Zoran Stanić

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MAIN EIGENVALUES OF REAL SYMMETRIC MATRICES WITH APPLICATION TO SIGNED GRAPHS

Zoran Stanić, Belgrade

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Abstract. An eigenvalue of a real symmetric matrix is called main if there is an associated eigenvector not orthogonal to the all-1 vector $\mathbf{j}$. Main eigenvalues are frequently considered in the framework of simple undirected graphs. In this study we generalize some results and then apply them to signed graphs.

Keywords: main angle; signed graph; adjacency matrix; Laplacian matrix; Gram matrix

MSC 2020: 05C22, 05C50

1. Introduction

Given a graph $G = (V(G), E(G))$, let $\sigma\colon\ E(G) \to \{-1, +1\}$. Then $\dot{G} = (G, \sigma)$ is a *signed graph* derived from its *underlying graph* G. The edge set of a signed graph is composed of subsets containing positive and negative edges, respectively. We interpret an (unsigned) graph as a signed graph with all the edges being positive.

The *degree* d_i of a vertex i of $\dot{G}$ is the number of edges incident with i. We also write d_i^+ (or d_i^-) for the number of positive (or negative) edges incident with i. We say that $\dot{G}$ is *regular* if the degree is a constant on the vertex set. The difference $d_i^+ - d_i^-$ is called the *net-degree* of i. We say that $\dot{G}$ is *net-regular* if the net-degree is a constant on the vertex set. The *negation* of $\dot{G}$, denoted by $-\dot{G}$, is obtained by reversing the sign of every edge of $\dot{G}$. The subgraph induced by negative edges is denoted by $\dot{G}^-$.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the $(0,1)$-adjacency matrix of the underlying graph G by reversing the sign of all 1's which correspond to negative

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edges. The eigenvalues of $\dot{G}$ are identified to be the eigenvalues of $A_{\dot{G}}$, they form the *spectrum* of $\dot{G}$. The *Laplacian matrix* of $\dot{G}$ is defined by $L_{\dot{G}} = D - A_{\dot{G}}$, where D is the diagonal matrix of vertex degrees. The *Laplacian eigenvalues* of $\dot{G}$ are identified to be the eigenvalues of $L_{\dot{G}}$. Needless to add, in the case of graphs, $L_{\dot{G}}$ reduces to the standard Laplacian matrix. We also recall that the *signless Laplacian matrix* of a graph G is defined by $Q_G = D + A_G$.

The signed graphs $\dot{G}$ and $\dot{H}$ are said to be *switching equivalent* if there exists a diagonal matrix E of ± 1's such that $A_{\dot{H}} = E^{-1} A_{\dot{G}} E$. This is an equivalence relation that preserves the eigenvalues of $A_{\dot{G}}$ and $L_{\dot{G}}$.

A *walk* in a signed graph is a sequence of alternate vertices and edges such that consecutive vertices are incident with the corresponding edge. Such a walk is *positive* if the number of its negative edges (counted with their multiplicity if there are repeated edges) is not odd. Otherwise, it is *negative*. The difference between positive and negative walks of length k starting at i is denoted by $w_k(i)$.

Introduce the vertex-edge orientation $\eta\colon V(\dot{G}) \times E(\dot{G}) \to \{1, 0, -1\}$ formed by obeying the following rules:

(1) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$,

(2) $\eta(i, ij) = 1$ or $\eta(i, ij) = -1$,

(3) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$.

The vertex-edge incidence matrix B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$, respectively, such that its (i, e)-entry is equal to $\eta(i, e)$. An *oriented signed graph* $\dot{G}_\eta$ is the ordered pair $(\dot{G}, \eta)$. Then we have

$$B_\eta^\top B_\eta = 2I + A_{L(\dot{G}_\eta)},$$

where $L(\dot{G}_\eta)$ is taken to be the *signed line graph* of $\dot{G}_\eta$. Observe that different vertex-edge orientations result in switching equivalent signed line graphs making up the switching class $\mathcal{L}(\dot{G})$ also called a *signed line graph* of the (unoriented) signed graph $\dot{G}$: for some different concepts of signed line graphs, see [13].

We denote the (standard) line graph of a graph G by $\mathrm{Line}(G)$. Since G can be interpreted as a signed graph, there exists the switching class $\mathcal{L}(G)$. We remark that, in general, $\mathrm{Line}(G)$ does not need to belong to $\mathcal{L}(G)$.

