Denote by $\mathcal{F}$ the class of all monounary algebras $\mathcal{A} = (A, f)$ such that the set $f^{-1}(a)$ is finite for every $a \in C_{\mathcal{A}}$. Thus $\mathcal{I} \subset \mathcal{F}$. In this section, we describe all algebras with EKP from the class $\mathcal{F}$. We will see that the non injective ones are exactly the algebras of the form $\mathcal{B} + \mathcal{D}$, where $\mathcal{B} \in \mathcal{I}$ has EKP and $\mathcal{D}$ is connected with EKP.

The next statement follows from the definition of $\mathcal{F}$ immediately.

Lemma 5.1. *Let $\mathcal{A} = (A, f) \in \mathcal{F}$. Then*

(1) *if $A \neq \bigcup\limits_{b \in C_{\mathcal{A}}} {\downarrow} b$, then $\mathcal{A}$ contains a component without a cycle,*

(2) *if $C^*_{\mathcal{A}} \neq \emptyset$ and $D = \bigcup\limits_{b \in C^*_{\mathcal{A}}} {\downarrow} b$, then one of the following cases occurs:*

  (a) *$D = A$;*

  (b) *$\mathcal{A} = (D, f) + \mathcal{B}$, where $C^*_{\mathcal{B}} = \emptyset$.*

We denote by $\mathcal{F}^\circ$ the class of all monounary algebras $\mathcal{A} \in \mathcal{F}$ such that every component of $\mathcal{A}$ has a 1-element cycle. Thus, $\mathcal{A} \in \mathcal{F}^\circ$ if and only if $C^*_{\mathcal{A}} \neq \emptyset$ and equality (2)(a) from Lemma 5.1 is valid.

We denote by $\mathcal{F}^*$ the class of all monounary algebras $\mathcal{A}$ such that $\mathcal{A} = (A, f) \in \mathcal{F}^\circ$ with $\mathcal{A} = \mathcal{B} \oplus \mathcal{D}$, where $\mathcal{B}$ is basic and $\mathcal{D}$ has a constant operation. The next assertion is obvious.

Lemma 5.2. *Let $\mathcal{A} = (A, f) \in \mathcal{F}^*$. Then*

(1) *$\mathcal{A}$ is connected,*

(2) *if $c \in C^*_{\mathcal{A}}$, then $f^{-1}(c) = A$ or $A \setminus f^{-1}(c)$ is a chain of $\mathcal{A}$.*

The next lemma says that connected monounary algebras with EKP from $\mathcal{F}$ are precisely the algebras of $\mathcal{F}^*$.

Theorem 5.1. *Let $\mathcal{A} \in \mathcal{F}$ be connected. Then the following conditions are equivalent*

(i) *$\mathcal{A}$ has EKP,*
(ii) *$\mathcal{A} \in \mathcal{F}^*$.*

Proof. Let $\mathcal{A}$ be nontrivial. The implication (ii) $\Rightarrow$ (i) follows from Lemma 3.1.

Suppose that (i) is valid and (ii) fails to hold. Then there is $c \in A$ such that $f(c) = c$ according to Lemma 3.5.

Let $d \in A \setminus \{c\}$ be such that $f^{-1}(d)$ has at least two elements. Then there exists $a \in A \setminus \{c\}$ such that

(1) $\|f^{-1}(a)\| > 1$,
(2) if $n \in \mathbb{N} \setminus \{1\}$ is such that $f^n(a) = c$ and $f^{n-1}(a) \neq c$, then for every $a_0 \in A \setminus \{c\}$ such that $f^{n-1}(a_0) = c$, the equality $\|f^{-1}(a_0)\| = 1$ is valid.

Take $B = (A \setminus \uparrow a) \cup \{c\}$ and for $x \in B$ put

$$g(x) = \begin{cases} c & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise.} \end{cases}$$

The algebra (B, g) is a homomorphic image of $\mathcal{A}$. The relationship $\|g^{-1}(c)\| > \|f^{-1}(c)\|$ is satisfied. Therefore $(B, g) \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction. We obtain that the set $f^{-1}(d)$ possesses at most one element. This is equivalent to injectivity of f on the set $A \setminus \{c\}$.

