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Somayeh Hadjirezaei; Sina Hedayat

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DECOMPOSITION OF FINITELY GENERATED MODULES USING FITTING IDEALS

Somayeh Hadjirezaei, Rafsanjan, Sina Hedayat, Kerman

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Abstract. Let R be a commutative Noetherian ring and M be a finitely generated R-module. The main result of this paper is to characterize modules whose first nonzero Fitting ideal is a product of maximal ideals of R, in some cases.

Keywords: Fitting ideal; torsion submodule; regular element

MSC 2020: 13C05, 13D05

1. Introduction

Let R be a commutative Noetherian ring with identity. Given any finitely generated R-module M, we can associate with M a sequence of ideals of R known as the Fitting invariants or Fitting ideals of M. The Fitting ideals are named after Fitting who investigated their properties in [4] in 1936.

Fitting ideal can provide us with useful information about the structure of a module. We will see that in some cases, if we know the Fitting ideals of a module, then we can determine the structure of the R-module completely. Even when this is not the case, the Fitting information can still help us to understand some interesting properties of modules.

For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus n} \xrightarrow{\varphi} M \longrightarrow 0\,, \tag{1.1}$$

where $R^{\oplus n}$ is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$ with λ in some index set Λ. Assume that A be the

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following matrix:

$$\begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}.$$

We call A the matrix presentation of the sequence (1.1). Let $\mathrm{Fitt}_i(M)$ be the ideal of R generated by the minors of size $n-i$ of matrix A. For $i \geqslant n$, $\mathrm{Fitt}_i(M)$ is defined as R and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is the invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M, see [4]. The ideal $\mathrm{Fitt}_i(M)$ will be called the ith Fitting ideal of the module M. It follows from the definition that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$ for every i. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $\mathrm{I}(M)$.

Fitting ideals are also used in mathematical physics. Einsiedler and Ward show how the dynamical properties of the system may be deduced from the Fitting ideals and they prove the entropy and expansiveness related with only the first Fitting ideal. This gives an easy computation instead of computing syzygy modules, see [3].

A partial list of important contributors to the theory of Fitting ideals includes the mathematicians Fitting, Buchsbaum, Lipman, Huneke, Katz, Northcott, Eisenbud (for references of each author see [1], [2], [4], [8], [9], [11]). Some recent works on Fitting ideals, due to the authors, are [5], [6] and [7].

In [6], it is shown that when $\mathrm{I}(M) = Q$ is a regular maximal ideal of R, then $M \cong R/Q \oplus P$ for some projective R-module P of constant rank if and only if $\mathrm{T}(M) \nsubseteq QM$ and so if M is an Artinian R-module and $\mathrm{I}(M) = Q$, then $M \cong R/Q$. In this paper we characterize modules whose first nonzero Fitting ideals are product of maximal ideals.

Throughout this paper, an element of R is called *regular* if it is a nonzero divisor and an ideal of R is *regular* if it contains a regular element. Let M be a finitely generated R-module. The *torsion submodule* of M, $\mathrm{T}(M)$, is the submodule of M consisting of all elements of M that are annihilated by a regular element of R. If $\mathrm{T}(M) = 0$, then M is called a *torsionfree R-module.*

2. Module whose first nonzero Fitting ideal is a product of maximal ideals

Let (R, P) be a local ring. In this section we investigate how useful Fitting ideals are in determining the underlying module. We study the behaviour of a module when the first nonzero Fitting ideal of it is a power of the maximal ideal P.

One of the frequent use of Fitting ideals is to say something about the annihilator of a module. See the following proposition.

Proposition 2.1. *If M is a finitely generated R-module which can be generated by n elements, then*

$$(\mathrm{Ann}(M))^n \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}(M).$$

Proof. See [2], Proposition 20.7. □

This basic theorem is generalized in [7], Lemma 2.5 as follows:

Lemma 2.2. *Let M be a finitely generated R-module. Then*

$$\mathrm{I}(M) \subseteq \mathrm{Ann}(\mathrm{T}(M)).$$

In what follows, $S(P^i)$ is the set of all elements of M, where $\mathrm{Ann}(x) = P^i$ for some positive integer i.

