Rahul Kumar; Atul Gaur

Avoidance principle and intersection property for a class of rings

Czechoslovak Mathematical Journal, Vol. 70 (2020), No. 4, 1191–1196

Persistent URL: http://dml.cz/dmlcz/148422

Czechoslovak Mathematical Journal, 70 (145) (2020), 1191–1196

AVOIDANCE PRINCIPLE AND INTERSECTION PROPERTY FOR A CLASS OF RINGS

Rahul Kumar, Atul Gaur, Delhi

Received August 8, 2019. Published online April 16, 2020.

Abstract. Let R be a commutative ring with identity. If a ring R is contained in an arbitrary union of rings, then R is contained in one of them under various conditions. Similarly, if an arbitrary intersection of rings is contained in R, then R contains one of them under various conditions.

Keywords: intersection property; avoidance principle

MSC 2020: 13A99, 13B30

1. Introduction

All rings considered below are commutative with nonzero identity. Let $\overline{S}$ denote the saturation of the multiplicative closed subset S of a ring R, let $\mathrm{Nil}(R)$ denote the set of all nilpotent elements in R, let $Z(R)$ denote the set of all zero divisors in R, and let $T(R)$ denote the total quotient ring of R. In this note, we extend the work of Gottlieb, see [1] from finite unions to infinite unions of overrings of an integral domain. The main theme of Gottlieb's paper was the following: If $A, A_1, A_2, \ldots, A_n$ are overrings of an integral domain R, where $A = R_{\bigcup_{i=1}^{m} \mathfrak{p}_i}$ for some prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ of R and where each $A_i = S_i^{-1}R$ for some multiplicatively closed subset S_i of R such that $A \subseteq \bigcup\limits_{i=1}^{n} A_i$, then $A \subseteq A_i$ for some i. In our first result, we show that there is no need for the following assumptions:

(1) A_i's are overrings of R.

(2) $A_i = S_i^{-1}R$ for some multiplicatively closed subset S_i of R for all i.

The first author was supported by the SRF grant from UGC India, Sr. No. 2061440976 and the second author was supported by the MATRICS grant from DST-SERB, No. MTR/2018/000707.

DOI: 10.21136/CMJ.2020.0360-19

(3) $A = S^{-1}R$, where S is the complement of finite union of prime ideals.

Interestingly, our proof is much simpler, slicker and direct. In particular, we prove that [1], Corollary 8 holds for arbitrary domain, that is, the assumption of Bézout domain can be dropped.

Our next motive is to discuss the question raised by Gottlieb in [1], Example 7. In [1], Proposition 5, Gottlieb proved that if A is a local overring of an integral domain R such that $A \supseteq R_{p_1} \cap R_{p_2} \cap \ldots \cap R_{p_n}$, then $A \supseteq R_{p_i}$ for some i. This motivated him to raise the question whether we can replace R_p's by $S^{-1}R$ in [1], Proposition 5. In [1], Example 7, he showed that the answer is no. Thus, the natural question arises when this $S^{-1}R$ form will work. Motivated by the work of Smith in [2], we give a condition under which the replacement is possible.

2. Results

We begin this section with the following avoidance theorem for integral domains.

Theorem 2.1. *Let R be an integral domain and $A_1, A_2, \ldots, A_s$ be integral domains containing R such that each A_i is a subring of a ring T. If A is any subring of T of the form $S^{-1}R$ for some multiplicative closed subset S of R such that $A \subseteq \bigcup_{i=1}^{s} A_i$, then $A \subseteq A_i$ for some i.*

Proof. If possible, suppose that A is not contained in any A_i. Then for all i there exists a maximal ideal $\mathfrak{m}_i$ of A_i such that A is not contained in $(A_i)_{\mathfrak{m}_i}$. Set $\mathfrak{n}_i = \mathfrak{m}_i \cap R$ for all i. Then $S \subseteq R \setminus \bigcap_{i=1}^{s} \mathfrak{n}_i$ as $A \subseteq \bigcup_{i=1}^{s} A_i$. Note that $\overline{S} = R \setminus \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$, where $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ is the family of all prime ideals of R which do not meet S. It follows that $\bigcap_{i=1}^{s} \mathfrak{n}_i \subseteq \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ because if $x \in \bigcap_{i=1}^{s} \mathfrak{n}_i \setminus \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$, then $x \in \overline{S}$, that is, $xt \in S$ for some $t \in R$, a contradiction. Now, it is easy to see that there exists a prime ideal $\mathfrak{p}$ of R such that $\bigcap_{i=1}^{s} \mathfrak{n}_i \subseteq \mathfrak{p} \subseteq \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$. Consequently, $\mathfrak{n}_j \subseteq \mathfrak{p} \subseteq \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ for some j. Therefore $S \subseteq \overline{S} \subseteq R \setminus \mathfrak{n}_j$ and thus $A = S^{-1}R \subseteq R_{\mathfrak{n}_j} \subseteq (A_j)_{\mathfrak{m}_j}$, a contradiction. □