Assume that $A \setminus f^{-1}(c) \neq \emptyset$ and it is not a chain of $\mathcal{A}$. Then there are $a, b \in f^{-1}(c) \setminus \{c\}$ such that $a \neq b$, $f^{-1}(a) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$. Take $B' = A \setminus \{a\}$ and put for $x \in B'$

$$h(x) = \begin{cases} b & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise.} \end{cases}$$

The algebra (B', h) is a homomorphic image of the algebra $\mathcal{A}$ that is not isomorphic to a subalgebra of $\mathcal{A}$ according to Lemma 2.2 (3), a contradiction. $\square$

Lemma 5.3. *Let $\mathcal{A} \in \mathcal{F}^\circ$ be not connected. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ has EKP,*
(ii) *$\mathcal{A} = \mathcal{B} + \mathcal{D}$, where $\mathcal{B} = (B, \mathrm{id})$ and $\mathcal{D} \in \mathcal{F}^*$.*

P r o o f. Every algebra satisfying (ii) has EKP according to Lemmas 5.1 and 3.2. Let $\mathcal{A}$ have EKP. Assume that $\mathcal{A}$ contains more than one nontrivial component. Remark that $C_{\mathcal{A}} = C^*_{\mathcal{A}}$ since $\mathcal{A} \in \mathcal{F}^\circ$. Let $c \notin A$. Put $B = (A \cup \{c\}) \setminus C_{\mathcal{A}}$. For $x \in A$ define

$$\varphi(x) = \begin{cases} c & \text{if } x \in C_{\mathcal{A}}, \\ x & \text{otherwise} \end{cases}$$

and for $y \in \varphi(A)$ define

$$h(y) = \begin{cases} c & \text{if } y = c \text{ or } f(y) \in C_{\mathcal{A}}, \\ f(y) & \text{otherwise.} \end{cases}$$

Then $(\varphi(A), h)$ is a homomorphic image of $\mathcal{A}$. This algebra is a c-root. In view of Lemma 2.2 (2) we obtain that $\varphi(A) = B$ and $(B, h) \notin \mathbf{S}(\mathcal{A})$, a contradiction. Thus, $\mathcal{A}$ has exactly one nontrivial component $\mathcal{D} = (D, f)$.

Let $\mathcal{D}$ do not have EKP. That means $\mathcal{D}$ can be mapped by a homomorphism onto an algebra $\mathcal{D}'$ such that $\mathcal{D}' \notin \mathbf{S}(\mathcal{D})$ according to Lemma 3.1. The algebra $\mathcal{D}'$ is a connected element of $\mathcal{F}^\circ$. Therefore the operation of $\mathcal{D}'$ is not injective. Further, $\mathcal{D}'$ is a homomorphic image of $\mathcal{A}$ too and it is not isomorphic to a subalgebra of $\mathcal{A}$ according to Lemma 2.2 (5), a contradiction. We obtain $\mathcal{D} \in \mathcal{F}^*$ by Theorem 5.1. Therefore (ii) is valid. □

Theorem 5.2. *Let $\mathcal{A} \in \mathcal{F}$ be not connected. Then the following statements are equivalent:*

(i) *The algebra $\mathcal{A}$ has EKP.*

(ii) *Denote $D_1 = \bigcup_{b \in C^*_{\mathcal{A}}} \downarrow b$. Then*

  (a) *every component of $\mathcal{A}$ has a cycle,*

  (b) *$(C_{\mathcal{A}}, f)$ has EKP,*

  (c) *$C^*_{\mathcal{A}} \neq \emptyset$ and the algebra (D_1, f) has EKP,*

  (d) *if $\mathcal{A} \notin \mathcal{F}^\circ$, then $C_{\mathcal{A}} \setminus C^*_{\mathcal{A}} = A \setminus D_1$.*

(iii) *$\mathcal{A} = \mathcal{B} + \mathcal{D}$, where*

  (a) *$\mathcal{B} \in \mathcal{I}$,*

  (b) *$\mathcal{B}$ has EKP,*

  (c) *$\mathcal{D} \in \mathcal{F}^*$.*

P r o o f. If $\mathcal{A} \in \mathcal{I}$, then properties (i), (ii) and (iii) are equivalent according to Theorem 4.1. We suppose that $\mathcal{A} \notin \mathcal{I}$.

