Czechoslovak Mathematical Journal

Milica Anđelić; Zhibin Du; Carlos M. da Fonseca; Slobodan K. Simić
Tridiagonal matrices and spectral properties of some graph classes

Czechoslovak Mathematical Journal, Vol. 70 (2020), No. 4, 1125–1138

Persistent URL: `http://dml.cz/dmlcz/148416`

Czechoslovak Mathematical Journal, 70 (145) (2020), 1125–1138

TRIDIAGONAL MATRICES AND SPECTRAL PROPERTIES OF SOME GRAPH CLASSES

Milica Anđelić, Safat, Zhibin Du, Guangdong,
Carlos M. da Fonseca, Safat, Slobodan K. Simić, Belgrade

Received April 17, 2019. Published online April 23, 2020.

Abstract. A graph is called a chain graph if it is bipartite and the neighbourhoods of the vertices in each colour class form a chain with respect to inclusion. In this paper we give an explicit formula for the characteristic polynomial of any chain graph and we show that it can be expressed using the determinant of a particular tridiagonal matrix. Then this fact is applied to show that in a certain interval a chain graph does not have any nonzero eigenvalue. A similar result is provided for threshold graphs.

Keywords: tridiagonal matrix; threshold graph; chain graph; eigenvalue-free interval

MSC 2020: 05C50

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph (without loops or multiple edges) of order $n = |V(G)|$ and size $m = |E(G)|$; $A = A(G)$ is its $(0,1)$-adjacency matrix, i.e., $a_{ij} = 1$ if i and j are adjacent and 0 otherwise. Since A is symmetric and its eigenvalues are real we assume $\lambda_1(G) \geqslant \ldots \geqslant \lambda_n(G)$.

In this paper we consider chain graphs, i.e., $\{K_2, C_3, C_5\}$-free graphs. In the spectral graph theory they feature as the bipartite graphs whose largest eigenvalue within the connected bipartite graphs of fixed order and size is maximal.

The vertex set of any chain graph G consists of two-colour classes also known as cocliques. To specify the nesting, both of them are partitioned into h nonempty cells $\bigcup_{i=1}^{h} U_i$ and $\bigcup_{i=1}^{h} V_i$, respectively; all vertices in U_s are joined (by cross edges) to all vertices in $\bigcup_{k=s}^{h} V_k$ for $s = 1, 2, \ldots, h$. Denote by $N_G(w)$ the set of neighbours of a vertex w. Hence, if $u' \in U_{s+1}$ and $u'' \in U_s$, $v' \in V_{t+1}$ and $v'' \in V_t$, then

DOI: 10.21136/CMJ.2020.0182-19

$N_G(u') \subset N_G(u'')$ and $N_G(v'') \subset N_G(v')$, and this makes the double nesting property precise from both sides (from left to right and from right to left).

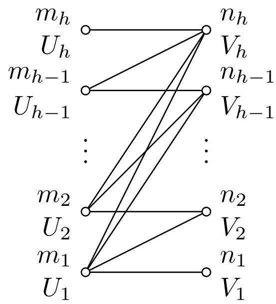

Figure 1. The chain graph $G = \mathrm{DNG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$.

If $m_s = |U_s|$ and $n_s = |V_s|$ $(s = 1, 2, \ldots, h)$, then G is denoted by

$$\mathrm{DNG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h).$$

For $m_1 = \ldots = m_h = n_1 = \ldots = n_h = 1$, the corresponding $\mathrm{DNG}(1, \ldots, 1; 1, \ldots, 1)$ is called a *half graph* and is denoted by H_h.

Given a graph G, the partition $W_1 \cup W_2 \cup \ldots \cup W_k$ of its vertex set is an *equitable partition* $\mathcal{D}$ if every vertex in W_i has the same number of neighbours in W_j, say d_{ij} for all $i, j \in \{1, 2, \ldots, k\}$. The matrix $[d_{ij}]$ is called the *divisor matrix* of $\mathcal{D}$ (for more details, see [7], page 83).

