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A CERTAIN TENSOR ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

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Abstract. In a nonflat complex space form (namely, a complex projective space or a complex hyperbolic space), real hypersurfaces admit an almost contact metric structure (φ, ξ, η, g) induced from the ambient space. As a matter of course, many geometers have investigated real hypersurfaces in a nonflat complex space form from the viewpoint of almost contact metric geometry. On the other hand, it is known that the tensor field h $(= \frac{1}{2}\mathcal{L}_\xi\varphi)$ plays an important role in contact Riemannian geometry. In this paper, we investigate real hypersurfaces in a nonflat complex space form from the viewpoint of the parallelism of the tensor field h.

Keywords: nonflat complex space form; real hypersurface; Hopf hypersurface; ruled real hypersurface; the tensor field h

MSC 2020: 53B25, 53C40, 53D15

1. Introduction

Let $\widetilde{M}_n(c)$ be a nonflat complex space form (namely, $\widetilde{M}_n(c)$ is congruent to either a complex projective space $\mathbb{C}P^n(c)$ of constant holomophic sectional curvature $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of holomophic sectional curvature $c < 0$). It is well-known that real hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$ admit *the almost contact metric structure* (φ, ξ, η, g), see Section 2. It is not too much to say that the theory of real hypersurfaces in $\widetilde{M}_n(c)$ have developed from the viewpoint of submanifold theory and almost contact metric geometry.

In contact Riemannian geometry, the tensor $h(= \frac{1}{2}\mathcal{L}_\xi\varphi)$ plays an important role, where $\mathcal{L}$ is the Lie derivative, see [1]. In fact, it is known that the condition $h = 0$ is equivalent to the condition of K*-contact manifolds* (namely, the characteristic vector field ξ on a contact manifold is a Killing vector field with respect to the metric g).

On the other hand, a real hypersurface M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$ satisfies $h = 0$ if and only if M^{2n-1} is locally congruent to a real hypersurface of type (A) in $\widetilde{M}_n(c)$, see [5].

The purpose of this paper is to investigate the following problems with respect to the parallelism of the tensor h of M^{2n-1}.

Problem 1.1. *Classify real hypersurfaces in $\widetilde{M}_n(c)$ satisfying the condition*

$$\nabla_\xi h = 0 \quad (\xi\text{-parallelism}). \tag{1.1}$$

Problem 1.2. *Classify real hypersurfaces in $\widetilde{M}_n(c)$ satisfying the condition*

$$\nabla_X h = 0 \quad (T^0M\text{-parallelism}) \tag{1.2}$$

for any tangent vector field X on M^{2n-1} orthogonal to the characteristic vector field ξ.

Condition (1.1) frequently appears in contact Riemannian geometry. In fact, condition (1.1) is equivalent to several conditions on contact Riemannian manifolds, see [18]. In particular, it is well-known that there exists the relationship between the *structure Jacobi operator* l and the tensor h, where l is the tensor of type $(1,1)$ such that $lX = R(X,\xi)\xi$ and R is the curvature tensor of the contact Riemannian manifold. Also, in the theory of real hypersurfaces in $\widetilde{M}_n(c)$, many geometers have studied real hypersurfaces from the aspect of the structure Jacobi operator, see [4]. Hence, it is natural that we investigate real hypersurfaces satisfying condition (1.1). Pérez, Santos and Suh investigated the condition $\nabla_X l = 0$ for any tangent vector field X orthogonal to the characteristic vector field ξ, see [17]. They showed the nonexistence of real hypersurfaces in $\mathbb{C}P^n(c)$ $(n \geqslant 3)$ satisfying this equation. Hence, Problem 1.2 is also a natural problem.

In the latter of this paper, we also consider *the η-parallelism* of the tensor h, that is,

$$g((\nabla_X h)Y, Z) = 0 \tag{1.3}$$

for all tangent vector fields X, Y and Z on M^{2n-1} orthogonal to the characteristic vector field ξ. We emphasize that there exist real hypersurfaces satisfying η-parallel condition of the tensor h but not those of type (A) in $\widetilde{M}_n(c)$. In fact, real hypersurfaces of type (B) in $\widetilde{M}_n(c)$ satisfy condition (1.3), see Theorem 5.1. We note that real hypersurfaces of type (A) and (B) are examples of Hopf hypersurfaces with constant principal curvatures in $\widetilde{M}_n(c)$. Moreover, *ruled real hypersurfaces* in $\widetilde{M}_n(c)$ have the η-parallelism of the tensor h. It is known that ruled real hypersurfaces in $\widetilde{M}_n(c)$ are typical examples of non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. We also give the classification of 3-dimensional real hypersurfaces in $\widetilde{M}_2(c)$ satisfying (1.3).

2. Preliminaries

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of a complex n-dimensional non-flat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature c. The *Levi-Civita connections* $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. Equation (2.1) is called *Gauss's formula* and equation (2.2) is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

In this paper, $V_\lambda^0 = \{X \in TM\colon\ AX = \lambda X,\ X \perp \xi\}$ is said to be a restricted principal distribution associated with principal curvature λ, where TM is the tangent bundle of M^{2n-1}.

It is known that M^{2n-1} admits the *almost contact metric structure* (φ,ξ,η,g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. The *characteristic vector field* ξ of M^{2n-1} is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\begin{aligned} &\varphi^2 = -I + \eta\otimes\xi, \quad \eta(X) = g(X,\xi), \quad \eta(\xi) = 1, \quad \varphi\xi = \eta(\varphi X) = 0,\\ &g(\varphi X, Y) = -g(X,\varphi Y) \quad \text{and} \quad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.3}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}. We call φ and η the *structure tensor* and the *contact form* of M^{2n-1}, respectively.

