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# MAXIMAL NON VALUATION DOMAINS IN AN INTEGRAL DOMAIN

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*Abstract.* Let $R$ be a commutative ring with unity. The notion of maximal non valuation domain in an integral domain is introduced and characterized. A proper subring $R$ of an integral domain $S$ is called a maximal non valuation domain in $S$ if $R$ is not a valuation subring of $S$, and for any ring $T$ such that $R \subset T \subset S$, $T$ is a valuation subring of $S$. For a local domain $S$, the equivalence of an integrally closed maximal non VD in $S$ and a maximal non local subring of $S$ is established. The relation between $\dim(R,S)$ and the number of rings between $R$ and $S$ is given when $R$ is a maximal non VD in $S$ and $\dim(R,S)$ is finite. For a maximal non VD $R$ in $S$ such that $R \subset R'^S \subset S$ and $\dim(R,S)$ is finite, the equality of $\dim(R,S)$ and $\dim(R'^S,S)$ is established.

*Keywords*: maximal non valuation domain; valuation subring; integrally closed subring

*MSC 2020*: 13B02, 13G05, 13F30, 13B22, 13B30

## 1. Introduction

All rings considered below are commutative with nonzero identity and all ring extensions are unital. By an overring of $R$, we mean a subring of the total quotient ring of $R$ containing $R$. By a local ring, we mean a ring with a unique maximal ideal. The symbol $\subseteq$ is used for inclusion, while $\subset$ is used for proper inclusion. Throughout this paper, $\mathrm{qf}(R)$ denotes the quotient field of an integral domain $R$ and $R'^S$ the integral closure of a subring $R$ in a ring $S$. For any ring extension $R \subset S$, by an intermediate ring, we mean a proper subring of $S$ properly containing $R$ and $[R,S] = \{T\colon R \subseteq T \subseteq S,\ T \text{ is a subring of } S\}$. Also, $\mathrm{Supp}(S/R) = \{P \in \mathrm{Spec}(R)\colon R_P \neq S_P\}$ is the support of the $R$-module $S/R$ and $\dim(R,S)$ denotes the number

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of terms of the longest maximal chains in $\mathrm{Supp}(S/R)$. Our work is motivated by [4] and [7]. Let $R \subset S$ be a ring extension of integral domains. Then $R$ is said to be a valuation subring of $S$ ($R$ is a VD in $S$ for short), see [4], if whenever $x \in S$, we have $x \in R$ or $x^{-1} \in R$. Note that if $S = \mathrm{qf}(R)$, then $R$ is a valuation domain. Thus, the concept of valuation subrings of a domain is the generalization of valuation domains. Moreover, if $R$ is not a valuation domain and each $T \in [R, S] \setminus \{R\}$ is a valuation domain, then $R$ is said to be a maximal non valuation subring of $S$, see [7]. It is obvious that if $R$ is a VD in $S$ and $T$ is a ring such that $R \subset T \subseteq S$, then $R$ is a VD in $T$ and $T$ is a VD in $S$. This motivates us to think of those extensions $R \subset S$ of integral domains such that $R$ is not a VD in $S$ and $R$ is maximal with this property, and $S$ is minimal with this property. Motivated by this idea, we introduce the notion of maximal non valuation domain in an integral domain which is a generalization of the concept of maximal non valuation subrings, see [7]. A proper subring $R$ of an integral domain $S$ is called a maximal non valuation domain in $S$ ($R$ is a maximal non VD in $S$, for short) if $R$ is not a valuation subring of $S$, and for any ring $T$ such that $R \subset T \subset S$, $T$ is a valuation subring of $S$. We establish some properties and characterizations of a maximal non VD in an integral domain. Also, we observe that no new class of ring extensions is obtained if $R$ is not a VD in $S$ and $S$ is minimal with this property, that is, $R$ is a VD in each proper subring of $S$ properly containing $R$, see Theorem 2.7.

We discuss the properties of a maximal non VD $R$ in an integral domain $S$ and characterize both $R$ and $S$. We prove that if $R$ is a maximal non VD in $S$, then either $R = R'^S$ or $R \subset R'^S$ has no intermediate ring, see Lemma 2.4. Also, $R$ has at most two maximal ideals if $S$ is local and $R$ is a maximal non VD in $S$, see Lemma 2.3. We also prove that if $R$ is a maximal non VD in $S$ such that $R$ is not a field, then $S$ is an overring of $R$, see Proposition 2.1. For a local domain $S$, the equivalence of an integrally closed maximal non VD in $S$ and a maximal non local subring of $S$ is established in Theorem 2.2. A pair $(R, S)$ is a normal pair (see [8]) if $R \subseteq S$ and $T$ is integrally closed in $S$ for all $T \in [R, S]$. In Theorem 2.3, we prove that $R$ is not local, $(R, S)$ is a normal pair, and either $|[R, S]| = 1 + \dim(R, S)$ or $|[R, S]| = 3 + \dim(R, S)$ for an integrally closed maximal non VD $R$ in $S$ such that $\dim(R, S)$ is finite. Also, when $R$ is not integrally closed then either $|[R, S]| = 1 + \dim(R, S)$ or $|[R, S]| = 2 + \dim(R, S)$, see Theorem 2.6.