Theorem 2.3. *Let (R, P) be a Noetherian local ring and M be a finitely generated R-module with $\mathrm{T}(M) \neq 0$ and $\mathrm{I}(M) = P^n$ for a positive integer n. Then the following conditions are equivalent:*

(1) *$M = N \oplus \mathrm{T}(M)$ for a submodule N of M and $\mathrm{T}(M) \cong (R/P)^{\oplus m}$ for a positive integer m;*
(2) *$\mathrm{T}(M) \cap PM = 0$;*
(3) *$S(P) \cap PM = \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Let $M = N \oplus \mathrm{T}(M) \cong N \oplus (R/P)^{\oplus m}$ for a positive integer m. Since $PM = PN$, $\mathrm{T}(M) \cap PM \subseteq \mathrm{T}(M) \cap N = 0$.

(2) $\Rightarrow$ (3): Since $S(P) \subseteq \mathrm{T}(M)$, it is clear that $S(P) \cap PM = \emptyset$.

(3) $\Rightarrow$ (1): Let $\mathrm{I}(M) = P^n$ and $S(P) \cap PM = \emptyset$. By Lemma 2.2, $\mathrm{I}(M) = P^n \subseteq \mathrm{Ann}(\mathrm{T}(M))$. So

$$P = \sqrt{\mathrm{Ann}(\mathrm{T}(M))} = \bigcap_{\mathrm{Ann}(\mathrm{T}(M)) \subseteq Q} Q,$$

where $Q \in \mathrm{Ass}(\mathrm{T}(M))$. Since P is a maximal ideal, $\mathrm{Ass}(\mathrm{T}(M)) = \{P\}$. Therefore there exists an element $x_1 \in \mathrm{T}(M)$ such that $\mathrm{Ann}(x_1) = P$. Since $S(P) \cap PM = \emptyset$, $\{x_1\}$ can be extended to a minimal generating set for $\mathrm{T}(M)$. So there exists a submodule N_1 of M such that $\mathrm{T}(M) = \langle x_1 \rangle \oplus N_1 \cong R/P \oplus N_1$.

We have $P^n\mathrm{T}(M) = P^nN_1 = 0$. Thus, if $N_1 \neq 0$, by the same argument as above, we have $\mathrm{Ass}(N_1) = \{P\}$. So there exists an element $x_2 \in N_1$ such that $\mathrm{Ann}(x_2) = P$. Since $S(P) \cap PM = \emptyset$, there exists a submodule N_2 of M such that

$$N_1 = \langle x_2 \rangle \oplus N_2 \cong R/P \oplus N_2.$$

Continuing this process, we have $\mathrm{T}(M) = \langle x_1, \ldots, x_t \rangle \cong (R/P)^{\oplus t}$ for a positive integer t.

Let $\{x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+s}\}$ be a generating set for M such that for every i, $1 \leqslant i \leqslant s$, $x_{t+i} \notin \{x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+(i-1)}, x_{t+(i+1)}, \ldots, x_{t+s}\}$. Put $N = \{x_{t+1}, \ldots, x_{t+s}\}$. Let

$$r_1x_1 + \ldots + r_tx_t + r_{t+1}x_{t+1} + \ldots + r_{t+s}x_{t+s} = 0.$$

Since $x_{t+i} \notin \{x_1, \ldots, x_t, x_{t+1}, \ldots, x_{t+(i-1)}, x_{t+(i+1)}, \ldots, x_{t+s}\}$, $r_{t+i} \in P$ for every i, $1 \leqslant i \leqslant s$. So $r_1x_1 + \ldots + r_tx_t \in PM$. Since $S(P) \cap PM = \emptyset$ and

$$P \subseteq \mathrm{Ann}(r_1x_1 + \ldots + r_tx_t), r_1x_1 + \ldots + r_tx_t = 0.$$

Therefore $\langle x_1, \ldots, x_t \rangle \cap N = 0$. So $M = N \oplus \mathrm{T}(M)$. □

Let M be an R-module. A proper submodule N of M is said to be a *prime submodule* of M if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that $a \in N$ or $rM \subseteq N$.