The following equations are fundamental tools in the theory of real hypersurfaces in $\widetilde{M}_n(c)$:

$$(\nabla_X\varphi)Y = \eta(Y)AX - g(AX,Y)\xi \tag{2.4}$$

and

$$\nabla_X\xi = \varphi AX \tag{2.5}$$

for any X and Y tangent to M^{2n-1}. The tensor h of M^{2n-1} is given by

$$hX = \frac{1}{2}(\mathcal{L}_\xi\varphi)X = \frac{1}{2}(\eta(X)A\xi - \varphi A\varphi X - AX), \tag{2.6}$$

where $\mathcal{L}$ is the Lie derivative. The *Codazzi equation* is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}. \tag{2.7}$$

Let R be the curvature tensor of M^{2n-1} in $\widetilde{M}_n(c)$. Namely,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The *Gauss equation* is given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y \\ &- 2g(\varphi X,Y)\varphi Z\} + g(AY,Z)AX - g(AX,Z)AY \end{aligned} \tag{2.8}$$

for all vectors X, Y and Z tangent to M^{2n-1}.

A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. The following lemma gives a useful property of Hopf hypersurfaces in $\widetilde{M}_n(c)$.

Lemma 2.1. *Let M^{2n-1} be a Hopf hypersurface M^{2n-1} with the principal curvature α corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$. Then M^{2n-1} has the following properties:*

(1) *The principal curvature α is locally constant on M^{2n-1}.*

(2) *If X is a tangent vector of M^{2n-1} perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\varphi X = (\alpha\lambda + \frac{1}{2}c)\varphi X$.*

In $\widetilde{M}_n(c)$, Hopf hypersurfaces with constant principal curvatures are standard examples. The following classification theorems are well known.

Theorem 2.1 ([3], [9], [19])**.** *Let M^{2n-1} be a Hopf hypersurface in a complex projective space $\mathbb{C}P^n(c)$ $(n \geqslant 2)$. Then M^{2n-1} has constant principal curvatures if and only if M^{2n-1} is locally congruent to one of the following:*

(A_1) *a geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$,*

(A_2) *a tube of radius r around a totally geodesic $\mathbb{C}P^l(c)$ $(1 \leqslant l \leqslant n-2)$, where $0 < r < \pi/\sqrt{c}$,*

(B) *a tube of radius r around a complex hyper quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$,*

(C) *a tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n ($n \geqslant 5$) is odd,*

(D) *a tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;*

(E) *a tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.*

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). In this paper, summing up real hypersurfaces of type (A_1) and (A_2), we call them real hypersurfaces of type (A). The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given in the following table, see [15]:

	(A_1)	(A_2)	(B)	(C), (D), (E)
λ_1	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r)$	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r)$	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r - \frac{1}{4}\pi)$	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r - \frac{1}{4}\pi)$
λ_2	—	$-\frac{1}{2}\sqrt{c}\tan(\frac{1}{2}\sqrt{c}r)$	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r + \frac{1}{4}\pi)$	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r + \frac{1}{4}\pi)$
λ_3	—	—	—	$\frac{1}{2}\sqrt{c}\cot(\frac{1}{2}\sqrt{c}r)$
λ_4	—	—	—	$-\frac{1}{2}\sqrt{c}\tan(\frac{1}{2}\sqrt{c}r)$
α	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$

Theorem 2.2 ([2], [13]). *Let M^{2n-1} be a Hopf hypersurface in a complex hyperbolic space $\mathbb{C}H^n(c)$ ($n \geqslant 2$). Then M^{2n-1} has constant principal curvatures if and only if M^{2n-1} is locally congruent to one of the following:*

(A_0) *a horosphere in $\mathbb{C}H^n(c)$,*

(A_{10}) *a geodesic sphere of radius r, where $0 < r < \infty$,*

(A_{11}) *a tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$,*

(A_2) *a tube of radius r around a totally geodesic $\mathbb{C}H^l(c)$ ($1 \leqslant l \leqslant n-2$), where $0 < r < \infty$,*

(B) *a tube of radius r around a totally real totally geodesic $\mathbb{R}H^n\frac{1}{4}c$, where $0 < r < \infty$.*

These real hypersurfaces are said to be of types (A_0), (A_{10}), (A_{11}), (A_2) and (B). Here, type (A_1) implies either type (A_{10}) or type (A_{11}). Summing up real hypersurfaces of type (A_1) and (A_2), we call them real hypersurfaces of type (A). The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given in the table below, see [15].

	(A_0)	(A_{10})	(A_{11})	(A_2)	(B)
λ_1	$\frac{1}{2}\sqrt{\lvert c\rvert}$	$\frac{1}{2}\sqrt{\lvert c\rvert}\coth(\frac{1}{2}\sqrt{\lvert c\rvert}r)$	$\frac{1}{2}\sqrt{\lvert c\rvert}\tanh(\frac{1}{2}\sqrt{\lvert c\rvert}r)$	$\frac{1}{2}\sqrt{\lvert c\rvert}\coth(\frac{1}{2}\sqrt{\lvert c\rvert})$	$\frac{1}{2}\sqrt{\lvert c\rvert}\coth(\frac{1}{2}\sqrt{\lvert c\rvert}r)$
λ_2	—	—	—	$\frac{1}{2}\sqrt{\lvert c\rvert}\tanh(\frac{1}{2}\sqrt{\lvert c\rvert}r)$	$\frac{1}{2}\sqrt{\lvert c\rvert}\tanh(\frac{1}{2}\sqrt{\lvert c\rvert}r)$
α	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

It is known that real hypersurfaces of type (A) have many properties. In particular, the following conditions (2) and (3) give the characterization of real hypersurfaces of type (A) in $\widetilde{M}_n(c)$.

Lemma 2.2 ([5], [14], [16]). *Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Then the following two conditions an equivalent:*

(1) M^{2n-1} *is locally congruent to a real hypersurface of type* (A),

(2) $\varphi A = A\varphi$ *on* M^{2n-1},

(3) $h = 0$.