In view of the above definition, for any $G = \mathrm{DNG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$, the partition

$$\mathcal{D} = (U_1 \cup U_2 \cup \ldots \cup U_h) \cup (V_h \cup V_{h-1} \cup \ldots \cup V_1)$$

of its vertex set is an equitable partition since every vertex in U_i (V_i) has the same number of neighbours in U_j (V_j) for all $i, j \in \{1, 2, \ldots, h\}$. Let A_D be its divisor matrix with respect to $\mathcal{D}$. Then

$$A_D = \left(\begin{array}{ccccc|ccccc}
 & & & & & n_h & n_{h-1} & \ldots & n_2 & n_1 \\
 & & & & & n_h & n_{h-1} & \ldots & n_2 & 0 \\
 & & 0 & & & \vdots & \vdots & \iddots & \iddots & \vdots \\
 & & & & & n_h & n_{h-1} & \iddots & & \vdots \\
 & & & & & n_h & 0 & \ldots & 0 & 0 \\
\hline
m_1 & m_2 & \ldots & m_{h-1} & m_h & & & & & \\
m_1 & m_2 & \ldots & m_{h-1} & 0 & & & & & \\
\vdots & \vdots & \iddots & \iddots & \vdots & & & 0 & & \\
m_1 & m_2 & \iddots & & \vdots & & & & & \\
m_1 & 0 & \ldots & 0 & 0 & & & & &
\end{array}\right).$$

For

$$(1.1)\qquad N=\begin{pmatrix} n_h & \dots & n_2 & n_1\\ n_h & \dots & n_2 & 0\\ \vdots & \iddots & & \vdots\\ n_h & \dots & 0 & 0\end{pmatrix}\quad\text{and}\quad M=\begin{pmatrix} m_1 & \dots & m_{h-1} & m_h\\ m_1 & \dots & m_{h-1} & 0\\ \vdots & \iddots & & \vdots\\ m_1 & \dots & 0 & 0\end{pmatrix},$$

matrix A_D can be rewritten as

$$A_D=\begin{pmatrix} 0 & N\\ M & 0\end{pmatrix}.$$

Notice that $A_D^2=\left(\begin{smallmatrix} NM & 0\\ 0 & MN\end{smallmatrix}\right)$. The spectra of NM and MN coincide (since M and N are square matrices). Therefore the eigenvalues of G are

$$(1.2)\qquad \lambda_i(G)=\sqrt{\lambda_i(NM)},\ 0^{n-2h},\ \lambda_{n-i+1}(G)=-\sqrt{\lambda_i(NM)},\quad i=1,\dots,h.$$

A chain graph can be defined by a binary sequence, say $b_1\dots b_n$, by the following generating procedure.

- ▷ $i=1$: We start with an isolated vertex, for example white.
- ▷ $i>1$: If $b_i=b_{i-1}$ and the previously added vertex was white, then we add an isolated white vertex; otherwise, if the last added vertex was black, we add a black vertex adjacent to all white vertices in the graph obtained until then.
  If $b_i\neq b_{i-1}$ and the previously added vertex was white, we add a black vertex adjacent to all white vertices in the graph obtained so far. Otherwise, if the last added vertex was black we add an isolated white vertex.

Note that by choosing white for the first added vertex, we have avoided isolated vertices coloured in black. Also, if the last added vertex is black, then we obtain a connected chain graph.

Example 1.1. If $b=(0^{t_1}1^{s_1})\dots(0^{t_h}1^{s_h})$ with $t_i,s_i>0$, then we obtain a chain graph $\mathrm{DNG}(t_1,\dots,t_h;s_1,\dots,s_h)$. The same chain graph can be also generated by a sequence $(0^{s_h}1^{t_h})\dots(0^{t_1}1^{s_1})$ for $t_i,s_i>0$. A generating sequence of a half graph is $\underbrace{0101\dots01}_{2h}$.

The paper is organized in the following way. After discussing a formula for the determinant of a monic Jacobi matrix, we establish a general formula of the characteristic polynomial of a chain graph. We propose a conjecture on the nonexistence of nonisomorphic cospectral chain graphs. In the last two sections, we find intervals without any eigenvalue of a connected chain graph and of a connected threshold graph.

2. Determinant of certain tridiagonal matrices

Let us consider the $n \times n$ monic Jacobi matrix

$$J_n(a_1 \dots a_n) = \begin{pmatrix} a_1 & 1 & & & \\ b & a_2 & 1 & & \\ & b & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b & a_n \end{pmatrix},$$

which can be seen as the weighted adjacency matrix of a path with vertices $1, 2, \dots, n$ and arcs $(i, i+1)$ and $(i+1, i)$ for $i = 1, 2, \dots, n-1$, and a loop in each vertex. We understand that the weight 1 is assigned to the arc $(i, i+1)$, the weight b to $(i+1, i)$, and the weight a_i to the ith loop.