Let (i) be fulfilled. Then (ii)(b) is true according to Corollary 4.1. Therefore $C^*_{\mathcal{A}} \neq \emptyset$ in view of Theorem 4.1. We have $C_{\mathcal{A}} \setminus C^*_{\mathcal{A}} \subseteq A \setminus D_1$. Let a_i, $i \in I$, $c \notin A$

and $\mathcal{D}$ be as in Lemma 2.4. If (ii)(a) is not valid, then $I \neq \emptyset$ and $\sum_{i\in I} \|f^{-1}(a_i)\| > 0$. Take $b \in C_{\mathcal{A}}$. We obtain

$$\|g^{-1}(c)\| = \|f^{-1}(b)\| + \sum_{x\in C_{\mathcal{A}}\setminus\{b\}} (\|f^{-1}(x)\| - 1) + \sum_{i\in I} \|f^{-1}(a_i)\| > \|f^{-1}(b)\|.$$

That means that $\mathcal{D} \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction. If (ii)(d) is not valid, then there is $a \in C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}$ such that $\|f^{-1}(a)\| > 1$. Take $b_1 \in C^*_{\mathcal{A}}$. We obtain

$$\begin{aligned}\|g^{-1}(c)\| &= \|f^{-1}(b_1)\| + \sum_{x\in C_{\mathcal{A}}\setminus\{b_1\}} (\|f^{-1}(x)\| - 1) + \sum_{i\in I} \|f^{-1}(a_i)\| \\ &\geqslant \|f^{-1}(b_1)\| + (\|f^{-1}(a)\| - 1) + \sum_{x\in C_{\mathcal{A}}\setminus\{a,b_1\}} (\|f^{-1}(x)\| - 1) > \|f^{-1}(b_1)\|.\end{aligned}$$

That means that $\mathcal{D} \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction.

Let $\mathcal{D}_0$ be a homomorphic image of (D_1, f). Then $\mathcal{D}_0$ is a homomorphic image of $\mathcal{A}$ by Lemma 2.3 (5). Therefore $\mathcal{D}_0 \in \mathbf{S}(\mathcal{A})$. That means $\mathcal{D}_0 \in \mathbf{S}(D_1, f)$ in view of Lemma 5.1 (2). Hence (ii)(c) holds.

Now let (ii) be valid. Property (c) and Lemma 5.3 imply that there exist $B_1, D \subset D_1$ such that $(D_1, f) = (B_1, \mathrm{id}) + (D, f)$ and $(D, f) \in \mathcal{F}^*$. Denote $\mathcal{D} = (D, f)$; thus (iii)(c) is valid. If $\mathcal{A} \in \mathcal{F}^\circ$, then we denote $\mathcal{B} = (B_1, \mathrm{id})$, else we denote $\mathcal{B} = (B_1, \mathrm{id}) + (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}, f)$. We obtain

$$B_1 \cup (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}) \cup D = (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}) \cup D_1 = A$$

according to (d). The algebra $\mathcal{B}$ consists of all components of the algebra $(C_{\mathcal{A}}, f)$ except one 1-element cycle. Thus, $\mathcal{B}$ has EKP according to (ii)(b) and Theorem 4.1. We have shown that (ii) implies (iii).

If (iii) is satisfied, then $\mathcal{A}$ has EKP according to Lemma 5.2 and Theorem 4.2. □

References

[1] *J. Araújo, J. Konieczny*: Centralizers in the full transformation semigroup. Semigroup Forum *86* (2013), 1–31. zbl MR doi

[2] *T. S. Blyth, J. Fang, H. J. Silva*: The endomorphism kernel property in finite distributive lattices and De Morgan algebras. Commun. Algebra *32* (2004), 2225–2242. zbl MR doi

[3] *T. S. Blyth, J. Fang, L.-B. Wang*: The strong endomorphism kernel property in distributive double p-algebras. Sci. Math. Jpn. *76* (2013), 227–234. zbl MR

[4] *T. S. Blyth, H. J. Silva*: The strong endomorphism kernel property in Ockham algebras. Commun. Algebra *36* (2008), 1682–1694. zbl MR doi