Remark 2.1. Obviously, real hypersurfaces of type (A) in $\widetilde{M}_n(c)$ satisfy the condition $\nabla h = 0$.

Next, we define ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. It is known that ruled real hypersurfaces are examples of non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} is called a *ruled real hypersurface* of a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqslant 2)$ if the holomorphic distribution T^0M defined by $T^0M = \{X \in TM\colon \eta(X) = 0\}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface $\widetilde{M}_{n-1}(c)$ of $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following way: Given an arbitrary regular real smooth curve γ in $\widetilde{M}_n(c)$ which is defined on an interval I, we have at each point $\gamma(t)$ $(t \in I)$ a totally geodesic complex hypersurface $\widetilde{M}^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we have a ruled real hypersurface $M^{2n-1} = \bigcup_{t\in I} \widetilde{M}^{(t)}_{n-1}(c)$ in $\widetilde{M}_n(c)$. The following lemma is a well-known characterization of ruled real hypersurfaces concerning the shape operator A, see [10] and [15].

Lemma 2.3 ([10], [15]). *Let M^{2n-1} be a real hypersurface M^{2n-1} in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Then the following three conditions are mutually equivalent:*

(1) M^{2n-1} *is a ruled real hypersurface.*

(2) *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1}\colon \beta(x) \neq 0\}$ with a unit vector field U orthogonal to ξ:*

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad AX = 0$$

for an arbitrary tangent vector X orthogonal to ξ and U, where α, β are differentiable functions on M_1 by $\alpha = g(A\xi, \xi)$ and $\beta = \|A\xi - \alpha\xi\|$.

(3) *The shape operator A of M^{2n-1} satisfies $g(AX, Y) = 0$ for arbitrary tangent vectors $X, Y \in T^0M$.*

3. ξ-PARALLELISM

In this section, we investigate real hypersurfaces satisfying condition (1.1). First, we prepare a fundamental tool. By (2.3), (2.4), (2.5) and (2.6), the equation

$$(\nabla_X h)Y = 0 \quad \forall X, Y \in TM$$

is equivalent to saying that

$$\begin{aligned} (3.1) \quad & g(\varphi AX, Y)A\xi + \eta(Y)(\nabla_X A)\xi + \eta(Y)A\varphi AX - \eta(A\varphi Y)AX + g(AX, A\varphi Y)\xi \\ & -\varphi(\nabla_X A)\varphi Y - \eta(Y)\varphi A^2 X + g(AX, Y)\varphi A\xi - (\nabla_X A)Y = 0 \end{aligned}$$

for all vectors X and Y tangent to M^{2n-1}.

By using this equation, we obtain the following result:

Theorem 3.1. *Let M^{2n-1} be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Then M^{2n-1} satisfies $\nabla_\xi h = 0$ if and only if M^{2n-1} is locally congruent to one of the following:*

(i) *a real hypersurface of type* (A) *in $\widetilde{M}_n(c)$,*

(ii) *a non-homogeneous Hopf hypersurface with $A\xi = 0$ in $\widetilde{M}_n(c)$.*

P r o o f. By using (3.1), we obtain

$$\begin{aligned} (3.2) \qquad (\nabla_\xi h)X = {} & g(\varphi A\xi, X)A\xi + \eta(X)(\nabla_\xi A)\xi + \eta(X)A\varphi A\xi - \eta(A\varphi X)A\xi \\ & + g(A\xi, A\varphi X)\xi - \varphi(\nabla_\xi A)\varphi X - \eta(X)\varphi A^2\xi + g(A\xi, X)\varphi A\xi \\ & - (\nabla_\xi A)X = 0. \end{aligned}$$

Suppose that there exists a non-Hopf hypersurface in $\widetilde{M}_n(c)$ satisfying $\nabla_\xi h = 0$. Then the shape operator A forms $A\xi = \alpha\xi + \beta U$, where the function β satisfies $\beta \neq 0$ and a unit vector U is orthogonal to the characteristic vector field ξ.

Putting $X = U$ in (3.2), we have

$$(3.3) \qquad g(A\xi, A\varphi U)\xi - \varphi(\nabla_\xi A)\varphi U + \beta^2\varphi U - (\nabla_\xi A)U = 0.$$

Taking an inner product of equation (3.3) with the vector φU, we obtain

$$(3.4) \qquad \beta^2 = 2g((\nabla_\xi A)U, \varphi U).$$

Next, we set $X = \varphi U$ in (3.2). Then we have

$$(3.5) \qquad g(\varphi A\xi, \varphi U)A\xi - \eta(A\varphi^2 U)A\xi + g(A\xi, A\varphi^2 U)\xi - \varphi(\nabla_\xi A)\varphi^2 U - (\nabla_\xi A)\varphi U = 0.$$

Taking an inner product of equation (3.5) with the vector U, we obtain

$$\beta^2 = g((\nabla_\xi A)U, \varphi U). \tag{3.6}$$

This, combined with equation (3.4), yields $\beta^2 = 0$, which is a contradiction.

Hence, there is no non-Hopf hypersurface satisfying $\nabla_\xi h = 0$ in $\widetilde{M}_n(c)$.