Therefore, the *determinant of* $J_n(a_1 \dots a_n)$ can be defined as

$$\det J_n(a_1 \dots a_n) = (-1)^n \sum (-1)^{|q|} C(q),$$

where the sum is over all cycle partitions q of $J_n(a_1 \dots a_n)$. This result can be in fact stated in more general terms (see [12]). For the path defined above, a cycle partition contains only cycles of the type $(i, i+1, i)$ or loops (i, i). Obviously, the number of such partitions is the nth Fibonacci number F_n, corresponding to the number of terms of the determinant.

For a certain partition q, let us denote by c_q the subset of cycles in q and l_q the subset of loops. Then

$$\det J_n(a_1 \dots a_n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k b^k \sum_{|c_q|=k} \prod_{i \in l_q} a_i. \tag{2.1}$$

In this section we also give an explicit formula for the determinant of the tridiagonal matrix

$$T_n^x(a_1 \dots a_n) = \begin{pmatrix} a_1 & x & & & \\ x & a_2 & x & & \\ & x & \ddots & \ddots & \\ & & \ddots & \ddots & x \\ & & & x & a_n \end{pmatrix},$$

where $a_1 \dots a_n$ is a sequence of n real numbers.

If $[n] = \{1, 2, \dots, n\}$, then $I_{n,l}$ denotes the set of all strictly increasing sequences in $[n]$ of length l such that any two consecutive terms have different parities and the last term is of the same parity as l. For example $I_{5,3} = \{123, 125, 145, 345\}$. For a given sequence $a = a_1 \dots a_n$ of real numbers we define

$$\gamma_{n,l}(a) = \sum_{i_1 \dots i_l \in I_{n,l}} a_{i_1} \dots a_{i_l}.$$

We set $\gamma_{n,0}(a_1 \dots a_n) = 1$. The function $\gamma_{n,l}(a_1 \dots a_n)$ was introduced in [11] and the following property was given.

If l and $n+1$ have the same parity then

$$\gamma_{n+1,l}(a_1 \dots a_{n+1}) = \begin{cases} a_{n+1}\gamma_{n,l-1}(a_1 \dots a_n) + \gamma_{n-1,l}(a_1 \dots a_{n-1}) & \text{if } 1 \leqslant l \leqslant n-1, \\ a_{n+1}\gamma_{n,l-1}(a_1 \dots a_n) & \text{if } l = n+1, \end{cases}$$

otherwise

$$\gamma_{n+1,l}(a_1 a_2 \dots a_{n+1}) = \gamma_{n,l}(a_1 a_2 \dots a_{n+1}).$$

Using the mathematical induction on n, the following formula for the determinant of $T_n^x(a_1 \dots a_n)$ can be easily proven:

$$\det(T_n^x(a_1 \dots a_n)) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} (-1)^i \gamma_{n,n-2i}(a_1 \dots a_n) x^{2i}. \tag{2.2}$$

Remark 2.1. Having in mind (2.1) we point out that $\sum\limits_{|c_q|=k} \prod\limits_{i \in l_q} a_i = \gamma_{n,l_q}(a_1 \dots a_n)$ as defined in [12].

3. The characteristic polynomial of a chain graph

The characteristic polynomial of a chain graph was studied in [6]. There, a fast algorithm for its calculation was provided. The algorithm is based on a certain recurrence relation and, in some cases, can be carried out in linear time. In this section we give an explicit formula for the characteristic polynomial of chain graphs deduced from its generating binary code. The characteristic polynomial of a graph G is denoted by $\phi(x; G)$.

Theorem 3.1. *Let G be a chain graph associated to a binary sequence*

$$(0^{t_1}1^{s_1})(0^{t_2}1^{s_2}) \dots (0^{t_h}1^{s_h}), \quad t_i, s_i > 0.$$

Then

$$\phi(x; G) = x^{n-2h} \sum_{i=0}^{h} (-1)^i \gamma_{2h,2h-2i}(t_1 s_1 \dots t_h s_h) x^{2i}, \tag{3.1}$$

where $n = \sum\limits_{i=1}^{h} (t_i + s_i)$.