For a matrix M, we use $\mathrm{sum}(M)$ to denote the sum of its entries. The characteristic polynomial of M is denoted by Φ_M. For the adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ we refer to $\Phi_{A_{\dot{G}}}$ as $\Phi_{\dot{G}}$.

Our results are announced in the Abstract. In Section 2 we consider the main eigenvalues of real symmetric matrices. It appears that we can say more in the case of Gram matrices, and so they are dealt with separately in Section 3. The main eigenvalues of the matrices associated with signed graphs are considered in Section 4.

2. General results

For easier reading, one may bear in mind that the matrix under consideration is one of the standard matrices associated with a (signed) graph. Results of this section will be used in Section 4.

Given an $n \times n$ real symmetric matrix M with (distinct) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$, let P_i represent the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\lambda_i)$ with respect to the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. Then the *spectral decomposition* of M is given by

$$M = \sum_{i=1}^{d} \lambda_i P_i.$$

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t\}$ is an orthonormal basis of $\mathcal{E}(\lambda_i)$, then $P_i = \sum_{i=1}^{t} \mathbf{x}_i \mathbf{x}_i^\top$, and therefore the matrices $P_1, P_2, \dots, P_d$ are symmetric, idempotent and mutually orthogonal (in the sense that $P_i P_j = 0$ for $i \neq j$). Accordingly, for any polynomial p over $\mathbb{R}$ we have

$$p(M) = \sum_{i=1}^{d} p(\lambda_i) P_i. \tag{2.1}$$

We also have

$$\mathrm{sum}(M^k) = \mathbf{j}^\top M^k \mathbf{j} = \sum_{i=1}^{d} \lambda_i^k \|P_i \mathbf{j}\|^2. \tag{2.2}$$

The numbers $\beta_i = \|P_i \mathbf{j}\| / \sqrt{n}$ are called the *main angles* of M. Precisely, they are the cosines of the angles between the eigenspaces $\mathcal{E}(\lambda_i)$ and the main direction $\mathbf{j}$. An eigenvalue of M is called *main* if the corresponding main angle is nonzero. Equivalently, it is main if there is an associated eigenvector not orthogonal to $\mathbf{j}$. We denote the main eigenvalues of M by $\mu_1, \mu_2, \dots, \mu_s$ and set

$$m(x) = \sum_{i=1}^{s} (x - \mu_i).$$

Let $\{\Delta_1, \Delta_2, \dots, \Delta_k\}$ be a partition of $\{1, 2, \dots, n\}$ which determines a blocking $M = (M_{i,j})$ such that each block $M_{i,j}$ has a constant row sum, say $f_{i,j}$. Observe that such a blocking exists for any M, as we can always take $k = n$. The corresponding partition is called an *equitable partition* and the $k \times k$ matrix $(f_{i,j})$ is denoted by F.

We transfer some results, along with slight modifications of their proofs, concerning the main eigenvalues of (the adjacency matrix) of a graph, which can be found in [10], the reader can also consult in [3].

Theorem 2.1. *Let M be an $n\times n$ real symmetric matrix with precisely s main eigenvalues and for $1\leqslant i\leqslant s$ let the main eigenvalues and the corresponding main angles be denoted by μ_i and β_i, respectively. The following statements hold.*

(i) $\beta_i=\sqrt{\operatorname{sum}(P_i)/n}$ *for* $1\leqslant i\leqslant s$.

(ii) $s\neq 0$, $\sum\limits_{i=1}^{s}\beta_i^2=1$ *and*

$$\operatorname{sum}(M^k)=n\sum_{i=1}^{s}\mu_i^k\beta_i^2. \tag{2.3}$$

(iii) *For a polynomial p over $\mathbb{R}$, $p(M)\mathbf{j}=0$ if and only if the polynomial m divides p. In particular, $m(M)\mathbf{j}=\mathbf{0}$.*

(iv) *The largest k such that the vectors $\mathbf{j}$, $M\mathbf{j},\dots,M^k\mathbf{j}$ are linearly independent is equal to $s-1$.*

(v) *The polynomial m divides Φ_F.*

(vi) *The spectrum of F is contained in the spectrum of M (taking into account the repetition of eigenvalues).*

(vii) *$s=1$ if and only if $\mathbf{j}$ is an eigenvector of M. The unique main eigenvalue is associated with $\mathbf{j}$.*