Next, we suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_n(c)$. We take a vector $V \in T^0M$ with $AV = \lambda V$. By (3.2), we can see that

$$(\nabla_\xi h)V = -\varphi(\nabla_\xi A)\varphi V - (\nabla_\xi A)V = 0. \tag{3.7}$$

From the Codazzi equation (2.7) we can see

$$\begin{aligned}(\nabla_\xi A)\varphi V &= (\nabla_{\varphi V} A)\xi - \frac{c}{4}V = \nabla_{\varphi V}(A\xi) - A\nabla_{\varphi V}\xi - \frac{c}{4}V \\ &= \alpha\varphi A\varphi V - A\varphi A\varphi V - \frac{c}{4}V \quad \text{(using (2.5) and Lemma 2.1).}\end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}(\nabla_\xi A)V &= (\nabla_V A)\xi + \frac{c}{4}\varphi V = \nabla_V(A\xi) - A\nabla_V\xi + \frac{c}{4}\varphi V \\ &= \alpha\lambda\varphi V - A\varphi AV + \frac{c}{4}\varphi V \quad \text{(using (2.5) and Lemma 2.1).}\end{aligned} \tag{3.9}$$

It follows from (2) of Lemma 2.1, (3.8) and (3.9) that

$$(2\lambda - \alpha)(\nabla_\xi h)V = -\alpha\Big(2\lambda^2 - 2\alpha\lambda - \frac{c}{2}\Big)\varphi V. \tag{3.10}$$

From (3.7) we can see that

$$\alpha\Big(2\lambda^2 - 2\alpha\lambda - \frac{c}{2}\Big) = 0. \tag{3.11}$$

If $2\lambda^2 - 2\alpha\lambda - \frac{1}{2}c = 0$, then we have $(2\lambda - \alpha)\lambda = \alpha\lambda + \frac{1}{2}c$. If $2\lambda - \alpha \neq 0$, then we can see that $\lambda = (\alpha\lambda + \frac{1}{2}c)/(2\lambda - \alpha)$. This implies that $\varphi V_\lambda = V_\lambda$. This, combined with $\varphi A\xi = 0 = A\varphi\xi$, yields $\varphi A = A\varphi$. By Lemma 2.2, M^{2n-1} is locally congruent to a real hypersurface of type (A) in $\widetilde{M}_n(c)$. If $2\lambda - \alpha = 0$, then M^{2n-1} is nothing but a horosphere in $\mathbb{C}H^n(c)$. This is included in the class of real hyperusrfaces of type (A) in $\widetilde{M}_n(c)$.

If $\alpha = 0$, M^{2n-1} is locally congruent to either a real hypersurface of type (A) of radius $r = \pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$ or a non-homogeneous Hopf hypersurface with $A\xi = 0$ in $\widetilde{M}_n(c)$. Needless to say, the former is included in the class of real hypersurfaces of type (A) in $\mathbb{C}P^n(c)$.

Conversely, if M^{2n-1} is a Hopf hypersurface with $\alpha = 0$, then $\lambda \neq 0$ (see (2) of Lemma 2.1). Hence, from relation (3.10), Hopf hypersurfaces with $\alpha = 0$ in $\widetilde{M}_n(c)$ satisfy $(\nabla_\xi h)X = 0$ for any vector $X \in T^0M$. Clearly, Hopf hypersurfaces also satisfy $(\nabla_\xi h)\xi = 0$. Therefore Hopf hypersurfaces with $\alpha = 0$ in $\widetilde{M}_n(c)$ satisfy $\nabla_\xi h = 0$. □

Remark 3.1. There exist non-homogeneous Hopf hypersurfaces with $A\xi = 0$ in $\widetilde{M}_n(c)$ (for detail, see [4]).

Remark 3.2. By the work of [18], for contact Riemannian manifolds, the following four conditions are mutually equivalent:

(i) $\nabla_\xi h = 0$,

(ii) $\nabla_\xi l = 0$,

(iii) $\nabla_\xi \tau = 0$,

(iv) $\varphi l = l\varphi$,

where $lX = R(X, \xi)\xi$, R is the curvature tensor on the contact Riemannian manifold and $\tau = \mathcal{L}_\xi g$. In the theory of real hypersurfaces, Cho and Ki classified Hopf hypersurfaces in $\widetilde{M}_n(c)$ satisfying condition (ii), see [6]. Moreover, many geometers have investigated the classification of real hypersurfaces satisfying condition (ii) under an additional condition, see [4]. Recently, Ghosh studied real hypersurfaces in $\widetilde{M}_n(c)$ satisfying condition (iii), see [7]. Condition (iv) was investigated by many geometers under additional conditions, (for detail, see [20]).

4. T^0M PARALLELISM

In this section, we investigate condition (1.2). To prove Theorem 4.1, we prepare the following results with respect to *ruled real hypersurfaces.*

Lemma 4.1 ([11])**.** *Every ruled real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geqslant 2)$ admits the η-parallelism with respect to the shape operator A. Namely, M^{2n-1} satisfy the condition*

$$g((\nabla_X A)Y, Z) = 0$$

for all vectors X, Y and Z in T^0M.

Remark 4.1. In general, for a tensor field T of type (1, 1), the condition

$$g((\nabla_X T)Y, Z) = 0 \quad \text{for any } X, Y, Z \in T^0M$$

is equivalent to the condition

$$(\nabla_X T)Y \in \operatorname{span}\{\xi\} \quad \text{for any } X, Y \in T^0M.$$

Lemma 4.2. *None ruled real hypersurface in $\widetilde{M}_n(c)$ $(n \geqslant 2)$ does not satisfies condition* (1.2).

P r o o f. Suppose that there exists a ruled real hypersurface in $\widetilde{M}_n(c)$ satisfying condition (1.2). From case (2) of Lemma 2.3, we set $X = U$ and $Y = \varphi U$ in (3.1). By using the properties of ruled real hypersurfaces (see [8], page 404), we have

$$\begin{aligned} 0 &= -\varphi(\nabla_U A)\varphi^2 U - (\nabla_U A)\varphi U \\ &= -(\nabla_U A)\varphi U \quad \text{(from Lemma 4.1)} \\ &= A\varphi\nabla_U U \quad \text{(from (2.4))} \\ &= -\Big(\beta^2 - \frac{c}{4}\Big)U. \end{aligned}$$