P r o o f. Let U_i (or V_i) denote the set of vertices corresponding to t_i (or s_i, respectively). Then $V(G) = U_1 \cup V_1 \cup \ldots \cup U_h \cup V_h$ is an equitable partition with the corresponding divisor matrix

$$D = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ \vdots \\ U_h \\ V_h \end{array} \begin{array}{c} \begin{array}{ccccccc} U_1 & V_1 & U_2 & V_2 & \ldots & U_h & V_h \end{array} \\ \begin{pmatrix} 0 & s_1 & 0 & s_2 & \ldots & 0 & s_h \\ t_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & s_2 & \ldots & 0 & s_h \\ t_1 & 0 & t_2 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & s_h \\ t_1 & 0 & t_2 & 0 & \ldots & t_h & 0 \end{pmatrix} \end{array}$$

According to [6], $\phi(x;G) = (-1)^h x^{n-2h} \det(D - xI)$. To the matrix

$$D - xI = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ \vdots \\ U_h \\ V_h \end{array} \begin{array}{c} \begin{array}{ccccccc} U_1 & V_1 & U_2 & V_2 & \ldots & U_h & V_h \end{array} \\ \begin{pmatrix} -x & s_1 & 0 & s_2 & \ldots & 0 & s_h \\ t_1 & -x & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -x & s_2 & \ldots & 0 & s_h \\ t_1 & 0 & t_2 & -x & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -x & s_h \\ t_1 & 0 & t_2 & 0 & \ldots & t_h & -x \end{pmatrix} \end{array}$$

we apply the following elementary row operations:

▷ $R_{2i-1} \leftarrow R_{2i-1} - R_{2i+1}$ for $i = 1, \ldots, h-1$,

▷ $R_{2i} \leftarrow R_{2i} - R_{2i-2}$ for $i = 2, \ldots, h$,

▷ $R_{2i-1} \leftrightarrow R_{2i}$ for $i = 1, \ldots, h$.

Consequently, $\det(D - xI) = (-1)^h \det D_x$, where

$$D_x = \begin{array}{c} \\ U_1 \\ V_1 \\ U_2 \\ V_2 \\ \vdots \\ U_h \\ V_h \end{array} \begin{array}{c} \begin{array}{ccccccc} U_1 & V_1 & U_2 & V_2 & \ldots & U_h & V_h \end{array} \\ \begin{pmatrix} t_1 & x & 0 & 0 & \ldots & 0 & 0 \\ x & s_1 & x & 0 & \ldots & 0 & 0 \\ 0 & x & t_2 & x & \ldots & 0 & 0 \\ 0 & 0 & x & s_2 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & t_h & x \\ 0 & 0 & 0 & 0 & \ldots & x & s_h \end{pmatrix} \end{array}$$

taking into account that the determinant of a tridiagonal matrix does not change if the corresponding ij and ji entries are replaced by their geometric mean. Now, by (2.2) the proof easily follows. $\square$

Remark 3.2. In [11] the authors provided the explicit formula for the characteristic polynomial of threshold graphs in a similar fashion. Then the formula is used to prove that there are no nonisomorphic cospectral threshold graphs with respect to the adjacency spectrum in the class of connected threshold graphs. We assume that the similar conclusion can be taken for chain graphs as well. Therefore, we conjecture the following statement.

Conjecture 3.3. *In a class of connected chain graphs there do not exist nonisomorphic cospectral chain graphs with respect to the adjacency spectrum or, equivalently, if*

$$\det T_n^x(a_1 \dots a_n) = \det T_n^x(b_1 \dots b_n) \quad \text{and} \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i,$$

then $(a_1, \dots, a_n) = (b_1, \dots, b_n)$ *or* $(a_1, \dots, a_n) = (b_n, \dots, b_1)$.

Observe that, if the chain graphs are not connected, then the previous conjecture does not hold. For example, the complete bipartite graphs $K_{1,4}$ and $K_{2,2} \cup K_1$ are nonisomorphic chain graphs that have the same characteristic polynomial $x^3(x^2 - 4)$.

4. Eigenvalue free interval for chain graphs

In [2] it was proven that any chain graph has no eigenvalues in the intervals $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Here we show that this interval can be extended if the generating sequence of a chain graph is known.

For the sake of completeness, we recall the following result from [4].

Proposition 4.1 ([4]). *Let*

$$A_n = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}$$

be a real symmetric tridiagonal matrix with positive diagonal entries. If

$$b_i^2 < \frac{1}{4} a_i a_{i+1} \frac{1}{\cos^2(\pi/(n+1))} \quad \text{for } i = 1, \dots, n-1, \tag{4.1}$$

then A_n is positive definite.

Theorem 4.2. *Let G be a chain graph associated to a binary sequence*

$$(0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\dots(0^{t_h}1^{s_h}),\quad t_i, s_i > 0,$$

and

$$M_h = \frac{1}{4\cos^2(\pi/(2h+1))}\min_{1\leqslant i\leqslant 2h-1} a_i a_{i+1},$$

where $a_1 \dots a_{2h} = t_1 s_1 \dots t_h s_h$. Then G has no eigenvalue in the intervals $(-\sqrt{M_h}, 0)$ and $(0, \sqrt{M_h})$.