(viii) *$s=2$ if and only if $\mathbf{j}$ is not an eigenvector of M and $(M^2-aM+bI)\mathbf{j}=\mathbf{0}$ holds for some $a,b\in\mathbb{R}$. The main eigenvalues μ_1, μ_2 are determined by $\mu_1+\mu_2=a$, $\mu_1\mu_2=b$.*

Proof. (i) We have

$$\beta_i=\frac{1}{\sqrt{n}}\|P_i\mathbf{j}\|=\sqrt{\frac{1}{n}(P_i\mathbf{j})^\top(P_i\mathbf{j})}=\sqrt{\frac{1}{n}\mathbf{j}^\top P_iP_i\mathbf{j}}=\sqrt{\frac{1}{n}\mathbf{j}^\top P_i\mathbf{j}}=\sqrt{\frac{1}{n}\operatorname{sum}(P_i)},$$

where the fourth equality follows since P_i is idempotent.

(ii) $s\neq 0$ follows by definition of main eigenvalue. As $\sum\limits_{i=1}^{s}P_i\mathbf{j}=\mathbf{j}$, we have $\sum\limits_{i=1}^{s}\beta_i^2=1$, while the equality (2.3) is deduced from (2.2).

(iii) By (2.1), we get $p(M)\mathbf{j}=\sum\limits_{i=1}^{s}p(\mu_i)P_i\mathbf{j}$. Since the vectors $P_1\mathbf{j},P_2\mathbf{j},\dots,P_s\mathbf{j}$ are linearly independent, we have $p(M)\mathbf{j}=0$ if and only if $p(\mu_i)=0$ for $1\leqslant i\leqslant s$. The particular case follows directly.

(iv) If $\sum\limits_{i=1}^{s-1}c_iM^i\mathbf{j}=0$, then by case (iii), $m(x)$ divides $\sum\limits_{i=1}^{s-1}c_ix^i$, which is possible only if $(c_1,c_2,\dots,c_{s-1})^\top=0$. Thus, $k\geqslant s-1$. If $k>s-1$, then the vectors $\mathbf{j}$, $M\mathbf{j},\dots,M^s\mathbf{j}$ are linearly independent, contradicting $m(M)\mathbf{j}=0$.

(v) If $\{\Delta_1, \Delta_2, \dots, \Delta_k\}$ is a partition of $\{1, 2, \dots, n\}$ which determines F, then for $1 \leqslant i \leqslant n$ we define $\mathbf{d}_i \in \mathbb{R}^n$ by

$$\mathbf{d}_i(j) = \begin{cases} 1 & \text{if } j \in \Delta_i, \\ 0 & \text{if } j \notin \Delta_i. \end{cases}$$

Observe that if $F\mathbf{x} = \mathbf{y}$ for some $\mathbf{x} = (x_1, x_2, \dots, x_k)^\top$, $\mathbf{y} = (y_1, y_2, \dots, y_k)^\top$, then $M \sum_{i=1}^{k} x_i \mathbf{d}_i = \sum_{i=1}^{k} y_i \mathbf{d}_i$.

Now $\Phi_F(F) = 0$, and $\Phi_F(F)\mathbf{j} = \mathbf{0}$ implies $\Phi_F(M) \sum_{i=1}^{k} \mathbf{d}_i = \mathbf{0}$. As $\sum_{i=1}^{k} \mathbf{d}_i = \mathbf{j}$, it follows that $\Phi_F(M)\mathbf{j} = \mathbf{0}$, and then by case (iii) we have that m divides Φ_F.

(vi) This result is known from the work of Haynsworth, see [6] or Petersdorf and Sachs, see [9], the corresponding result can also be found in [2], Theorem 0.12.

(vii) By case (iv), $s = 1$ if and only if $(M - aI)\mathbf{j} = \mathbf{0}$ for some $a \in \mathbb{R}$, and the result follows.

(viii) By cases (iv) and (vii), $s = 2$ if and only if $(M^2 - aM + bI)\mathbf{j} = \mathbf{0}$ for some $a, b \in \mathbb{R}$, and $\mathbf{j}$ is not an eigenvector of M. Observing that for $s = 2$, $m(x) = x^2 - ax + b$, we conclude the proof. $\square$

According to Theorem 2.1 (vi), matrix F is called the *divisor* (or the *front divisor*) of M. Note that if M is an integer matrix, then the coefficients a, b which appear in case (viii) are also integers.

The theory of main eigenvalues of the adjacency matrix of a simple graph is highly developed, which in particular means that many other results can be transferred as in the previous theorem. We restricted ourselves to those that will be used in the sequel.