Let R be a ring such that $\mathrm{Nil}(R) = Z(R)$. Then it is easy to see that R has a unique minimal prime ideal and $S^{-1}R \subseteq T(R)$ for all multiplicative closed subsets S of R. On the other hand, if there exists a ring T such that $S^{-1}R \subseteq T$ for all multiplicative closed subsets S of R, then $\mathrm{Nil}(R) = Z(R)$. To see this, if possible, suppose there exists $x \in Z(R) \setminus \mathrm{Nil}(R)$. Then there exists a prime ideal $\mathfrak{p}$ of R such that $x \notin \mathfrak{p}$. It follows that $x/1$ is a unit in $R_\mathfrak{p}$. Since $R_\mathfrak{p}$ is a subring of T, $x/1$ is a unit in T, a contradiction as $T(R) \subseteq T$ and so $x/1$ is a zero divisor in T. Thus, we conclude that for a ring R there exists a ring containing $S^{-1}R$ for all multiplicative closed subsets S

of R if and only if $\mathrm{Nil}(R) = Z(R)$. In particular, if $\mathrm{Nil}(R) = Z(R)$, then $T(R)$ is the smallest ring containing $S^{-1}R$ for all multiplicative closed subsets S of R. Motivated by this, we define the following:

Definition 2.1. Let R be a ring such that $\mathrm{Nil}(R) = Z(R)$. Then we say that

(i) R satisfies the intersection property of localizations if the following holds: Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of rings of the form $S_\alpha^{-1}R$, where S_α's are multiplicative closed subsets of R, and let A be a local ring of the form $S^{-1}R$ for some multiplicative closed subset S of R such that $\bigcap\limits_{\alpha\in\Lambda} A_\alpha \subseteq A$. Then $A_\alpha \subseteq A$ for some $\alpha \in \Lambda$.

(ii) R satisfies the avoidance principle for localizations if the following holds: Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of local rings of the form $S_\alpha^{-1}R$, where S_α's are multiplicative closed subsets of R, and let A be any ring of the form $S^{-1}R$ for some multiplicative closed subset S of R such that $A \subseteq \bigcup\limits_{\alpha\in\Lambda} A_\alpha$. Then $A \subseteq A_\alpha$ for some $\alpha \in \Lambda$.

The next theorem provides a necessary and sufficient condition for a ring R to satisfy the intersection property of localizations provided $\mathrm{Nil}(R) = Z(R)$.

Theorem 2.2. *Let R be a ring such that $\mathrm{Nil}(R) = Z(R)$. Then R satisfies the intersection property of localizations if and only if each prime ideal of R is the radical of some principal ideal of R.*

P r o o f. First suppose that each prime ideal of R is the radical of some principal ideal of R. Consider a local ring A and a family of rings $\{A_\alpha\}_{\alpha\in\Lambda}$ as defined in Definition 2.1 (i) such that $\bigcap\limits_{\alpha\in\Lambda} A_\alpha \subseteq A$. Let $\mathfrak{m}$ be the maximal ideal of A and $\{\mathfrak{m}_{\alpha\beta}\colon \mathfrak{m}_{\alpha\beta}$ is a prime ideal of $A_\alpha\}$ be the family of all prime ideals of A_α for all α. Let $\mathfrak{p}$ be the contraction of $\mathfrak{m}$ in R and $\mathfrak{p}_{\alpha\beta}$ be the contraction of $\mathfrak{m}_{\alpha\beta}$ in R for all β and for all α. Now, we assert that $\mathfrak{p} \subseteq \bigcup\limits_{\alpha,\beta} \mathfrak{p}_{\alpha\beta}$. If possible, suppose that $x \in \mathfrak{p}$ but not in any $\mathfrak{p}_{\alpha\beta}$. Then $x \in \overline{S}_\alpha$ for all α. It follows that $x/1$ is a unit in A_α for all α. Since each A_α is a subring of $R_{\mathrm{Nil}(R)}$, $1/x \in A_\alpha$ for all α and hence $1/x \in A$, a contradiction as $x/1 \in \mathfrak{m}$. Thus, our assertion holds. Since $\mathfrak{p} = \mathrm{Rad}(r)$ for some $r \in R$, $\mathfrak{p} \subseteq \mathfrak{p}_{\alpha\beta}$ for some α, β. Consequently, $A_\alpha \subseteq A$.