Proof. Let $\varphi(x) = \det D_x$, where D_x is given in (3.2). According to [3] all nonzero eigenvalues of G are eigenvalues of its divisor matrix and, consequently, by Theorem 3.1, the nonzero eigenvalues of G are roots of $\varphi(x) = \sum_{i=0}^{h}(-1)^i\gamma_{2h,2h-2i}(t_1 s_1 \dots t_h s_h)x^{2i}$. Now, by Proposition 4.1 for any real number a such that

$$a^2 < \frac{1}{4\cos^2(\pi/(2h+1))}\min_{1\leqslant i\leqslant 2h-1} a_i a_{i+1},$$

we have $\varphi(a) > 0$, i.e., $\varphi(a) > 0$ for any $a \in (-\sqrt{M_h}, \sqrt{M_h})$. This completes the proof. □

Bearing in mind that $\min\limits_{1\leqslant i\leqslant 2h-1} a_i a_{i+1} \geqslant 1$ we easily conclude the following statement.

Corollary 4.3. *Let G be a chain graph associated to a binary sequence*

$$(0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\dots(0^{t_h}1^{s_h}),\quad t_i, s_i > 0,$$

then G has no eigenvalue in the intervals

$$\Big(-\frac{1}{2\cos(\pi/(2h+1))}, 0\Big) \quad \text{and} \quad \Big(0, \frac{1}{2\cos(\pi/(2h+1))}\Big).$$

For chain graphs, the tridiagonal matrices arise also in another context. For the matrices N and M given in (1.1) we have $\det NM = n_1 \dots n_h m_1 \dots m_h$, i.e., NM is nonsingular. Furthermore, the inverse of NM is the tridiagonal matrix

$$T_h = \begin{pmatrix} a_1 & -a_1 & & & & \\ -b_1 & a_2+b_1 & -a_2 & & & \\ & -b_2 & a_3+b_2 & -a_3 & & \\ & & -b_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & -a_{h-1} \\ & & & & -b_{h-1} & a_h+b_{h-1} \end{pmatrix},$$

where

$$a_i = \frac{1}{m_i n_i} \quad \text{for } i = 1, 2, \dots, h$$

and

$$b_i = \frac{1}{m_{i+1} n_i} = \frac{m_i}{m_{i+1}} a_i \quad \text{for } i = 1, 2, \dots, h-1.$$

In fact, observing that the inverse of N is

$$\begin{pmatrix} & & & \frac{1}{n_h} \\ & & \frac{1}{n_{h-1}} & -\frac{1}{n_{h-1}} \\ & \iddots & \iddots & \\ \frac{1}{n_1} & -\frac{1}{n_1} & & \end{pmatrix}$$

and the inverse of M has the same form, it follows that $M^{-1}N^{-1}$ is T_h.

For a half graph H_h, the inverse is the tridiagonal matrix

$$T'_h = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

The eigenvalues of T'_h are given by the expression (see, e.g., [8])

$$\mu_i = -2\cos\left(\frac{2i-1}{2h+1}\pi\right) + 2, \tag{4.2}$$

where $i = 1, \dots, h$. Obviously, $\mu_1 < \dots < \mu_h$.

Consequently, the eigenvalues of H_h are

$$\begin{aligned} \pm 1 \Big/ \sqrt{-2\cos\left(\frac{2i-1}{2h+1}\pi\right) + 2} &= \pm\frac{1}{2}\sin\left(\frac{2i-1}{2(2h+1)}\pi\right), \\ &= \pm\frac{1}{2}\cos\left(\frac{h-i+1}{2h+1}\pi\right), \quad i = 1, \dots, h. \end{aligned}$$

Remark 4.4. Note that

$$\lambda_h(H_h) = \frac{1}{2\cos(\pi/(2h+1))},$$

i.e., the eigenvalue free interval for chain graphs, according to Corollary 4.3, can be seen as $(-\lambda_h(H_h), 0) \cup (0, \lambda_h(H_h))$.