Hence, there is a possibility that the case when $\beta^2 = \frac{1}{4}c$ satisfies (1.2). We only check whether a ruled real hypersurface having $\beta^2 = \frac{1}{4}c$ satisfies condition (1.2) or not. Again, from case (2) of Lemma 2.3, we put $X = \varphi U$ and $Y = U$ on the left side of (3.1). By using the properties of ruled real hypersurfaces (see [8], page 404), we have

$$-(\nabla_{\varphi U} A)U = -\nabla_{\varphi U}(\beta\xi) = -\Big(\beta^2 + \frac{c}{4}\Big)\xi = -\frac{c}{2}\xi \neq 0.$$

Therefore ruled real hypersurfaces in $\widetilde{M}_n(c)$ do not satisfy (1.2). $\square$

Theorem 4.1. *Let M^{2n-1} be a real hypersurface in a non-flat complex space form $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Then M^{2n-1} satisfies $\nabla_X h = 0$ for any vector $X \in T^0M$ if and only if M^{2n-1} is locally congruent to a real hypersurface of type* (A) *in $\widetilde{M}_n(c)$.*

P r o o f. Suppose that there exists a non-Hopf hypersurface in $\widetilde{M}_n(c)$ satisfying $\nabla_X h = 0$ for any vector $X \in T^0M$. Then the shape operator A satisfies $A\xi = \alpha\xi + \beta U$, where the function β satisfies $\beta \neq 0$ and the unit vector U is orthogonal to the characteristic vector field ξ. For any vector $X \in T^0M$ we set $Y = U$ in (3.1). Then we have

$$g(\varphi AX, U)A\xi + g(AX, A\varphi U)\xi - \varphi(\nabla_X A)\varphi U + g(AX, U)\varphi A\xi - (\nabla_X A)U = 0$$

for any vector $X \in T^0M$. We take the inner product of this equation with U and φU, respectively. Then we can see that

$$\beta g(\varphi AX, U) + g((\nabla_X A)\varphi U, \varphi U) - g((\nabla_X A)U, U) = 0, \tag{4.1}$$

$$2g((\nabla_X A)\varphi U, U) = \beta g(X, AU) \tag{4.2}$$

for any vector $X \in T^0M$. Similarly, we set $Y = \varphi U$ in (3.1) and take the inner product with U and φU, respectively. Then we have

$$g((\nabla_X A)U, \varphi U) = \beta g(X, AU), \tag{4.3}$$

$$2\beta g(X, A\varphi U) + g((\nabla_X A)U, U) - g((\nabla_X A)\varphi U, \varphi U) = 0 \tag{4.4}$$

for any vector $X \in T^0M$. From equations (4.2) and (4.3) we have $g(AU, X) = 0$ for any vector $X \in T^0M$. This implies that

$$AU = \beta\xi. \tag{4.5}$$

Next, from equations (4.1) and (4.4) we have $g(A\varphi U, X) = 0$ for any vector $X \in T^0M$. Noting that $g(A\varphi U, \xi) = 0$, we obtain

$$A\varphi U = 0. \tag{4.6}$$

We take a unit vector $V \in T^1M = T^0M \cap \operatorname{span}\{U, \varphi U\}^\perp$ such that $AV = \lambda V$. For any vector $X \in T^0M$ we set $Y = \xi$ in (3.1). Then we have

$$A\varphi AX - \varphi A^2X + \beta^2 g(X, U)\varphi U = 0 \tag{4.7}$$

for any vector $X \in T^0M$. Putting $X = V$ in (4.7), we get

$$\lambda(A\varphi V - \lambda\varphi V) = 0.$$

This equation implies the following two cases:
(1) $A\varphi V = \lambda\varphi V$ $(\lambda \neq 0)$,
(2) $\lambda = 0$.

Now we shall show that case (1) does not occur. We suppose that $A\varphi V = \lambda\varphi V$ $(\lambda \neq 0)$. By using the Codazzi equation, we get

$$(\nabla_V A)\varphi V - (\nabla_{\varphi V} A)V = -\frac{c}{2}\xi.$$

On the other hand, by (2.4), we have

$$\begin{aligned}(\nabla_V A)\varphi V - (\nabla_{\varphi V} A)V = (V\lambda)\varphi V + (\alpha\lambda - \lambda^2)\xi + (\lambda I - A)\varphi\nabla_V V \\ + \beta\lambda U - (\varphi V\lambda)V - (\lambda I - A)\nabla_{\varphi V} V.\end{aligned}$$

These two equations yield

$$\alpha\lambda - \lambda^2 + \beta g(\nabla_V V, \varphi U) + (\alpha - \lambda)g(\nabla_{\varphi V} V, \xi) + \beta g(\nabla_{\varphi V} V, U) = -\frac{c}{2}. \tag{4.8}$$

Next, we compute $g(\nabla_V V, \varphi U)$, $g(\nabla_{\varphi V} V, \xi)$ and $g(\nabla_{\varphi V} V, U)$ one by one. By using (2.5), we have

$$g(\nabla_{\varphi V} V, \xi) = -g(V, \nabla_{\varphi V}\xi) = -g(V, \varphi A\varphi V) = \lambda. \tag{4.9}$$

We put $X = \varphi V$ and $Y = V$ in (3.1), and take the inner product with ξ. By using (4.9), we can see that

$$-\alpha\lambda + \lambda^2 - \lambda g(\nabla_{\varphi V} V, \xi) + g(A\nabla_{\varphi V} V, \xi) = \beta g(\nabla_{\varphi V} V, U) = 0.$$

Since $\beta \neq 0$, we have

$$g(\nabla_{\varphi V} V, U) = 0. \tag{4.10}$$

By using (2.4), we then have

$$g(\nabla_V V, \varphi U) = -g(V, (\nabla_V \varphi)V + \varphi\nabla_V U) = -g(V, \varphi\nabla_V U) = g(\nabla_V U, \varphi V).$$

Hence, we shall calculate $g(\nabla_V U, \varphi V)$. By the Codazzi equation, we obtain

$$(\nabla_V A)U - (\nabla_U A)V = 0.$$

On the other hand, we have

$$(\nabla_V A)U - (\nabla_U A)V = (V\beta)\xi + \beta\lambda\varphi V - A\nabla_V U - (U\lambda)V - (\lambda I - A)\nabla_U V.$$