Conversely, suppose that R satisfies the intersection property of localizations. If possible, suppose that there is a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \neq \mathrm{Rad}(r)$ for all $r \in R$. Then for each $r \in \mathfrak{p}$ there exists a prime ideal $\mathfrak{p}_r$ of R containing r such that $\mathfrak{p}$ is not contained in $\mathfrak{p}_r$. Clearly, we have $\mathfrak{p} \subseteq \bigcup\limits_{r\in\mathfrak{p}} \mathfrak{p}_r$. Set $A = R_\mathfrak{p}$ and $A_r = R_{\mathfrak{p}_r}$ for all $r \in \mathfrak{p}$. Then $\bigcap\limits_{r\in\mathfrak{p}} A_r \subseteq A$ but A_r is not contained in A for any $r \in \mathfrak{p}$, a contradiction. $\square$

Note that if R is an integral domain, then the intersection property of localizations has the following compact form:

Let R be an integral domain. Then we say that R satisfies the intersection property of localizations if the following holds: Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of rings of the form $S_\alpha^{-1}R$, where S_α's are multiplicative closed subsets of R and let A be a local overring of R such that $\bigcap\limits_{\alpha\in\Lambda} A_\alpha \subseteq A$. Then $A_\alpha \subseteq A$ for some $\alpha \in \Lambda$.

Now, as an immediate consequence of Theorem 2.2, we have the following corollary.

Corollary 2.1. *Let R be an integral domain. Then R satisfies intersection property of localizations if and only if each prime ideal of R is the radical of some principal ideal of R.*

Let R be a ring. Then R is said to satisfy condition $(*)$ if the following holds: Let $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ be any family of prime ideals of R. If $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \subseteq \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R, then $\mathfrak{p}_\alpha \subseteq \mathfrak{p}$ for some $\alpha \in \Lambda$. In the next theorem, we prove the equivalence of condition $(*)$ and the avoidance principle for localizations.

Theorem 2.3. *Let R be a ring such that $\operatorname{Nil}(R) = Z(R)$. Then R satisfies the avoidance principle for localizations if and only if R satisfies condition $(*)$.*

P r o o f. First suppose that R satisfies condition $(*)$. Consider a ring A and a family of local rings A_α as defined in Definition 2.1 (ii) such that $A \subseteq \bigcup\limits_{\alpha\in\Lambda} A_\alpha$. Let $\mathfrak{m}_\alpha$ be the maximal ideal of A_α and $\mathfrak{p}_\alpha$ be the contraction of $\mathfrak{m}_\alpha$ in R for all α. Further, suppose that $\{\mathfrak{q}_\beta\}_{\beta\in\Delta}$ is the family of all prime ideals of R which do not meet S. Then $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \subseteq \bigcup\limits_{\beta\in\Delta} \mathfrak{q}_\beta$, because as if $x \in \bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \setminus \bigcup\limits_{\beta\in\Delta} \mathfrak{q}_\beta$, then $x \in \overline{S} = R \setminus \bigcup\limits_{\beta\in\Delta} \mathfrak{q}_\beta$. Consequently, there exists $t \in R$ such that $xt \in S$, that is, $1/xt \in A$. It follows that $1/xt \in A_\alpha$ for some α, a contradiction as $xt/1 \in \mathfrak{m}_\alpha$. Note that there is a prime ideal $\mathfrak{p}$ of R such that $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \subseteq \mathfrak{p} \subseteq \bigcup\limits_{\beta\in\Delta} \mathfrak{q}_\beta$. By hypothesis, it follows that $\mathfrak{p}_\alpha \subseteq \bigcup\limits_{\beta\in\Delta} \mathfrak{q}_\beta$ for some $\alpha \in \Lambda$. Thus, $A \subseteq A_\alpha$.