5. Eigenvalue free intervals for threshold graphs

For a given binary sequence $b = b_1b_2\dots b_n$ with $b_i \in \{0,1\}$, the associated threshold graph $G(b)$ is constructed as follows:

(i) for $i = 1$, $G_1 = G(b_1) = K_1$, i.e., a single vertex;

(ii) for $i = 2,\dots,n$, with $G_{i-1} = G(b_0\dots b_{i-1})$ already constructed, $G_i = G(b_0\dots b_{i-1}b_i)$ is formed by adding an isolated vertex to G_{i-1} if $b_i = 0$ (that is, a vertex nonadjacent to any vertex in G_{i-1}), or by adding a dominating vertex to G_{i-1} if $b_i = 1$ (that is, a vertex adjacent to all vertices in G_{i-1}).

Clearly, $G(b) = G_n$. Note first that the resulting threshold graph is independent of b_1 due to (i) above. So we use the convention that the binary sequences always start with 0. Conversely, it is easy to see that a threshold graph on n vertices gives rise to a binary sequence $b = 0b_2\dots b_n$, where $b_i = 0$ if the vertex added at the ith step to the graph is isolated, or $b_i = 1$ if it is dominating ($i = 2,\dots,n$). Thus, there is a bijection between the set of threshold graphs on n vertices and binary sequences of length n starting with 0 at the first position. Moreover, if $b_n = 1$ in the binary sequence b, then the constructed threshold graph $G(b)$ is connected; otherwise, it is disconnected due to isolated vertices.

For simplicity, it is convenient to write a binary sequence b in a shorthand notation so that the maximal run of t consecutive 0's (or s consecutive 1's) is written as 0^t (or 1^s, respectively). Then, for any connected threshold graph, b has the form

$$b = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\dots(0^{t_h}1^{s_h}), \quad t_i, s_i > 0.$$

In [11] an explicit formula for calculating the characteristic polynomial of a threshold graph G with a given binary sequence was obtained. It reads

$$\phi(x;G) = x^{\sum_{i=1}^h t_i - h}(x+1)^{\sum_{i=1}^h s_i - h}\det N_x,$$

where

$$N_x = \begin{pmatrix} x+t_1 & x+1 & & & & \\ x & s_1 & x & & & \\ & x+1 & t_2 & x+1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & x+1 & t_h & x+1 \\ & & & & x & s_h \end{pmatrix}. \tag{5.1}$$

In view of (2.2), the determinant of N_x can be obtained as

$$\det N_x = T_{2h}^{\sqrt{x(x+1)}}(t_1s_1\dots t_hs_h) + xT_{2h-1}^{\sqrt{x(x+1)}}(s_1t_2\dots t_hs_h).$$

The eigenvalues of G different from 0 and -1 are the eigenvalues of the corresponding divisor matrix whose characteristic polynomial is equal to $\det N_x$.

In [10] it was shown that no threshold graph has eigenvalues in the interval $(-1,0)$. Note that $x(x+1)>0$ for any $x\notin[-1,0]$. Later, in [1] it was conjectured that this interval can be extended to $(\frac{1}{2}(-1-\sqrt{2}),0)\cup(0,\frac{1}{2}(-1+\sqrt{2}))$. The proof of this conjecture is given in [9]. Here, we go one step further. We show that for a threshold graph with a given binary sequence, the previous interval can be extended. As a corollary we prove the conjecture given in [1] in a different way.

Theorem 5.1. *Let G be a threshold graph associated to the binary sequence*

$$(0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\dots(0^{t_h}1^{s_h}),\quad t_i,s_i>0,$$

and let

$$N_h=\frac{1}{\cos^2(\pi/(2h+1))}\min_{2\leqslant i\leqslant 2h-1}a_ia_{i+1}\quad\text{and}\quad c_1=\frac{s_1}{4\cos^2(\pi/(2h+1))},$$

where $a_1a_2\dots a_{2h-1}a_{2h}=t_1s_1\dots t_hs_h$. Then:

▷ *If $t_1=1$, then G has no eigenvalue in*

$$\Bigl(\frac{1}{2}(-1-\sqrt{1+N_h}),\min\Bigl\{\frac{1}{2}(-1+\sqrt{1+N_h}),c_1\Bigr\}\Bigr),$$

except possibly -1 and 0.

▷ *If $t_1\neq1$, then G has no eigenvalue in (l_h,r_h), where*

$$\begin{aligned}l_h&=\max\Bigl\{\frac{1}{2}(-1-\sqrt{1+N_h}),\frac{1}{2}\bigl(-1+c_1-\sqrt{(-1+c_1)^2+4c_1t_1}\bigr)\Bigr\}\\ r_h&=\min\Bigl\{\frac{1}{2}(-1+\sqrt{1+N_h}),\frac{1}{2}\bigl(-1+c_1+\sqrt{(-1+c_1)^2+4c_1t_1}\bigr)\Bigr\},\end{aligned}$$

except possibly -1 and 0.