3. Gram matrices

Results of this section will be used in Subsection 4.4.

A *Gram matrix* M of a set of vectors $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m \in \mathbb{R}^m$ is the $m \times m$ inner product matrix whose (i, j)-entry is $\langle \mathbf{s}_i, \mathbf{s}_j \rangle$. Setting $S = (\mathbf{s}_1 | \mathbf{s}_2 | \dots | \mathbf{s}_m)$, we get $M = S^\top S$. It is known that both $S^\top S$ and $SS^\top$ are symmetric positive semidefinite and that they share the same nonzero eigenvalues (taken with their repetition).

If 0 is an eigenvalue of $S^\top S$, then a nonzero vector $\mathbf{x} \in \mathbb{R}^m$ is associated with 0 if and only if $S\mathbf{x} = \mathbf{0}$. Indeed, $S\mathbf{x} = \mathbf{0}$ obviously gives $S^\top S\mathbf{x} = \mathbf{0}$; on the contrary, $S^\top S\mathbf{x} = \mathbf{0}$ implies $\mathbf{x}^\top S^\top S\mathbf{x} = 0$, that is $\|S\mathbf{x}\| = 0$, i.e., $S\mathbf{x} = \mathbf{0}$. Consequently, if $S\mathbf{x} \neq \mathbf{0}$ for all nonzero $\mathbf{x} \in \mathbb{R}^m$, then $S^\top S$ is non-singular.

If $\mathbf{x}$ is an eigenvector associated with a nonzero eigenvalue λ of $S^\top S$, then $S^\top S\mathbf{x} = \lambda\mathbf{x}$ gives $SS^\top S\mathbf{x} = \lambda S\mathbf{x}$, i.e., $S\mathbf{x}$ is associated with the same eigenvalue in $SS^\top$.

Assume further that for $1 \leqslant i \leqslant m$, $\operatorname{sum}(\mathbf{s}_i) \in \{0, c\}$, where $c \in \mathbb{R} \setminus \{0\}$.

Theorem 3.1. *Let $S = (\mathbf{s}_1|\mathbf{s}_2|\dots|\mathbf{s}_m)$ be an $n \times m$ real matrix such that*

$$\operatorname{sum}(\mathbf{s}_i) = \begin{cases} c & \text{if } i \leqslant l, \\ 0 & \text{if } i > l, \end{cases}$$

for a fixed l ($0 \leqslant l \leqslant m$). If $\|P_i\mathbf{j}\|/\sqrt{m}$ is the main angle corresponding to the nonzero eigenvalue λ_i of $S^\top S$, then the main angle corresponding to the same eigenvalue of $SS^\top$ is $c\sqrt{\operatorname{sum}(P_i^{(l)})/n}$, where $P_i^{(l)}$ is the $l \times l$ top-left block of P_i.

Proof. If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t\}$ is an orthonormal basis of $\mathcal{E}(\lambda_i)$ (in $S^\top S$), then $P_i = \sum\limits_{i=1}^{t} \mathbf{x}_i\mathbf{x}_i^\top$. The orthogonal projection (say P_i') onto $\mathbb{R}^n$ of the eigenspace of λ_i in $SS^\top$ is given by

$$P_i' = \sum_{i=1}^{t} S\mathbf{x}_i(S\mathbf{x}_i)^\top = \sum_{i=1}^{t} S\mathbf{x}_i\mathbf{x}_i^\top S^\top = S\left(\sum_{i=1}^{t} \mathbf{x}_i\mathbf{x}_i^\top\right)S^\top = SP_iS^\top.$$

Due to Theorem 2.1 (i), for the corresponding main angle (say β_i'), we have

$$\beta_i' = \sqrt{\frac{1}{n}\operatorname{sum}(P_i')} = \sqrt{\frac{1}{n}\operatorname{sum}(SP_iS^\top)} = \sqrt{\frac{1}{n}\mathbf{j}^\top SP_iS^\top\mathbf{j}} \tag{3.1}$$

By the assumption on S, we have

$$\mathbf{j}^\top S = (S^\top\mathbf{j})^\top = (\underbrace{c, c, \dots, c}_{l}, 0, 0, \dots, 0),$$

which together with (3.1) gives the result. □

In the previous theorem we considered the matrix S in which the first l columns coincide with vectors whose sum is equal to c. This assumption does not essentially affect the result, as a rearrangement of columns produces the permutation matrix which simultaneously permutes rows and columns of $S^\top S$ and coordinates of associated eigenvectors. As a result, $P_i^{(l)}$ is the $l \times l$, but not necessarily the top-left, submatrix of P_i.