Lemma 2.4. *Let M be an R-module and Q be a maximal ideal of R. If $QM \neq M$, then QM is a prime submodule of M.*

P r o o f. See [10], §1, Proposition 2. □

Now we try to generalize Theorem 2.3 to global cases. Note that the formation of Fitting ideal commutes with base change; that is for any map of rings $R \to S$, $\mathrm{Fitt}_j(M \otimes_R S) = (\mathrm{Fitt}_j M)S$. Therefore, for every prime ideal P of R and every i we have $\mathrm{Fitt}_i(M_P) = \mathrm{Fitt}_i(M)_P$. But the construction of $\mathrm{I}(M) = \mathrm{I}_{\mathrm{rank}\,\varphi}(\varphi)$ does not in general commute with localization because the rank of φ may decrease when we localize. However, if $\mathrm{I}(M)$ contains a regular element, then $\mathrm{rank}(\varphi) = \mathrm{rank}(\varphi_P)$ and so $\mathrm{I}(M_P) = \mathrm{I}(M)_P$ for every prime ideal P of R.

An R-module M is said to be *regular* if $\mathrm{I}(M)$ is a regular ideal.

Theorem 2.5. *Let R be a Noetherian ring and M be a finitely generated, regular R-module. Assume that $P_1, \ldots, P_m$ are distinct maximal ideals of R such that $\mathrm{T}(M_{P_i}) \neq 0$ for every $1 \leqslant i \leqslant m$ and $\mathrm{I}(M) = P_1^{n_1} \ldots P_m^{n_m}$ for some positive integers n_i, $1 \leqslant i \leqslant m$. Then the following conditions are equivalent:*

(1) $M = N \oplus \mathrm{T}(M)$ *for a submodule* N *of* M *and*

$$\mathrm{T}(M) \cong (R/P_1)^{\oplus n_1} \oplus \ldots \oplus (R/P_m)^{\oplus n_m}.$$

(2) $\mathrm{T}(M) \cap P_1P_2 \ldots P_mM = 0$.
(3) $S(P_i) \cap P_iM = \emptyset$, $i = 1, \ldots, m$.

P r o o f. (1) $\Rightarrow$ (2): Assume that

$$M = N \oplus \mathrm{T}(M) \cong N \oplus (R/P_1)^{\oplus n_1} \oplus \ldots \oplus (R/P_m)^{\oplus n_m}$$

for an R-submodule N and positive integer n_i, $1 \leqslant i \leqslant m$. Then

$$P_1P_2 \ldots P_mM = P_1P_2 \ldots P_mN.$$

Since $N \cap \mathrm{T}(M) = 0$, $\mathrm{T}(M) \cap P_1P_2 \ldots P_mM = 0$, $i = 1, \ldots, m$.

(2) $\Rightarrow$ (3): By contrapositive, fix i, $i = 1, \ldots, m$, and suppose there exists $x \in S(P_i) \cap P_iM$. Then $\mathrm{Ann}(x) = P_i$. Since $P_1, \ldots, P_m$ are distinct maximal ideals of R, one may choose $r \in P_1 \ldots P_{i-1}P_{i+1} \ldots P_m \setminus P_i$. So $\mathrm{Ann}(rx) = P_i$ and hence $0 \neq rx \in \mathrm{T}(M) \cap P_1 \ldots P_mM$.

(3) $\Rightarrow$ (1): Let $\mathrm{S}(P_i) \cap P_iM = \emptyset$ for $i = 1, \ldots, m$. Since $\mathrm{I}(M) = P_1^{n_1} \ldots P_m^{n_m}$ for every $i = 1, \ldots, m$, we have $\mathrm{I}(M_{P_i}) = \mathrm{I}(M)_{P_i} = P_i^{n_i}R_{P_i}$. We claim that $S(P_iR_{P_i}) \cap P_iM_{P_i} = \emptyset$. Let $x/1 \in S(P_1R_{P_1}) \cap P_1M_{P_1}$ and $P_1 = \langle a_1, \ldots, a_k \rangle$.

For $i = 1, \ldots, k$ we have

$$\frac{a_i}{1}\frac{x}{1} = \frac{0}{1}.$$

So there exist $t_i \in R \setminus P_1$ such that $t_ia_ix = 0$. Put $t = t_1 \ldots t_k$. Since $(x/1) \neq (0/1)$, $tx \neq 0$ and so $P_1 = \mathrm{Ann}(tx)$.

On the other hand, $x/1 \in P_1M_{P_1}$. Thus there exist $s_i \in R \setminus P_1$ and $x_i \in M$ such that

$$\frac{x}{1} = \frac{a_1}{s_1}\frac{x_1}{1} + \ldots + \frac{a_k}{s_k}\frac{x_k}{1}.$$

Therefore there exists $s \in R \setminus P_1$ such that $sx \in P_1M$. If $M = P_1M$, then $M_{P_1} = P_1M_{P_1}$. So by Nakayama's Lemma, $M_{P_1} = 0$, a contradiction because $\mathrm{I}(M_{P_1}) = P_1R_{P_1}$. Hence, by Lemma 2.4, P_1M is a prime submodule of M, so $x \in P_1M$ or $sM \subseteq P_1M$.