Proof. If $t_1=1$, then $x=-1$ is an eigenvalue of N_x. Also, in this case $\det N_x$ is equal to

$$(x+1)\det\begin{pmatrix}1&1&&&&\\ x&s_1&x&&&\\ &x+1&t_2&x+1&&\\ &&\ddots&\ddots&\ddots&\\ &&&x+1&t_h&x+1\\ &&&&x&s_h\end{pmatrix},$$

i.e., to

$$(x+1)\det\begin{pmatrix} 1 & \sqrt{x} & & & & \\ \sqrt{x} & s_1 & \sqrt{x(x+1)} & & & \\ & \sqrt{x(x+1)} & t_2 & \sqrt{x(x+1)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{x(x+1)} & t_h & \sqrt{x(x+1)} \\ & & & & \sqrt{x(x+1)} & s_h \end{pmatrix}.$$

For all real values x such that

$$x(x+1) < \frac{1}{4\cos^2(\pi/(2h+1))}a_i a_{i+1}, \quad i \geqslant 2, \quad \text{and} \quad x < \frac{s_1}{4\cos^2(\pi/(2h+1))},$$

the matrix N_x is positive definite, i.e., for all

$$x \in \Bigl(\frac{1}{2}\bigl(-1-\sqrt{1+N_h}\bigr), \min\Bigl\{\frac{1}{2}\bigl(-1+\sqrt{1+N_h}\bigr), c_1\Bigr\}\Bigr).$$

If $t_1 \neq 1$, then for all $x \in \mathbb{R}$ such that

$$x+t_1>0, \quad x(x+1) < \frac{1}{4\cos^2(\pi/(2h+1))}\min_{2\leqslant i\leqslant 2h-1} a_i a_{i+1}$$
$$x(x+1) < \frac{1}{4\cos^2(\pi/(2h+1))}(x+t_1)s_1,$$

i.e., for all $x \in I_1 \cap I_2 \cap I_3$, where

$$\begin{aligned} I_1 &= \Bigl(\frac{1}{2}\bigl(-1-\sqrt{1+N_h}\bigr), \frac{1}{2}\bigl(-1+\sqrt{1+N_h}\bigr)\Bigr), \\ I_2 &= \Bigl(\frac{1}{2}\bigl(-1+c_1-\sqrt{(-1+c_1)^2+4c_1t_1}\bigr), \frac{1}{2}\bigl(-1+c_1+\sqrt{(-1+c_1)^2+4c_1t_1}\bigr)\Bigr), \\ I_3 &= (-t_1, \infty), \end{aligned}$$

the matrix N_x is positive definite. Since $\frac{1}{2}\bigl(-1+c_1-\sqrt{(-1+c_1)^2+4c_1t_1}\bigr) \geqslant -t_1$, $I_1 \cap I_2 \cap I_3 = (l_h, r_h)$. Also, it can be easily verified that $\frac{1}{2}\bigl(-1+\sqrt{1+N_h}\bigr) > \frac{1}{2}\bigl(-1+c_1-\sqrt{(-1+c_1)^2+4c_1t_1}\bigr)$ as well as $\frac{1}{2}\bigl(-1+c_1+\sqrt{(-1+c_1)^2+4c_1t_1}\bigr) > \frac{1}{2}\bigl(-1-\sqrt{1+N_h}\bigr)$, which implies $r_h > l_h$, i.e., the interval (l_h, r_h) is always nonempty.

This completes the proof. □

Remark 5.2. Taking into account results on the inertia of threshold graphs (for example, see [2], [5]), the results of Theorem 5.1 can be reformulated in the following way:

▷ If $t_1 = 1$, then $\lambda_{n-h} < \frac{1}{2}(-1-\sqrt{1+N_h})$ and $\lambda_h > \min\{\frac{1}{2}(-1+\sqrt{1+N_h}), c_1\}$.
▷ If $t_1 \neq 1$, then $\lambda_{n-h+1} < l_h$ and $\lambda_h > r_h$.

Corollary 5.3. *Let G be a threshold graph associated to a binary sequence*

$$(0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\dots(0^{t_h}1^{s_h}),\quad t_i, s_i > 0.$$

Then G has no eigenvalue in the interval

$$\Bigl(-\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{1}{\cos^2(\pi/(2h+1))}}, -\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{1}{\cos^2(\pi/(2h+1))}}\Bigr),$$

except possibly -1 and 0.