Recall from [14] that a ring extension $R \subseteq T$ is said to be a $\lambda$-extension (equivalently, $T$ is a $\lambda$-extension of $R$) if the set of all subrings of $T$ containing $R$ is linearly ordered by inclusion. If $T = \mathrm{qf}(R)$, then $R$ is said to be a $\lambda$-domain. In Theorem 2.4, we prove that if $R$ is integrally closed in a domain $S$ such that $\dim(R, S)$ is finite, then $|[R, S]| = 1 + \dim(R, S)$ if and only if $R \subset S$ is a $\lambda$-extension and $\mathrm{Supp}(S/R)$ is finite with a unique maximal element. For a maximal non VD $R$ in $S$ such that

$R \subset R'^S \subset S$ and $\dim(R,S)$ is finite, the equality of $\dim(R,S)$ and $\dim(R'^S,S)$ is established in Proposition 2.4.

For any ring $R$, $\operatorname{Spec}(R)$ denotes the set of all prime ideals of $R$; $\operatorname{Max}(R)$ the set of all maximal ideals of $R$. As usual, $|X|$ denotes the cardinality of a set $X$.

## 2. Maximal non valuation domains

We begin the section by defining a maximal non valuation domain in an integral domain formally.

**Definition 2.1.** A proper subring $R$ of an integral domain $S$ is called a maximal non valuation domain in $S$ ($R$ is a maximal non VD in $S$ for short) if $R$ is not a valuation subring of $S$, and for any ring $T$ such that $R \subset T \subset S$, $T$ is a valuation subring of $S$.

Recall from [11] that a ring extension $R \subseteq S$ is said to be residually algebraic if for any prime ideal $Q$ of $S$, $S/Q$ is algebraic over $R/(Q \cap R)$. Moreover, if for any ring $T$ in $[R,S]$, the ring extension $R \subseteq T$ is residually algebraic, then $(R,S)$ is said to be a residually algebraic pair, see [5]. It is trivial to see that if $R$ is a VD in $S$, then $(R,S)$ is a residually algebraic pair, see the proof of Theorem 2.1. However, in general, it is not true for non valuation subrings of an integral domain. Now, we will show that if $R$ is a maximal non VD in $S$, where $R$ is not a field, then $(R,S)$ is a residually algebraic pair which is a generalization of [7], Lemma 1 (iii). In the next lemma, we first show that $R \subset S$ is an algebraic extension which is a generalization of [7], Lemma 1 (i). For the sake of completeness, we are giving the proof.

**Lemma 2.1.** *Let $R \subset S$ be an extension of integral domains where $R$ is not a field. If $R$ is a maximal non VD in $S$, then $R \subset S$ is a residually algebraic extension.*

Proof. Let $Q \in \operatorname{Spec}(S)$ and set $P = Q \cap R$. If $S/Q$ is not algebraic over $R/P$, then there exists $t \in S$ such that $\bar{t} = t + Q \in S/Q$ is transcendental over $R/P$. Now, consider $T = (R/P)[\bar{t}^2]$. Then $R \subset U \subset S$, where $T = U/(Q \cap U)$. Therefore, $U$ is a VD in $S$ as $R$ is a maximal non VD in $S$. Thus, either $t \in U$ or $t^{-1} \in U$, which is a contradiction. □

**Theorem 2.1.** *Let $R \subset S$ be an extension of integral domains, where $R$ is not a field. If $R$ is a maximal non VD in $S$, then $(R,S)$ is a residually algebraic pair.*

Proof. Let $R \subset T \subseteq S$. Then either $R$ is a maximal non VD in $T$ or $R$ is a VD in $T$. If $R$ is a maximal non VD in $T$, then the result follows from Lemma 2.1. Now, assume that $R$ is a VD in $T$. Let $Q \in \operatorname{Spec}(T)$ and set $P = Q \cap R$. If possible, suppose that $T/Q$ is not algebraic over $R/P$. Then there exists $t \in T$ such that $\bar{t} = t + Q \in T/Q$ is transcendental over $R/P$. Now, consider $T' = (R/P)[\bar{t}^2]$. Then $R \subset U \subset T$, where $T' = U/(Q \cap U)$. Therefore, $U$ is a VD in $T$. Thus, either $t \in U$ or $t^{-1} \in U$, which is a contradiction. □

Recall from [4], Remark 1.1 (3) that if $R \subset S$ is an extension of integral domains and if $R$ is a VD in $S$, then $\mathrm{qf}(R) = \mathrm{qf}(S)$. Clearly, this may not be true if $R$ is not a VD in $S$. However, if $R$ is a maximal non VD in $S$, where $R$ is not a field, then $\mathrm{qf}(R) = \mathrm{qf}(S)$ as we have the next proposition which is a generalization of [7], Lemma 1 (ii). The proof is similar to that of [7], Lemma 1 (ii) and thus we omit it.