Corollary 3.2. *Under the assumptions of Theorem 3.1:*

(i) *If $l = 0$, then 0 is the unique main eigenvalue of $SS^\top$.*

(ii) *If $l = m$, then a nonzero eigenvalue of $S^\top S$ is main if and only if it is a main eigenvalue of $SS^\top$.*

Proof. Assume that λ is a nonzero eigenvalue of $S^\top S$. Then it is an eigenvalue of $SS^\top$ as well.

(i) By Theorem 3.1, the main angle of λ in $SS^\top$ is zero, and thus λ is non-main. Since λ is arbitrary, all nonzero eigenvalues are non-main. On the other hand, by Theorem 2.1 (ii), there must be at least one main eigenvalue, and the result follows.

(ii) By Theorem 3.1, the main angle of λ in $SS^\top$ is zero if and only if the main angle of λ in $S^\top S$ is zero. $\square$

4. Main eigenvalues of signed graphs

We consider some applications of the foregoing results.

4.1. Counting walks. Given a signed graph with n vertices, let $N_k = \sum\limits_{i=1}^{n} w_k(i)$, i.e., N_k denotes the difference of the numbers of positive and negative walks of length k.

Lemma 4.1. *Let $\dot{G}$ be a signed graph with n vertices and main eigenvalues $\mu_1, \mu_2, \ldots, \mu_s$. The following statements hold.*

(i) $N_k = n \sum\limits_{i=1}^{s} \mu_i^k \beta_i^2$.

(ii) *An eigenvalue λ of $\dot{G}$ is main if and only if $-\lambda$ is a main eigenvalue of $-\dot{G}$. Moreover, they share the same main angle.*

(iii) *If λ is a non-main eigenvalue of $\dot{G}$, then there exists a switching equivalent signed graph in which λ is main.*

Proof. (i) Since $N_k = \mathrm{sum}(A_{\dot{G}}^k)$, the result follows by (2.3).

(ii) This follows since λ (of $\dot{G}$) and $-\lambda$ of $(-\dot{G})$ share the same eigenspace.

(iii) If $A_{\dot{G}}\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$, and $E = (e_{ij})$ is the diagonal matrix determined by

$$e_{ii} = \begin{cases} -1 & \text{if } x_i < 0, \\ 1 & \text{if } x_i \geqslant 0, \end{cases}$$

then $E\mathbf{x}$ is an eigenvector associated with λ in $E^{-1}A_{\dot{G}}E$ since

$$(E^{-1}A_{\dot{G}}E)E\mathbf{x} = EA_{\dot{G}}\mathbf{x} = \lambda E\mathbf{x}.$$

Moreover, the coordinates of $E\mathbf{x}$ are non-negative, and we are done. $\square$

In the particular case of graphs, Lemma 4.1 (i) reduces to the well-known result expressing the number of walks of length k in terms of main eigenvalues and main angles, see [10].

Ordering graphs with respect to a fixed structural or spectral parameter is a frequently studied topic; a number of results can be found in [11]. Since signed graphs which belong to a fixed switching class share the same spectrum, it seems natural to consider their ordering by some other parameter, which is, in some sense, close to the spectrum. Inspired by the previous lemma, we propose the ordering with respect to N_k (k being fixed). We say more if a signed graph has 2 eigenvalues.

Lemma 4.2. *If a signed graph $\dot{G}$ has precisely 2 eigenvalues, λ and μ with $\lambda \neq -\mu$, then the ordering by N_k ($k \geqslant 1$) of signed graphs in the switching class of $\dot{G}$ coincides with the ordering by the number of positive edges.*

Proof. Assume that λ is positive then μ must be negative. Without loss of generality, we may also assume that $\lambda > -\mu$, as otherwise, by Lemma 4.1 (ii) we can consider $-\dot{G}$ instead.