Conversely, assume that R satisfies the avoidance principle for localizations. Let $\{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ be any family of prime ideals of R such that $\bigcap\limits_{\alpha\in\Lambda} \mathfrak{p}_\alpha \subseteq \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. Then $A \subseteq \bigcup\limits_{\alpha\in\Lambda} A_\alpha$, where $A = R_{\mathfrak{p}}$ and $A_\alpha = R_{\mathfrak{p}_\alpha}$ for all $\alpha \in \Lambda$. Thus, $A \subseteq A_\alpha$ for some $\alpha \in \Lambda$ and so $\mathfrak{p}_\alpha \subseteq \mathfrak{p}$. □

Let R be a ring. Then we say that R satisfies the intersection property of subrings if each subring V of R has the following property:

If $\{V_\alpha\}_{\alpha\in\Lambda}$ is any family of subrings of R such that $\bigcap\limits_{\alpha\in\Lambda} V_\alpha \subseteq V$, then $V_\alpha \subseteq V$ for some α.

Theorem 2.4. *A ring R satisfies the intersection property of subrings if and only if for each subring V of R there exists $x \in R \setminus V$ such that all the subrings not containing x are contained in V.*

Proof. Let R satisfy the intersection property of subrings. If possible, suppose there exists a subring V of R such that for each $x \in R \setminus V$ there exists a subring V_x of R not containing x such that V_x is not contained in V. Now, we assert that $\bigcap_{x \in R \setminus V} V_x \subseteq V$. If possible, take $y \in \Big(\bigcap_{x \in R \setminus V} V_x\Big) \setminus V$. Then $y \in V_y$, a contradiction. Thus, our assertion holds, which again contradicts the intersection property of subrings.

Conversely, assume that for each subring V of R there exists $x \in R \setminus V$ such that all subrings not containing x are contained in V. Let V be a subring of R and $\{V_\alpha\}_{\alpha \in \Lambda}$ be any family of subrings of R such that $\bigcap_{\alpha \in \Lambda} V_\alpha \subseteq V$. Then $x \notin \bigcap_{\alpha} V_\alpha$ and so $x \notin V_\alpha$ for some α. Thus, by assumption $V_\alpha \subseteq V$. □

Let R be a ring and V be a subring of R. Then we say that V *satisfies the intersection property of subrings in* R if the following holds:

If $\{V_\alpha\}_{\alpha \in \Lambda}$ is any family of subrings of R such that $\bigcap_{\alpha \in \Lambda} V_\alpha \subseteq V$, then $V_\alpha \subseteq V$ for some α.

Next, we offer the following companion for Theorem 2.4.

Corollary 2.2. *A subring V of a ring R satisfies the intersection property of subrings in R if and only if there exists $x \in R \setminus V$ such that all the subrings not containing x are contained in V.*

Example 2.1. Let $R = \mathbb{Q}$ and $V = \mathbb{Z}_{p\mathbb{Z}}$ for some prime p. Then it is easy to see that all the subrings of R not containing $1/p$ are contained in V. Thus, by Corollary 2.2, V satisfies the intersection property of subrings in R.

Let V be a subring of a ring R. Then we say that V is compact in R if the following holds: If $\{V_\alpha\}_{\alpha \in \Lambda}$ is any family of subrings in R such that $V \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$, then $V \subseteq V_\alpha$ for some α.

Theorem 2.5. *Let V be a subring of a ring R. Then V is compact in R if and only if there exists $x \in V$ such that the subrings of R which contains x must contain V.*

Proof. Let V be compact in R. Assume for each $x \in V$ there exists a subring V_x of R which contains x such that V is not contained in V_x. Clearly, we have $V \subseteq \bigcup_{x \in V} V_x$, which contradicts the compactness of V.

Conversely, assume that there exists $x \in V$ such that the subrings of R which contain x must contain V. Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be any family of subrings of R such that

$V \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$. Then $x \in V_\alpha$ for some α. Thus, by assumption, $V \subseteq V_\alpha$ and hence V is compact. $\square$

References

[1] *C. Gottlieb*: Finite unions of overrings of an integral domain. Available at https://projecteuclid.org/euclid.jca/1543654843.

[2] *W. W. Smith*: A covering condition for prime ideals. Proc. Am. Math. Soc. *30* (1971), 451–452. zbl MR doi

Authors' address: Rahul Kumar, Atul Gaur (corresponding author), Department of Mathematics, University of Delhi, New Academic Block, University Enclave, Delhi, 110007, India, e-mail: rahulkmr977@gmail.com, gaursatul@gmail.com.