If $sM \subseteq P_1M$, then since $s \in R \setminus P_1$, $M_{P_1} = P_1M_{P_1}$, a contradiction. Thus $x \in P_1M$. Hence $tx \in S(P_1) \cap P_1M$, a contraction. Thus

$$S(P_1R_{P_1}) \cap P_1M_{P_1} = \emptyset.$$

Similarly, $S(P_iR_{P_i}) \cap P_iM_{P_i} = \emptyset$ for $i = 2, \ldots, m$. By hypothesis, $\mathrm{T}(M_{P_i}) \neq 0$ and therefore by Theorem 2.3, $M_{P_i} = (N_i)_{P_i} \oplus (R_{P_i}/P_iR_{P_i})^{\oplus n_i}$ for a submodule N_i of M.

Now we show that $\mathrm{T}(M_{P_1}) = \mathrm{T}(M)_{P_1}$. It is clear that $\mathrm{T}(M)_{P_1} \subseteq \mathrm{T}(M_{P_1})$. So let

$$\frac{0}{1} \neq \frac{x}{1} \in \mathrm{T}(M_{P_1}).$$

By Lemma 2.2, $P_1R_{P_1} = \mathrm{I}(M_{P_1}) \subseteq \mathrm{Ann}(\mathrm{T}(M_{P_1}))$. Thus $\mathrm{Ann}(x/1) = P_1R_{P_1}$. So there exists an element $t \in R \setminus P_1$ such that $\mathrm{Ann}(tx) = P_1$. Let $r \in P_1$ be a regular element. Hence $r(tx) = 0$. So $tx \in \mathrm{T}(M)$. Therefore

$$\frac{x}{1} = \frac{tx}{t} \in \mathrm{T}(M)_{P_1}.$$

This means that $\mathrm{T}(M_{P_1}) = \mathrm{T}(M)_{P_1}$.

Let

$$\mathrm{T}(M_{P_i}) = \Bigl\langle \frac{x_{1i}}{1}, \ldots, \frac{x_{n_i i}}{1} \Bigr\rangle, \quad 1 \leqslant i \leqslant m,$$

where

$$\mathrm{Ann}\Bigl(\frac{x_{ji}}{1}\Bigr) = P_iR_{P_i}.$$

By the same argument as above, there exist $t_{ji} \in R \setminus P_i$ such that $\mathrm{Ann}(t_{ji}x_{ji}) = P_i$ for every $1 \leqslant j \leqslant n_i$, $1 \leqslant i \leqslant m$.

Assume that $Q \neq P_1, \ldots, P_m$ is a maximal ideal of R. We have

$$\mathrm{I}(M_Q) = \mathrm{I}(M)_Q = R_Q.$$

So by [1], Lemma 1, M_Q is a free R_Q-module.

Thus $\mathrm{T}(M)_P = \langle t_{ji}x_{ji} \colon 1 \leqslant j \leqslant n_i, 1 \leqslant i \leqslant m \rangle_P$ for every maximal ideal P of R. Since $\mathrm{Ann}(t_{ji}x_{ji}) = P_i$,

$$\mathrm{T}(M) = \langle t_{ji}x_{ji} \colon 1 \leqslant j \leqslant n_i, 1 \leqslant i \leqslant m \rangle \cong (R/P_1)^{\oplus n_1} \oplus \ldots \oplus (R/P_m)^{\oplus n_m}.$$

Put

$$N = P_2P_3 \ldots P_mN_1 + \ldots + P_1P_2 \ldots P_{m-1}N_m + P_1 \ldots P_mM.$$

Since $M_{P_i} = (N_i)_{P_i} \oplus (R_{P_i}/P_iR_{P_i})^{\oplus n_i}$, $P_iM_{P_i} = P_i(N_i)_{P_i}$. Therefore for every i, $1 \leqslant i \leqslant m$ we have

$$N_{P_i} = P_i(N_1)_{P_i} + \ldots + (N_i)_{P_i} + \ldots + P_i(N_m)_{P_i} + P_iM_{P_i} = (N_i)_{P_i}.$$