Here, $N_k = n(\lambda^k \beta_\lambda^2 + \mu^k \beta_\mu^2)$, where the β-notation is clear from the context. As $\beta_\mu^2 = 1 - \beta_\lambda^2$ (by a part of Theorem 2.1 (ii)), we have

$$N_k = n(\beta_\lambda^2(\lambda^k - \mu^k) + \mu^k).$$

If $\dot{G}_1$ and $\dot{G}_2$ are switching equivalent to $\dot{G}$, then as $\lambda > -\mu$, we have $N_k(\dot{G}_1) \leqslant N_k(\dot{G}_2)$ if and only if $\beta_\lambda(\dot{G}_1) \leqslant \beta_\lambda(\dot{G}_2)$. Consequently, $N_k(\dot{G}_1) \leqslant N_k(\dot{G}_2)$ if and only if $N_1(\dot{G}_1) \leqslant N_1(\dot{G}_2)$ (the ordering does not depend on k). The result follows as N_1 is the difference of the numbers of positive and negative edges in a signed graph. □

For $\lambda = -\mu$ and k even, we get $N_k = n\mu^k$ for every signed graph in a switching class.

Example 1. The ordering by N_k in the switching class of K_n is determined by the following rule: If $A_{\dot{G}_1} = E_1^{-1} A_{K_n} E_1$ and $A_{\dot{G}_2} = E_2^{-1} A_{K_n} E_2$, then $N_k(\dot{G}_1) \leqslant N_k(\dot{G}_2)$ if and only if $|\mathrm{sum}(E_1)| \leqslant |\mathrm{sum}(E_2)|$. Indeed, the number of positive edges of G_i for $i \in \{1, 2\}$ is $\binom{n}{2} - l_i(n - l_i)$, where l_i is the number of -1's in E_i. On the other hand, $l_i = \frac{1}{2}(n - \mathrm{sum}(E_i))$, giving $l_i(n - l_i) = \frac{1}{4}(n^2 - \mathrm{sum}(E_i)^2)$, and so the number of positive edges increases with $|\mathrm{sum}(E_i)|$.

4.2. Signed graphs with at most 2 main eigenvalues.

Lemma 4.3. *A signed graph $\dot{G}$ has exactly one main eigenvalue if and only if $\dot{G}$ is net-regular (with net-degree as the main eigenvalue). Similarly, $\dot{G}$ has exactly one main Laplacian eigenvalue if and only if $\dot{G}^-$ is regular (with $2d_i^-$ as the main eigenvalue).*

P r o o f. By Theorem 2.1, any matrix associated with a signed graph has exactly one main eigenvalue if and only if $\mathbf{j}$ is its eigenvector.

For the adjacency matrix this occurs if and only if $\dot{G}$ is net-regular, which is an easy exercise, see [12], [13]. For the Laplacian, we have $(D - A_{\dot{G}})\mathbf{j} = \lambda\mathbf{j}$, giving $d_i^+ + d_i^- - (d_i^+ - d_i^-) = \lambda$ for every vertex i, i.e., $2d_i^- = \lambda$, which leads to the result. □

By Theorem 2.1 (v)–(vi), if a real symmetric matrix, say M, admits an equitable partition which produces a $k \times k$ divisor, then M has at most k main eigenvalues. Here is a particular result.

Corollary 4.4. *If $\dot{G}$ is not net-regular and its adjacency matrix has a 2×2 divisor, then $\dot{G}$ has exactly 2 main eigenvalues; they are the eigenvalues of the corresponding divisor.*

P r o o f. This follows by Theorem 2.1 (v)–(vi) and Lemma 4.3. □

For graphs, the adjacency matrix has a 2×2 divisor if and only if there exists a vertex partition into 2 sets such that all the vertices of the same set has equal number of neigbours in it and also equal number of neighbours in the other set. If the graph is non-regular, then it has exactly 2 main eigenvalues. We note that all the graphs with 2 main eigenvalues obtained in [1] (and described in terms of so-called (κ, τ)-regular sets) admit aforementioned partition.

The reader may also observe that all the trees with 2 main eigenvalues (obtained in [8]) admit either the same vertex partition or a similar partition into 3 sets. More results on graphs with 2 main eigenvalues can be found in [7], [10].

4.3. Signed graphs in which all eigenvalues are main. We continue with some results on signed graphs described in the subtitle. By $\mathbf{x}|_U$ we denote the restriction of the eigenvector $\mathbf{x}$ on the vertex subset U.