These equations imply that $\beta\lambda - \lambda g(\nabla_V U, \varphi V) = 0$. Since $\lambda \neq 0$, we have

$$g(\nabla_V V, \varphi U) = g(\nabla_V U, \varphi V) = \beta. \tag{4.11}$$

Equations (4.8), (4.9), (4.10) and (4.11) give

$$-2\lambda^2 + 2\alpha\lambda + \beta^2 = -\frac{c}{2}. \tag{4.12}$$

By the Codazzi equation, we have

$$(\nabla_V A)\xi - (\nabla_\xi A)V = -\frac{c}{4}\varphi V.$$

On the other hand, we can see that

$$(\nabla_V A)\xi - (\nabla_\xi A)V = (V\alpha)\xi + (\alpha\lambda - \lambda^2)\varphi V + (V\beta)U + \beta\nabla_V U - (\xi\lambda)V - (\lambda I - A)\nabla_\xi V.$$

By using equation (4.11), these two equations yield

$$-2\lambda^2 + 2\alpha\lambda + 2\beta^2 = -\frac{c}{2}.$$

This, combined with (4.12), gives $\beta = 0$, which is a contradiction. Hence, case (1) does not occur. Namely, we only consider case (2) for the distribution T^1M. This implies that $AX = 0$ for any vector $X \in T^1M$. This, together with (4.6), gives $AX = 0$ for any vector $X \perp \xi, U$. Hence, this case means that M^{2n-1} is locally congruent to the ruled real hypersurface in $\widetilde{M}_n(c)$. However, by Lemma 4.2, ruled real hypersurfaces do not satisfy condition (1.2).

Finally, we consider the case of Hopf hypersurfaces in $\widetilde{M}_n(c)$. We suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_n(c)$. For any vector $X \in T^0M$ we put $Y = \xi$ in (3.1). Then we obtain

$$A\varphi AX - \varphi A^2 X = 0$$

for any vector $X \in T^0M$. We take a vector $V \in T^0M$ with $AV = \lambda V$. By Lemma 2.1 and the above equation, we can see that

$$\lambda\Big(2\lambda^2 - 2\alpha\lambda - \frac{c}{2}\Big) = 0.$$

This equation implies that the function λ is locally constant and $\lambda = 0$ or $2\lambda^2 - 2\alpha\lambda - \frac{1}{2}c = 0$. The former does not occur (see the tables in Section 2). By the discussion of Theorem 3.1, the latter gives that M^{2n-1} is locally congruent to a real hypersurface of type (A) in $\widetilde{M}_n(c)$. $\square$

5. η-PARALLEL CONDITION

Motivated by the discussion of Theorem 4.1, we would like to find the condition which ruled real hypersurfaces satisfying a certain parallelism of the tensor h. In this section, we investigate real hypersurfaces satisfying condition (1.3). We note that ruled real hypersurfaces in $\widetilde{M}_n(c)$ satisfy condition (1.3). First, we classify Hopf hypersurfaces in $\widetilde{M}_n(c)$ satisfying condition (1.3). In the latter half of this section, we classify 3-dimensional real hypersurfaces in $\widetilde{M}_2(c)$ satisfying condition (1.3).

From (3.1), condition (1.3) is equivalent to saying that

$$\begin{aligned} g(\varphi AX, Y)g(A\xi, Z) - \eta(A\varphi Y)g(AX, Z) - g(\varphi(\nabla_X A)\varphi Y, Z) \\ + g(AX, Y)g(\varphi A\xi, Z) - g((\nabla_X A)Y, Z) = 0 \end{aligned} \tag{5.1}$$

for any X, Y and Z orthogonal to the characteristic vector field ξ. It follows from equation (5.1) that we obtain the following lemma:

Lemma 5.1. *Let M^{2n-1} be a Hopf hypersurface in $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Suppose that M^{2n-1} satisfies condition* (1.3). *Then M^{2n-1} satisfies*

$$g((\nabla_X A)\varphi Y, \varphi Z) = g((\nabla_X A)Y, Z) \tag{5.2}$$

for any X, Y and Z orthogonal to the characteristic vector field ξ.

By virtue of this lemma, we shall classify Hopf hypersurfaces M^{2n-1} satisfying η-parallel condition of the tensor h of M^{2n-1} in $\widetilde{M}_n(c)$.

Theorem 5.1. *Let M^{2n-1} be a Hopf hypersurface in $\widetilde{M}_n(c)$ $(n \geqslant 2)$. Suppose that M^{2n-1} meets condition* (1.3). *Then M^{2n-1} is locally congruent to one of the following:*
(i) *a real hypersurface of type* (A) *in $\widetilde{M}_n(c)$,*
(ii) *a real hypersurface of type* (B) *in $\widetilde{M}_n(c)$.*

P r o o f. We suppose that M^{2n-1} admits condition (5.1). Then we shall show that M^{2n-1} is locally congruent to a Hopf hypersurface with constant principal curvatures in $\widetilde{M}_n(c)$.

Since M^{2n-1} is the Hopf hypersurface (with $A\xi = \alpha\xi$), we take a unit vector field $V \in T^0M$ with $AV = \lambda V$. First, we consider the case when $(2\lambda - \alpha)(p) \neq 0$ at some point p on M^{2n-1}. It follows from the continuity of the function λ that $2\lambda - \alpha \neq 0$ on some sufficiently small neighborhood $\mathcal{U}$ of the point p. By case (2) of Lemma 2.1, we can see that $A\varphi V = \mu\varphi V$ on $\mathcal{U}$, where $\mu = (\alpha\lambda + \frac{1}{2}c)/(2\lambda - \alpha)$.