Also for every maximal ideal $Q \neq P_1, \ldots, P_m$ we have

$$N_Q = (N_1)_Q + \ldots + (N_m)_Q + M_Q = M_Q.$$

Therefore for every maximal ideal P of R we have

$$M_P = N_P \oplus \mathrm{T}(M)_P = (N \oplus \mathrm{T}(M))_P.$$

So

$$M = N \oplus \mathrm{T}(M) \cong N \oplus (R/P_1)^{\oplus n_1} \oplus \ldots \oplus (R/P_m)^{\oplus n_m}.$$

□

Let (R, P) be a local ring. Now, we study the finitely generated R-module M which R/P^t is a direct summand of M for some positive integer t.

Further, we use the following lemmas.

Lemma 2.6. *For an exact sequence of finitely generated R-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and integers $r, s \geqslant 0$, we have $\mathrm{Fitt}_r(L)\mathrm{Fitt}_s(N) \subseteq \mathrm{Fitt}_{r+s}(M)$. Furthermore, if the above sequence splits, i.e. $M = L \oplus N$, then for each integer $t \geqslant 0$ we have $\mathrm{Fitt}_t(L \oplus N) = \sum\limits_{r+s=t} \mathrm{Fitt}_r(L)\mathrm{Fitt}_s(N)$, and consequently $\mathrm{I}(L \oplus N) = \mathrm{I}(L)\mathrm{I}(N)$.

Proof. See [11], pages 90–93. □

Lemma 2.7. *Let $\mathrm{T}(M)$ be a finitely generated R-submodule of M. Then $\mathrm{I}(\mathrm{T}(M)) = \mathrm{Fitt}_0(\mathrm{T}(M))$.*

Proof. Assume that $\mathrm{T}(M)$ is generated by n elements, so there exists a regular element q in R such that $q \in \mathrm{Ann}(\mathrm{T}(M))$. By Proposition 2.1, $q^n \in \mathrm{Fitt}_0(\mathrm{T}(M))$. Since q is a regular element, $q^n \neq 0$. Thus $\mathrm{I}(\mathrm{T}(M)) = \mathrm{Fitt}_0(\mathrm{T}(M))$. □

Theorem 2.8. *Let (R, P) be a Noetherian local ring and M be a regular R-module which is generated by n elements. Then $M \cong R^{\oplus(n-1)} \oplus R/P^t$, for a positive integer t, if and only if $\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = P^t$ and $S(P^t) \nsubseteq PM$. Furthermore, in the above case, $S(P^t) \cap PM = \emptyset$.*

Proof. Let $M \cong R^{\oplus(n-1)} \oplus R/P^t$ for a positive integer t. By Lemma 2.6 and [1], Lemma 1, it is easily seen that $\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = P^t$. We have

$$PM \cong P^{\oplus(n-1)} \oplus P/P^t.$$

Since M is a regular R-module, P contains a regular element. Thus $\mathrm{Ann}(b) = 0$ for every $b \in P^{\oplus(n-1)}$ and $P^{t-1} \subseteq \mathrm{Ann}(a + P^t)$ for every element $a \in P$. Thus, there exists no element of $S(P^t)$ in PM. Therefore $S(P^t) \cap PM = \emptyset$.

Now assume that $\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = P^t$ and $S(P^t) \nsubseteq PM$. So there exists an element $x_1 \in S(P^t) \setminus PM$. Since $x_1 \notin PM$, $\{x_1\}$ can be extended to a minimal generating set $\{x_1, \ldots, x_n\}$ for M. (Note that since $\mathrm{Fitt}_{n-1}(M) \neq R$, by [2], Proposition 20-6, M cannot be generated by less than n elements.) Let

$$R^{\oplus m} \xrightarrow{\varphi} R^{\oplus n} \xrightarrow{\psi} M \longrightarrow 0$$

be a free presentation of M, where $\psi(e_i) = x_i$, $1 \leqslant i \leqslant n$. Let $A = (a_{ij}) \in \mathrm{M}_{n \times m}(R)$ be the matrix presentation of φ. We have

$$0 \neq \mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = \mathrm{I}_{n-(n-1)}(\varphi) = \mathrm{I}_1(\varphi).$$

Thus $\mathrm{rank}(A) = \mathrm{rank}(\varphi) = 1$.