Theorem 4.5. *Let $\dot{G}$ be a net-regular signed graph and let V denote its vertex set. If there exists $U \subseteq V$ such that*

- ▷ *for every non-main eigenvalue there is an associated eigenvector $\mathbf{x}$ satisfying $\operatorname{sum}(\mathbf{x}|_U) \neq 0$ and*
- ▷ *for every main eigenvalue there is an associated eigenvector $\mathbf{x}$ non-orthogonal to $\mathbf{j}$ satisfying $\operatorname{sum}(\mathbf{x}|_U) \neq \operatorname{sum}(\mathbf{x}|_{V\setminus U})$,*

then $\dot{G}$ switches to a signed graph in which all eigenvalues are main.

P r o o f. Let $\dot{H}$ be obtained by switching with respect to U, that is, $A_{\dot{H}} = E^{-1}A_{\dot{G}}E$, where $e_{ii} = 1$ precisely when $i \in U$. Also, let λ be an eigenvalue of $\dot{G}$

and $\mathbf{x}$ an associated eigenvector which satisfies the assumptions of the theorem. Then $E\mathbf{x}$ is an eigenvector associated with λ in $\dot{H}$. Observe that $E\mathbf{x}|_U = \mathbf{x}|_U$ and $E\mathbf{x}|_{V\setminus U} = -\mathbf{x}|_{V\setminus U}$.

If λ is non-main in $\dot{G}$, then $\mathbf{x} \perp \mathbf{j}$, i.e., $\operatorname{sum}(\mathbf{x}) = 0$. Since $\operatorname{sum}(\mathbf{x}) = \operatorname{sum}(\mathbf{x}|_U) + \operatorname{sum}(\mathbf{x}|_{V\setminus U})$, we have $\operatorname{sum}(\mathbf{x}|_U) = -\operatorname{sum}(\mathbf{x}|_{V\setminus U})$. It follows that $\operatorname{sum}(E\mathbf{x}) = 2\operatorname{sum}(\mathbf{x}|_U) \neq 0$, hence $E\mathbf{x} \not\perp \mathbf{j}$, i.e., λ is main in $\dot{H}$.

If λ is main in $\dot{G}$, in a very similar way we get $\operatorname{sum}(E\mathbf{x}) = \operatorname{sum}(\mathbf{x}|_U) - \operatorname{sum}(\mathbf{x}|_{V\setminus U}) \neq 0$, hence λ is main in $\dot{H}$. □

Some consequences:

Corollary 4.6. *Let $\dot{G}$ be a net-regular signed graph and let V denote its vertex set. If there exists $U \subseteq V$ such that $2|U| \neq |V|$ and for every eigenvalue distinct from the net-degree there exists an associated eigenvector $\mathbf{x}$ satisfying $\operatorname{sum}(\mathbf{x}|_U) \neq 0$, then $\dot{G}$ switches to a signed graph in which all eigenvalues are main.*

Proof. The desired switching equivalent signed graph $\dot{H}$ is obtained as in the proof of Theorem 4.5, that is, by switching with respect to U. Indeed, apart from the net-degree, all the eigenvalues of $\dot{G}$ are non-main but the same eigenvalues of $\dot{H}$ are main, by the same theorem. The net-degree of $\dot{G}$ appears in the spectrum of $\dot{H}$ also as a main eigenvalue, since an associated eigenvector consists of ±1's, where exactly $|U|$ of its coordinates are 1's. □

Corollary 4.7. *Let $\dot{G}$ be a net-regular signed graph with at least 3 vertices containing a vertex-deleted subgraph which do not share any eigenvalue with $\dot{G}$. Then $\dot{G}$ switches to a signed graph in which all eigenvalues are main.*

Proof. Let $\dot{H}$ be obtained by switching with respect to a single vertex, say i, whose deletion results in a subgraph (say $\dot{G} - i$) described in the statement.

By Corollary 4.6, the net-degree of $\dot{G}$ is a main eigenvalue of $\dot{H}$. Let λ be one of the remaining eigenvalues. Observe that for an associated eigenvector $\mathbf{x}$ in $\dot{G}$, the coordinate which corresponds to i is nonzero. (Otherwise, we would have $A_{\dot{G}-i}\mathbf{x}|_{V\setminus\{i\}} = \lambda\mathbf{x}|_{V\setminus\{i\}}$, contradicting the assumption on eigenvalues of $\dot{G} - i$.) The remainder of the proof follows by Theorem 4.5, as $\{i\}$ satisfies the assumption (of that theorem) regarding non-main eigenvalues. □

4.4. Signed line graphs and line graphs. If B_η is the vertex-edge incidence matrix of an oriented signed graph $\dot{G}_\eta$, then according to the definition, the switching class $\mathcal{L}(\dot{G})$ is determined by $B_\eta^\top B_\eta - 2I$. Note that the matrix $B_\eta B_\eta^\top$ does not depend on η and $L_{\dot{G}} = B_\eta B_\eta^\top$.