**Proposition 2.1.** *Let $R \subset S$ be an extension of integral domains, where $R$ is not a field. If $R$ is a maximal non VD in $S$, then the following hold true:*
(i) $\mathrm{qf}(R) = \mathrm{qf}(S)$.
(ii) *If $S$ is a field, then $S$ is the quotient field of $R$.*

The next proposition is a generalization of [7], Proposition 1 whose proof is a routine.

**Proposition 2.2.** *Let $R \subset S$ be an extension of integral domains such that $R$ is a maximal non VD in $S$. Then the following statements hold true:*
(i) *For each multiplicatively closed subset $H$ of $R$, either $H^{-1}R$ is a VD in $H^{-1}S$ or $H^{-1}R$ is a maximal non VD in $H^{-1}S$.*
(ii) *For each $Q \in \mathrm{Spec}(S)$, either $R/(Q \cap R)$ is a VD in $S/Q$ or $R/(Q \cap R)$ is a maximal non VD in $S/Q$.*

In the above proposition, suppose that $H = R \setminus P$ for any $P \in \mathrm{Spec}(R)$. Then in the next proposition we show that $H^{-1}R$ is a VD in $H^{-1}S$ provided $R$ is integrally closed in $S$. Under the stated conditions, first we observe that $|\mathrm{Max}(R)| > 1$ in the next lemma.

**Lemma 2.2.** *Let $R \subset S$ be an extension of integral domains and $R$ be integrally closed in $S$. If $R$ is a maximal non VD in $S$, then $R$ is not local.*

P r o o f. Suppose $R$ is local. Since $R \subset S$ is an algebraic extension, $R$ is a VD in $S$ by [5], Theorem 2.5, which is a contradiction. □

**Proposition 2.3.** *Let $R \subset S$ be an extension of integral domains, where $R$ is integrally closed in $S$. If $R$ is a maximal non VD in $S$, then $R_P$ is a VD in $S_P$ for all $P \in \mathrm{Spec}(R)$.*

P r o o f. If $R_P$ is not a VD in $S_P$ for some $P \in \operatorname{Spec}(R)$, then $R_P$ is a maximal non VD in $S_P$ such that $R_P$ is integrally closed in $S_P$ by Proposition 2.2. Therefore, $R_P$ is not local by Lemma 2.2, which is absurd. □

**Remark 2.1.** It is easily seen that if $R$ is a VD in $S$ then $R_P$ is a VD in $S_P$ for all $P \in \operatorname{Spec}(R)$. The preceding proposition shows that the same is true if $R$ is integrally closed and a maximal non VD in $S$.

In Lemma 2.2, we have seen that $|\operatorname{Max}(R)| > 1$ for any integrally closed and maximal non VD $R$ in $S$. Now, if we remove the condition of being integrally closed, then $|\operatorname{Max}(R)| \leqslant 2$ provided $S$ is local. This we see in the next lemma.

**Lemma 2.3.** *Let $R \subset S$ be an extension of integral domains such that $S$ is local. If $R$ is a maximal non VD in $S$, then the following statements hold true:*
(i) $|\operatorname{Max}(R)| \leqslant 2$.
(ii) $|\operatorname{Max}(R'^S)| \leqslant 2$.

P r o o f. Let $R \neq R'^S$. Then either $S$ is integral over $R$ or $R'^S$ is a VD in $S$. Thus, $R'^S$ is local by [4], Corollary 1.6. Hence, $R$ is local. Now, assume that $R = R'^S$. Let $M$ be the maximal ideal of $S$. Then $S = S_M = R_{M\cap R}$ by [5], Lemma 2.9. Suppose that $N_1$, $N_2$, and $N_3$ are any three maximal ideals of $R$. Then $R \subset T = R_{N_1} \cap R_{N_2} \subset S$. Since $R$ is a maximal non VD in $S$, $T$ is a VD in $S$. Therefore, $T$ is local by [4], Corollary 1.6, which is a contradiction. □

Let $R \subset S$ be a ring extension. Then $R$ is said to be a *maximal non local subring* of $S$ if $R$ is not local but each subring of $S$ which contains $R$ properly is local, see [17].