Since $M/\mathrm{T}(M) = \langle x_1 + \mathrm{T}(M), \ldots, x_n + \mathrm{T}(M) \rangle$, we have the free presentation

$$R^{\oplus k} \xrightarrow{f} R^{\oplus n} \xrightarrow{\overline{\psi}} M/\mathrm{T}(M) \longrightarrow 0$$

of $M/\mathrm{T}(M)$, where $\overline{\psi}(e_i) = x_i + \mathrm{T}(M)$, $1 \leqslant i \leqslant n$. So let $B \in M_{n \times k}(R)$ be the matrix presentation of f. Thus $\ker(\overline{\psi}) = \mathrm{Im}(f) = \langle B \rangle$. Let $x \in \ker(\overline{\psi})$ be arbitrary. Then $\overline{\psi}(x) = 0$ and so $\psi(x) \in \mathrm{T}(M)$. Thus, there exists a regular element $q \in R$ such that $q\psi(x) = 0$. Hence $qx \in \ker(\psi) = \mathrm{Im}(\varphi)$ and so $qx \in \langle A \rangle$. Therefore $\langle qB \rangle \subseteq \langle A \rangle$. Since q is regular, $\mathrm{rank}(B) = \mathrm{rank}(qB) \leqslant \mathrm{rank}(A) = 1$. Thus $\mathrm{I}(M/\mathrm{T}(M)) = \mathrm{I}_1(f)$. Since $x_1 + \mathrm{T}(M) = \bar{x}_1 = \bar{0}$, $(1, 0, \ldots, 0)^t \in \langle B \rangle$. Hence

$$\mathrm{I}(M/\mathrm{T}(M)) = \mathrm{Fitt}_{n-1}(M/\mathrm{T}(M)) = R.$$

So by [1], Lemma 1, $M/\mathrm{T}(M)$ is a free R-module of rank $n-1$. Hence $M \cong M/\mathrm{T}(M) \oplus \mathrm{T}(M)$. By Lemma 2.6 and Lemma 2.7, $\mathrm{Fitt}_{n-1}(M) = \mathrm{Fitt}_0(\mathrm{T}(M)) = P^t$. Since every minimal generating set of M has n elements, $\mathrm{T}(M)$ is a cyclic R-module. Therefore

$$M \cong M/\mathrm{T}(M) \oplus \mathrm{T}(M) \cong R^{\oplus(n-1)} \oplus R/P^t.$$

□

Now we generalize Theorem 2.8 to global case.

Theorem 2.9. *Let R be a Noetherian ring and M be a regular R-module which is generated by n elements. Assume that $P_1, \ldots, P_m$ be distinct maximal ideals of R. Then $M = N \oplus \mathrm{T}(M)$ and $\mathrm{T}(M) \cong \left(R/\bigcap_{i=1}^{m} P_i^{n_i}\right)$ for a projective R-module N of constant rank $n-1$ and some positive integers n_i, $1 \leqslant i \leqslant m$, if and only if $\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = P_1^{n_1} \ldots P_m^{n_m}$ and $S(P_i^{n_i}) \nsubseteq P_i M$, $1 \leqslant i \leqslant m$.*

P r o o f. Let

$$M \cong N \oplus \left(R/\bigcap_{i=1}^{m} P_i^{n_i}\right)$$

for a projective R-module N of constant rank $n-1$. Hence by [1], Lemma 1, $\mathrm{I}(N) = \mathrm{Fitt}_{n-1}(N) = R$. So it is easily seen that

$$\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = \bigcap_{i=1}^{m} P_i^{n_i} = P_1^{n_1} \dots P_m^{n_m}$$

and $\mathrm{S}(P_i^{n_i}) \nsubseteq P_iM$, $1 \leqslant i \leqslant m$.

Conversely, let

$$\mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = P_1^{n_1} \dots P_m^{n_m}$$

and $\mathrm{S}(P_i^{n_i}) \nsubseteq P_iM$, $1 \leqslant i \leqslant m$. Thus $\mathrm{I}(M_{P_i}) = P_i^{n_i}R_{P_i}$ for every i, $1 \leqslant i \leqslant m$. Assume that i, $1 \leqslant i \leqslant n$, is fixed. We claim that $\mathrm{S}(P_i^{n_i}R_{P_i}) \nsubseteq P_iM_{P_i}$. Let $x \in \mathrm{S}(P_i^{n_i}) \setminus P_iM$. So $\mathrm{Ann}(x) = P_i^{n_i}$. Since $P_i^{n_i}$ is a P_i-primary ideal, $\mathrm{Ann}(x/1) = P_i^{n_i}R_{P_i}$. Now let $x/1 \in P_iM_{P_i}$. Similarly to the proof of Theorem 2.5, ($3 \Rightarrow 1$) because P_iM is a prime submodule of M, hence $x \in P_iM$, a contraction.