Proof. For any t_i, s_i, the interval

$$\Bigl(-\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{1}{\cos^2(\pi/(2h+1))}}, -\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{1}{\cos^2(\pi/(2h+1))}}\Bigr)$$

is contained in the eigenvalue free intervals obtained in the previous theorem. $\square$

Remark 5.4. We point out that

$$-1\pm\sqrt{1+\frac{1}{\cos^2(\pi/(2h+1))}}$$

are the least positive and the greatest negative eigenvalues of the antiregular graph given by the generating sequence $\underbrace{01\dots01}_{2h}$.

Since

$$1+\frac{1}{\cos^2(\pi/(2h+1))}\leqslant 2,$$

we have the following corollary.

Corollary 5.5. *Let G be a threshold graph. Then G has no eigenvalue in the interval $\bigl(\frac{1}{2}(-1-\sqrt{2}), \frac{1}{2}(-1+\sqrt{2})\bigr)$, except possibly -1 and 0.*

References

[1] *C. O. Aguilar, J.-Y. Lee, E. Piato, B. J. Schweitzer*: Spectral characterizations of anti-regular graphs. Linear Algebra Appl. *557* (2018), 84–104. zbl MR doi

[2] *A. Alazemi, M. Anđelić, S. K. Simić*: Eigenvalue location for chain graphs. Linear Algebra Appl. *505* (2016), 194–210. zbl MR doi

[3] *M. Anđelić, E. Andrade, D. M. Cardoso, C. M. da Fonseca, S. K. Simić, D. V. Tošić*: Some new considerations about double nested graphs. Linear Algebra Appl. *483* (2015), 323–341. zbl MR doi

[4] *M. Anđelić, C. M. da Fonseca*: Sufficient conditions for positive definiteness of tridiagonal matrices revisited. Positivity *15* (2011), 155–159. zbl MR doi

[5] *M. Anđelić, E. Ghorbani, S. K. Simić*: Vertex types in threshold and chain graphs. Discrete Appl. Math. *269* (2019), 159–168. zbl MR doi
[6] *M. Anđelić, S. K. Simić, D. Živković, E. Ć. Dolićanin*: Fast algorithms for computing the characteristic polynomial of threshold and chain graphs. Appl. Math. Comput. *332* (2018), 329–337. zbl MR doi
[7] *D. Cvetković, P. Rowlinson, S. Simić*: An Introduction to the Theory of Graph Spectra. London Mathematical Society Student Texts 75, Cambridge University Press, Cambridge, 2010. zbl MR doi
[8] *C. M. da Fonseca*: On the eigenvalues of some tridiagonal matrices. J. Comput. Appl. Math. *200* (2007), 283–286. zbl MR doi
[9] *E. Ghorbani*: Eigenvalue-free interval for threshold graphs. Linear Algebra Appl. *583* (2019), 300–305. zbl MR doi
[10] *D. P. Jacobs, V. Trevisan, F. Tura*: Eigenvalues and energy in threshold graphs. Linear Algebra Appl. *465* (2015), 412–425. zbl MR doi
[11] *J. Lazzarin, O. F. Márquez, F. C. Tura*: No threshold graphs are cospectral. Linear Algebra Appl. *560* (2019), 133–145. zbl MR doi
[12] *J. S. Maybee, D. D. Olesky, P. van den Driessche, G. Wiener*: Matrices, digraphs, and determinants. SIAM J. Matrix Anal. Appl. *10* (1989), 500–519. zbl MR doi

Authors' addresses: Milica Anđelić (corresponding author), Department of Mathematics, Faculty of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait, e-mail: `milica@sci.kuniv.edu.kw`; Zhibin Du, School of Software, South China Normal University, Foshan, Guangdong 528225, P. R. China, School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, Guangdong, P. R. China, e-mail: `zhibindu@126.com`; Carlos M. da Fonseca, Kuwait College of Science and Technology, Doha District, Block 4, P.O. Box 27235, Safat 13133, Kuwait, e-mail: `c.dafonseca@kcst.edu.kw`, University of Primorska, FAMNIT, Glagoljsaška 8, 6000 Koper, Slovenia, e-mail: `carlos.dafonseca@famnit.upr.si`; Slobodan K. Simić, Mathematical Institute SANU, Kneza Mihaila 36, 11 000 Belgrade, Serbia, e-mail: `sksimic@mi.sanu.ac.rs`.