Then we have

$$g((\nabla_X A)V, V) = g(\nabla_X(AV) - A\nabla_X V, V) = X\lambda \tag{5.3}$$

for any vector $X \in T^0M$. On the other hand, we can see that

$$\begin{aligned} g((\nabla_X A)\varphi V, \varphi V) &= g(\nabla_X(A\varphi V) - A\nabla_X(\varphi V), \varphi V) \\ &= X\mu = -X\lambda\frac{\alpha^2 + c}{(2\lambda - \alpha)^2} \end{aligned} \tag{5.4}$$

for any vector $X \in T^0M$. From (5.2), (5.3) and (5.4) we obtain

$$X\lambda(4\lambda^2 - 4\alpha\lambda + 2\alpha^2 + c) = 0$$

for any vector $X \in T^0M$. If $4\lambda^2 - 4\alpha\lambda + 2\alpha^2 + c = 0$, by case (1) of Lemma 2.1, λ is locally constant.

Next, we shall consider the case when

$$X\lambda = 0 \tag{5.5}$$

for any vector $X \in T^0M$. By the Codazzi equation, we get

$$(\nabla_\xi A)V - (\nabla_V A)\xi = \frac{c}{4}\varphi V.$$

On the other hand, by using Lemma 2.1, case (1), we have

$$(\nabla_\xi A)V - (\nabla_V A)\xi = (\xi\lambda)V + \lambda\nabla_\xi V - A\nabla_\xi V - \alpha\lambda\varphi V + \lambda\mu\varphi V.$$

Taking the inner product of these equations and the vector V, we have

$$\xi\lambda = 0. \tag{5.6}$$

Hence, equations (5.5) and (5.6) imply that the function λ is locally constant. Thus, M^{2n-1} is locally congruent to a Hopf hypersurface with constant principal curvatures $\widetilde{M}_n(c)$.

Next, we consider the case when $(2\lambda-\alpha)(p) = 0$ at some point p on M^{2n-1}. Then we can see that $2\lambda - \alpha = 0$ on some sufficiently small neighborhood $\mathcal{V}$ of the point p. Hence λ is constant on $\mathcal{V}$. Namely, in this case, M^{2n-1} is locally congruent to a Hopf hypersurface with constant principal curvatures $\widetilde{M}_n(c)$.

It is well-known that real hypersurfaces of types (A) and (B) have η-parallel shape operator A (see [3] and [15]). By this fact, we can see that real hypersurfaces of types (A) and (B) in $\widetilde{M}_n(c)$ satisfy condtion (5.1).

Finally, we shall show that real hypersurfaces of types (C), (D) and (E) in $\mathbb{C}P^n(c)$ do *not* satisfy condition (1.3). Let M^{2n-1} be a real hypersurface of either type (C), type (D) or type (E) in $\mathbb{C}P^n(c)$. Suppose that the operator h of M^{2n-1} is η-parallel. The holomorphic distribution is decomposed as $T^0M = V^0_{\lambda_1} \oplus V^0_{\lambda_2} \oplus V^0_{\lambda_3} \oplus V^0_{\lambda_4}$ (see the table of Section 2). Each principal distribution satisfies $\varphi V^0_{\lambda_1} = V^0_{\lambda_2}$, $\varphi V^0_{\lambda_2} = V^0_{\lambda_1}$, $\varphi V^0_{\lambda_3} = V^0_{\lambda_3}$ and $\varphi V^0_{\lambda_4} = V^0_{\lambda_4}$. We take vectors $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$. By the left side of equation (3.1), we can see that

$$(\nabla_X h)Y = \frac{1}{2}(-\varphi(\nabla_X A)\varphi Y - (\nabla_X A)Y) = \frac{1}{2}(((\lambda_2 - \lambda_1)I + A)\nabla_X Y + \varphi A\varphi\nabla_X Y).$$

Since h is η-parallel, $(\nabla_X h)Y \in \operatorname{span}\{\xi\}$ for any $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$ (see Remark 4.1). Here we note that $\varphi A\varphi\nabla_X Y \in T^0M$. The above implies that

$$((\lambda_2 - \lambda_1)I + A)\nabla_X Y \in \operatorname{span}\{\xi\}.$$

Thus, we have

$$\nabla_X Y \in \operatorname{span}\{\xi\} \tag{5.7}$$

for any $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$. Moreover, by equation (2.5), we can see that

$$g(\nabla_X Y, \xi) = -g(Y, \varphi A X) = -\lambda_3 g(Y, \varphi X) = 0$$

for any $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$. This, together with (5.7), yields

$$\nabla_X Y = 0 \tag{5.8}$$

for any $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$.

For any $X \in V^0_\lambda$ ($\lambda = \lambda_1, \lambda_2, \lambda_3$ or λ_4), from the Codazzi equation (2.7), we have

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\varphi X. \tag{5.9}$$

On the other hand, we obtain

$$\begin{aligned}(\nabla_X A)\xi - (\nabla_\xi A)X &= \nabla_X(A\xi) - A\varphi A X - \nabla_\xi(AX) + A\nabla_\xi X \\ &= \alpha\lambda\varphi X - \lambda\frac{\alpha\lambda + \frac{1}{2}c}{2\lambda - \alpha}\varphi X - (\lambda I - A)\nabla_\xi X.\end{aligned}$$

This, combined with (5.9), gives

$$(\lambda I - A)\nabla_\xi X = \Big(\lambda\Big(\alpha - \frac{\alpha\lambda + \frac{1}{2}c}{2\lambda - \alpha}\Big) + \frac{c}{4}\Big)\varphi X. \tag{5.10}$$

Now we take a unit vector $X \in V^0_{\lambda_1}$. From equation (5.10) we have

$$g(\nabla_\xi X, \varphi X) = \frac{\alpha}{2} g(\varphi X, \varphi X) = \frac{\alpha}{2}. \tag{5.11}$$