Therefore by Theorem 2.8,

$$M_{P_i} \cong R_{P_i}^{\oplus(n-1)} \oplus R_{P_i}/P_i^{n_i}R_{P_i}.$$

Assume that $Q \neq P_1, \dots, P_m$ is a maximal ideal of R. We have

$$\mathrm{I}(M_Q) = \mathrm{I}(M)_Q = R_Q.$$

So M_Q is a free R_Q-module. Now we show that $\mathrm{T}(M_{P_i}) = \mathrm{T}(M)_{P_i}$. It is clear that $\mathrm{T}(M)_{P_i} \subseteq \mathrm{T}(M_{P_i})$. Let

$$0 \neq \frac{z}{1} \in \mathrm{T}(M_{P_i}) \cong (R_{P_i}/P_i^{n_i}R_{P_i}).$$

Thus $P_i^{n_i}R_{P_i} \subseteq \mathrm{Ann}(z/1)$. Choose a regular element r_0 in $P_i^{n_i}$, so we have

$$\frac{r_0}{1}\frac{z}{1} = \frac{0}{1}.$$

Therefore there exists an element $t \in R \setminus P_i$ such that $tr_0z = 0$ and hence $tz \in \mathrm{T}(M)$. Thus

$$\frac{z}{1} = \frac{tz}{t} \in \mathrm{T}(M)_{P_i}.$$

This means that $\mathrm{T}(M_{P_i}) = \mathrm{T}(M)_{P_i}$. Thus

$$\left(\frac{M}{\mathrm{T}(M)}\right)_P = \frac{M_P}{\mathrm{T}(M_P)}$$

is free for every maximal ideal P of R. Therefore $M/\mathrm{T}(M)$ is a projective R-module and so $M \cong \mathrm{T}(M) \oplus M/\mathrm{T}(M)$. Put

$$A_i = \{\mathrm{Ann}(y)\colon\ \mathrm{T}(M_{P_i}) = \langle y/1\rangle\}$$

for $i = 1, \ldots, m$. Let $\mathrm{T}(M_{P_i}) = \langle x_i/1\rangle$ such that $\mathrm{Ann}_R(x_i)$ is maximal in A_i. We show that $\mathrm{Ann}(x_i) = P_i^{n_i}$. For every i, $1 \leqslant i \leqslant m$, let $r_i \in P_i^{n_i}$. Then

$$\frac{r_i}{1}\frac{x_i}{1} = 0.$$

So there exists $s \in R \setminus P_i$ such that $r_i s x_i = 0$. Since $\mathrm{T}(M_{P_i}) = \langle s x_i/1\rangle$ and $\mathrm{Ann}(x_i)$ is maximal in A_i, then $r_i \in \mathrm{Ann}(s x_i) = \mathrm{Ann}(x_i)$.

Now, let $r_i \in \mathrm{Ann}(x_i)$. So $r_i/1 \in P_i^{n_i} R_{P_i}$. Since $P_i^{n_i}$ is P_i-primary, then $r \in P_i^{n_i}$. Hence, $\mathrm{Ann}(x_i) = P_i^{n_i}$. Put $P = \bigcap\limits_{i=1}^{m} P_i^{n_i}$ and define

$$f\colon\ R/P \to \mathrm{T}(M);\ f(r+P) = r(x_1 + \ldots + x_m).$$

For every $j \neq i$, if $s_j \in P_j^{n_j} \setminus P_i$, then

$$\frac{x_j}{1} = \frac{s_j x_j}{s_j} = 0 \quad \text{in } M_{P_i}.$$

On the other hand, for every maximal ideal $Q \notin \{P_1, \ldots, P_m\}$, $f_Q = 0$ is an isomorphism between two zero modules. Thus, f_q is an isomorphism for every maximal ideal q of R. Hence $\mathrm{T}(M) \cong R/P$. So $M \cong N \oplus \left(R/\bigcap\limits_{i=1}^{m} P_i^{n_i}\right)$ for a projective R-module N of M. By Lemmas 2.6 and 2.7,

$$P_1^{n_1} \ldots P_m^{n_m} = \mathrm{I}(M) = \mathrm{Fitt}_{n-1}(M) = \mathrm{I}(N)\mathrm{Fitt}_0(\mathrm{T}(M)) = \mathrm{I}(N) P_1^{n_1} \ldots P_m^{n_m}.$$