Observe that for any signed graph with vertex set $V = \{1, 2, \ldots, n\}$ there exists exactly one vertex-edge orientation η satisfying $\eta(i, ij) = 1$ for $i < j$. Denote this orientation by η^+. Every column of B_{η^+} has the form $\mathbf{e}_i \pm \mathbf{e}_j$ $(1 \leqslant i < j \leqslant n)$. Moreover, an eigenvalue λ of $B_{\eta^+}^\top B_{\eta^+}$ shares the eigenspace with $\lambda-2$ of $B_{\eta^+}^\top B_{\eta^+}-2I$. Therefore, a relation between the main angles of $B_{\eta^+}^\top B_{\eta^+} - 2I$ and the main angles of $B_{\eta^+} B_{\eta^+}^\top$ is given by Theorem 3.1.

Corollary 4.8. *Let $\dot{G}$ be a signed graph all of whose edges are positive and $-\dot{G}$ be its negation. Then*

(i) 0 *is the unique main eigenvalue of $L_{\dot{G}}$,*

(ii) *a nonzero eigenvalue λ of $L(-\dot{G}_{\eta^+})$ is main if and only if λ is a main eigenvalue of $L_{-\dot{G}}$.*

Proof. (i) Since the sum of entries of every column of any vertex-edge incidence matrix associated with $\dot{G}$ is zero, the claim follows by Corollary 3.2 (i).

(ii) Similarly, since in this case, every column of B_{η^+} has the form $\mathbf{e}_i + \mathbf{e}_j$, the result follows by Corollary 3.2 (ii). □

The previous discussion can easily be adapted to the particular case of graphs. Indeed, if R is the standard vertex-edge incidence matrix of a graph G, then the signless Laplacian matrix of G is given by $Q_G = RR^\top$, and the adjacency matrix of its line graph is given by $A_{\mathrm{Line}(G)} = R^\top R - 2I$.

Corollary 4.9. *For a graph G:*

(i) 0 *is the unique main eigenvalue of L_G.*

(ii) *A nonzero eigenvalue λ of Q_G is main if and only if $\lambda - 2$ is a main eigenvalue of* $\mathrm{Line}(G)$.

(iii) *If G is connected bipartite, then 0 is a non-main eigenvalue of Q_G if and only if the colour classes of G are equal in size.*

(iv) -2 *is never a main eigenvalue of* $\mathrm{Line}(G)$.

Proof. (i) Observing that $\mathbf{j}$ is associated with 0, we get the result by Theorem 2.1 (vii).

(ii) This follows by Corollary 3.2 (ii), as an eigenvalue $\lambda - 2$ of $\mathrm{Line}(G)$ shares an eigenspace with λ of $R^\top R$.

(iii) First, if G is connected, then 0 belongs to its spectrum if and only if G is bipartite, see [11], Theorem 1.18. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ is an associated eigenvector, then $Q_G\mathbf{x} = \mathbf{0}$ implies $R^\top \mathbf{x} = \mathbf{0}$ (see the discussion at the beginning of this section), which holds if and only if $x_i = -x_j$ for every edge ij. As G is connected, we have that x_i is a constant on each colour class and $|x_i|$ is a constant on the entire vertex set. Consequently, 0 is non-main if and only if the colour classes are equal in size.

(iv) If -2 is an eigenvalue of $\mathrm{Line}(G)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)^\top$ is an associated eigenvector, then $R^\top R\mathbf{x} = \mathbf{0}$, i.e., $R\mathbf{x} = \mathbf{0}$. Therefore $0 = \mathrm{sum}(R\mathbf{x}) = 2\,\mathrm{sum}(\mathbf{x})$, and the result follows. $\square$

The result of Corollary 4.9 (iv) is obtained by Doob, see [5].

In [4], all the trees and all the unicyclic graphs whose signless Laplacian matrix has exactly 2 main eigenvalues are determined. By virtue of Corollary 4.9 (ii)–(iv), the line graph of any of them has at most 2 main eigenvalues and their number depends on whether the corresponding root is a bipartite graph whose colour classes are equal in size or not.

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Author's address: Zoran Stanić, Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11 000 Belgrade, Serbia, e-mail: `zstanic@math.rs`.