Next, we take unit vectors $X \in V^0_{\lambda_3}$ and $Y \in V^0_{\lambda_1}$. Then the Gauss equation (2.8) gives the following:

$$g(R(X, \varphi X)Y, \varphi Y) = -\frac{c}{2} g(\varphi X, \varphi X) g(\varphi Y, \varphi Y) = -\frac{c}{2}. \tag{5.12}$$

On the other hand, it follows from (5.8) that we have

$$\begin{aligned}R(X, \varphi X)Y &= \nabla_X\nabla_{\varphi X}Y - \nabla_{\varphi X}\nabla_X Y - \nabla_{[X,\varphi X]}Y \\ &= -\nabla_{\nabla_X(\varphi X)}Y + \nabla_{\nabla_{\varphi X}X}Y.\end{aligned} \tag{5.13}$$

We here remark that

$$\nabla_X(\varphi X), \nabla_{\varphi X} X \in V^0_{\lambda_3} \oplus \operatorname{span}\{\xi\}$$

for any $X \in V^0_{\lambda_3}$, see [12]. Hence, equation (5.13) is expressed as

$$R(X,\varphi X)Y = -\nabla_{(\nabla_X(\varphi X))_{\lambda_3}} Y - \nabla_{(\nabla_X(\varphi X))_\xi} Y + \nabla_{(\nabla_{\varphi X} X)_{\lambda_3}} Y + \nabla_{(\nabla_{\varphi X} X)_\xi} Y,$$

where $(*)_{\lambda_3}$ and $(*)_\xi$ are the $V^0_{\lambda_3}$-component and the ξ-component of $(*)$, respectively. This, together with equations (2.4), (2.5), (5.8) and (5.11), gives us

$$g(R(X,\varphi X)Y,\varphi Y) = 2\lambda_3 g(\nabla_\xi Y, \varphi Y) = \alpha\lambda_3. \tag{5.14}$$

By (5.12) and (5.14), we obtain $\cot^2(\frac{1}{2}\sqrt{c}r) = -1$, which is a contradiction. Therefore M^{2n-1} does not satisfy condition (1.3). □

In the rest of this paper, we consider the case of 3-dimensional non-Hopf hypersurfaces $\widetilde{M}_2(c)$. Obviously, by Lemma 2.3 and Lemma 4.1, ruled real hypersurfaces satisfy equation (5.1). Namely, we have the following:

Lemma 5.2. *Every ruled real hypersurface in $\widetilde{M}_n(c)$ admits the η-parallelism with respect to the tensor h.*

By using this lemma, we can establish the following proposition:

Proposition 5.1. *Let M^3 be a non-Hopf hypersurface in $\widetilde{M}_2(c)$. Then M^3 satisfies condition* (1.3) *if and only if M^3 is locally congruent to a ruled real hypersurface in $\widetilde{M}_2(c)$.*

Proof. We suppose that M^3 is a non-Hopf hypersurface satisfying condition (1.3) in $\widetilde{M}_2(c)$. Since M^3 is a non-Hopf hypersurface, we can take a local fields of orthonormal frame $\{\xi, U, \varphi U\}$ such that

$$\begin{cases} A\xi = \alpha\xi + \beta U, \\ AU = \beta\xi + \gamma U + \delta\varphi U, \\ A\varphi U = \delta U + \varepsilon\varphi U, \end{cases}$$

where $\beta \neq 0$.

Setting $X = U$, $Y = \varphi U$ and $Z = U$ in (5.1), we have

$$\beta\gamma = g((\nabla_U A)\varphi U, U). \tag{5.15}$$

On the other hand, putting $X = U$, $Y = U$ and $Z = \varphi U$ in (5.1), we have

$$\beta\gamma = 2g((\nabla_U A)\varphi U, U).$$

This, together with equation (5.15), gives $\gamma = 0$.

Similarly, we set $X = \varphi U$, $Y = \varphi U$ and $Z = U$ in (5.1), and get

$$\beta\delta = g((\nabla_{\varphi U} A)\varphi U, U). \tag{5.16}$$

On the other hand, we put $X = \varphi U$, $Y = U$ and $Z = \varphi U$ in (5.1), and we obtain

$$\beta\delta = 2g((\nabla_{\varphi U} A)\varphi U, U).$$

This, combined with equation (5.16), yields $\delta = 0$.

Moreover, we set $X = \varphi U$, $Y = U$ and $Z = U$ in (5.1), and we obtain

$$-\beta\varepsilon + g((\nabla_{\varphi U} A)\varphi U, \varphi U) - g((\nabla_{\varphi U} A)U, U) = 0. \tag{5.17}$$

On the other hand, we put $X = \varphi U$, $Y = \varphi U$ and $Z = \varphi U$ in (5.1), and we get

$$2\beta\varepsilon + g((\nabla_{\varphi U} A)U, U) - g((\nabla_{\varphi U} A)\varphi U, \varphi U) = 0.$$

This, together with equation (5.17), yields $\varepsilon = 0$. Hence, we have $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$ and $A\varphi U = 0$. These imply that M^3 is locally congruent to the ruled real hypersurface. □

By this proposition and Theorem 5.1, we have the following:

Corollary 5.1. *Let M^3 be a real hypersurface in $\widetilde{M}_2(c)$. Suppose that M^3 satisfies condition* (1.3). *Then M^3 is locally congruent to one of the following:*
(i) *a real hypersurface of type* (A) *in $\widetilde{M}_2(c)$,*
(ii) *a real hypersurface of type* (B) *in $\widetilde{M}_2(c)$,*
(iii) *a ruled real hypersurface in $\widetilde{M}_2(c)$.*

We do not know the case when $n \geqslant 3$. Hence, we pose the following problem:

Problem 5.2. *Does there exist a non-Hopf hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ ($n \geqslant 3$) satisfying condition* (1.3) *but being not a ruled real hypersurface in $\widetilde{M}_n(c)$?*

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