So by Nakayama's Lemma, $\mathrm{I}(N) = R$. Hence, N is projective of constant rank. Since $M_{P_i} \cong R_{P_i}^{\oplus(n-1)} \oplus R_{P_i}/P_i^{n_i} R_{P_i}$, $1 \leqslant i \leqslant m$, N is projective of constant rank $n-1$. $\square$

Example 2.10. Let $R=k[x,y]$ be the ring of polynomials over a field k. Set

$$P:=\langle x,y\rangle,\quad A:=\begin{pmatrix} x & x^2 & xy & y & 0\\ x^2 & x & 0 & x & x\\ xy & y & 0 & y & y\\ 0 & y & x & 0 & 0\end{pmatrix}\quad\text{and}\quad M:=\frac{R^{\oplus 4}}{\langle A\rangle}.$$

We have $\mathrm{I}(M)=P^3$. Assume that A_i is the ith column of the matrix A, $1\leqslant i\leqslant 5$. Let $(a,b,c,d)^t+\langle A\rangle\in PM$, where t denotes transpose. Thus, there exist r_{ij}, $1\leqslant i,j\leqslant 2$ such that $a=r_{11}x+r_{12}y$, $b=r_{21}x+r_{22}y$, $c=r_{31}x+r_{32}y$, $d=r_{41}x+r_{42}y$. It is easily seen that $(a,0,0,d)^t=(r_{11}-r_{42}x)A_1+r_{42}A_2+r_{41}A_3+(r_{12}-r_{41}x)A_4+(r_{41}x-r_{42}-r_{21}+r_{42}x^2-r_{11}x)A_5$. Thus $(a,b,c,d)^t+\langle A\rangle=(0,b,c,0)^t+\langle A\rangle$.

Now let $(a,b,c,d)^t+\langle A\rangle=(0,b,c,0)^t+\langle A\rangle\in\mathrm{T}(M)$. Therefore there exists a regular element $q\in R$ such that $q(0,b,c,0)^t\in\langle A\rangle$. So $q(b,c)^t=s(x,y)^t$ for some $s\in R$. Hence $qb=sx$ and $qc=sy$. If $s=0$, then $b=c=0$. Otherwise $qcb=syb=csx$ and so $yb=cx$. We have $x\mid yb$ and $y\mid cx$. Since $\mathrm{GCD}(x,y)=1$, $x\mid b$ and $y\mid c$. So there exist $t_1,t_2\in R$ such that $b=t_1x$ and $c=t_2y$. From $yb=cx$ we imply that $t_1=t_2$. Hence $(b,c)^t=t_1(x,y)^t$. Thus $(0,b,c,0)^t=t_1(0,x,y,0)^t\in\langle A\rangle$. Thus $PM\cap\mathrm{T}(M)=0$. So by Theorem 2.5, $M\cong\mathrm{T}(M)\oplus M/\mathrm{T}(M)$. In fact,

$$M\cong\Big(\frac{R}{P}\Big)^{\oplus 2}\oplus\frac{R^{\oplus 2}}{\Big\langle\begin{pmatrix}x\\ y\end{pmatrix}\Big\rangle}.$$

Example 2.11. Let $R=k[x,y]$ be the ring of polynomials over a field k. Set $P:=\langle x,y\rangle$ and

$$M:=\frac{R^{\oplus 2}}{\Big\langle\begin{pmatrix}x & y & x & 0\\ 0 & 0 & y & x\end{pmatrix}\Big\rangle}.$$

We have $\mathrm{I}(M)=P^2$ and $x(0,y)^t=(0,xy)^t=y(0,x)^t=0$. So $0\neq(0,y)^t\in\mathrm{T}(M)\cap PM$. Therefore by Theorem 2.5, $\mathrm{T}(M)$ does not split off.

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Authors' addresses: Somayeh Hadjirezaei (corresponding author), Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box 7718897111, Rafsanjan, Iran, e-mail: `s.hajirezaei@vru.ac.ir`; Sina Hedayat, Department of Pure Mathematics, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran, e-mail: `s-hedayat@uk.ac.ir`.