[5] *M. Černegová, D. Jakubíková-Studenovská*: Reconstructability of a monounary algebra from its second centralizer. Commun. Algebra *45* (2017), 4656–4666. zbl MR doi

[6] *G. Fang, J. Fang*: The strong endomorphism kernel property in distributive p-algebras. Southeast Asian Bull. Math. *37* (2013), 491–497. zbl MR
[7] *J. Fang, Z.-J. Sun*: Semilattices with the strong endomorphism kernel property. Algebra Univers. *70* (2013), 393–401. zbl MR doi
[8] *H. Gaitán, Y. J. Cortés*: The endomorphism kernel property in finite Stone algebras. JP J. Algebra Number Theory Appl. *14* (2009), 51–64. zbl MR
[9] *J. Guričan*: The endomorphism kernel property for modular p-algebras and Stone lattices of order n. JP J. Algebra Number Theory Appl. *25* (2012), 69–90. zbl MR
[10] *J. Guričan*: A note on the endomorphism kernel property. JP J. Algebra Number Theory Appl. *33* (2014), 133–139. zbl
[11] *J. Guričan*: Strong endomorphism kernel property for Brouwerian algebras. JP J. Algebra Number Theory Appl. *36* (2015), 241–258. zbl
[12] *J. Guričan, M. Ploščica*: The strong endomorphism kernel property for modular p-algebras and for distributive lattices. Algebra Univers. *75* (2016), 243–255. zbl MR doi
[13] *E. Halušková*: Strong endomorphism kernel property for monounary algebras. Math. Bohem. *143* (2018), 161–171. zbl MR doi
[14] *D. Jakubíková-Studenovská, J. Pócs*: Cardinality of retracts of monounary algebras. Czech. Math. J. *58* (2008), 469–479. zbl MR doi
[15] *D. Jakubíková-Studenovská, J. Pócs*: Monounary Algebras. Pavol Josef Šafárik University, Košice, 2009. zbl
[16] *D. Jakubíková-Studenovská, K. Potpinková*: The endomorphism spectrum of a monounary algebra. Math. Slovaca *64* (2014), 675–690. zbl MR doi
[17] *D. Jakubíková-Studenovská, M. Šuličová*: Centralizers of a monounary algebra. Asian-Eur. J. Math. *8* (2015), Article No. 1550007, 12 pages. zbl MR doi
[18] *D. Jakubíková-Studenovská, M. Šuličová*: Green's relations in the commutative centralizers of monounary algebras. Demonstr. Math. *48* (2015), 536–545. zbl MR doi
[19] *B. Jónsson*: Topics in Universal Algebra. Lecture Notes in Mathematics 250. Springer, Berlin, 1972. zbl MR doi
[20] *J. Konieczny*: Green's relations and regularity in centralizers of permutations. Glasg. Math. J. *41* (1999), 45–57. zbl MR doi
[21] *J. Konieczny*: Second centralizers of partial transformations. Czech. Math. J. *51* (2001), 873–888. zbl MR doi
[22] *J. Konieczny*: Centralizers in the semigroup of injective transformations on an infinite set. Bull. Aust. Math. Soc. *82* (2010), 305–321. zbl MR doi
[23] *J. Konieczny*: Infinite injective transformations whose centralizers have simple structure. Cent. Eur. J. Math. *9* (2011), 23–35. zbl MR doi
[24] *M. Novotný, O. Kopeček, J. Chvalina*: Homomorphic Transformations: Why and Possible Ways to How. Masaryk University, Brno, 2012. doi
[25] *J. Pitkethly, B. Davey*: Dualisability. Unary Algebras and Beyond. Advances in Mathematics 9. Springer, New York, 2005. zbl MR doi
[26] *B. V. Popov, O. V. Kovaleva*: On a characterization of monounary algebras by their endomorphism semigroups. Semigroup Forum *73* (2006), 444–456. zbl MR doi
[27] *I. V. Pozdnyakova*: Semigroups of endomorphisms of some infinite monounary algebras. Mat. Metody Fiz.-Mekh. Polya *55* (2012), 29–38 (In Russian.); translated in J. Math. Sci. *190*, 658–668. zbl MR doi

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