**Theorem 2.2.** *Let $R \subset S$ be an extension of integral domains. If $S$ is local, then the following statements are equivalent:*
(i) *$R$ is a maximal non VD in $S$ such that $R$ is integrally closed in $S$.*
(ii) *$R$ is a maximal non local subring of $S$.*

P r o o f. First suppose that $R$ is a maximal non VD in $S$ such that $R$ is integrally closed in $S$. Then by Lemma 2.2, $R$ is not local. Thus, if $R \subset S$ has no intermediate ring, then we are done. Now, assume that $T$ is a ring such that $R \subset T \subset S$. Then $T$ is a VD in $S$. Thus, $T$ is local by [4], Corollary 1.6. Hence, $R$ is a maximal non local subring of $S$.

Now, suppose that $R$ is a maximal non local subring of $S$. If $R$ is a VD in $S$, then $R$ is local by [4], Corollary 1.6, which is a contradiction. Thus, $R$ is not a VD in $S$. Now, if $R \subset S$ has no intermediate ring, then either $R$ is integrally closed in $S$ or $S$ is integral over $R$. If the latter condition holds, then $R$ is local, a contradiction. Thus, the former condition holds and we are done. Now, suppose that $T$ is a ring such that $R \subset T \subset S$. Then $T$ is local. Now, by [17], Lemma 2 we have that $(R, S)$ is a normal pair. Thus, $R$ and $T$ are integrally closed in $S$. Also, by [17], Lemma 1 we get that $(R, S)$ is a residually algebraic pair and hence $(T, S)$ is a residually algebraic pair. Therefore, $T$ is a VD in $S$ by [5], Theorem 2.5. Thus, $R$ is a maximal non VD in $S$. □

For any prime ideals $P \subset Q$ in $R$, let $[P, Q[$ denote the set of all prime ideals of $R$ containing $P$ which are properly contained in $Q$. The next corollary is a direct consequence of [17], Theorem 1 and Theorem 2.2.

**Corollary 2.1.** *Let $R \subset S$ be an extension of integral domains. If $R$ is integrally closed in $S$ and $S$ is local, then the following statements are equivalent:*

(i) *$R$ is a maximal non VD in $S$.*

(ii) *$(R, S)$ is a normal pair, $R$ is semi local with exactly two maximal ideals $N_1$ and $N_2$ and either:*

  (a) *$S = R_{N_1}$ and $[(0), N_2[ \subseteq [(0), N_1[$, or*

  (b) *$S = R_{N_2}$ and $[(0), N_1[ \subseteq [(0), N_2[$, or*

  (c) *there exists a prime ideal $Q$ of $R$ such that $Q \subset N_1 \cap N_2$, $S = R_Q$ and $[(0), N_1[ = [(0), N_2[$.*

**Remark 2.2.** In [3], Ayache introduced the notion of $\dim(R, S)$ as the number of terms of the longest maximal chains in $\mathrm{Supp}(S/R)$. Ayache, in [3], Proposition 6 (i), showed the following: If $R$ is integrally closed in $S$ and $\dim(R, S)$ is finite, then $(R, S)$ is a normal pair and $R$ is local if and only if $|[R, S]| = 1 + \dim(R, S)$.

One should note the above statement is not correct. For example, take $R = \mathbb{Z}$ and $S = \mathbb{Z}[1/p]$, where $p$ is a prime integer. Then $R$ is integrally closed in $S$ and $\mathrm{Supp}(S/R) = \{p\mathbb{Z}\}$. Clearly, there is no intermediate ring between $R$ and $S$. Thus, $(R, S)$ is a normal pair and $|[R, S]| = 1 + \dim(R, S)$. However, $R$ is not local. In the next theorem, we prove that there is a complete class of ring extensions which counters [3], Proposition 6 (i).

Recall from [10] that a prime ideal $Q$ of a ring $R$ is said to be a divided prime ideal if $QR_Q = Q$. In [1], Akiba characterized the divided prime ideal of $R$ as a prime ideal which is comparable to every ideal of $R$.

**Theorem 2.3.** *Let $R \subset S$ be an extension of integral domains. Assume that $R$ is integrally closed in $S$ and $\dim(R, S)$ is finite. If $R$ is a maximal non VD in $S$, then $R$ is not local, $(R, S)$ is a normal pair, and either*

(i) *$|[R, S]| = 1 + \dim(R, S)$, or*

(ii) *$|[R, S]| = 3 + \dim(R, S)$.*

P r o o f. Let $R$ be a maximal non VD in $S$. Then $R$ is not local by Lemma 2.2. If $|[R,S]| = 2$, then $(R,S)$ is a normal pair as $R$ is integrally closed in $S$. Also, we have $|\mathrm{Supp}(S/R)| = 1$ by [2], Lemma 5. Thus, $|[R,S]| = 1+\dim(R,S)$. Now, assume that $|[R,S]| > 2$. As $R$ is a maximal non VD in $S$, there is a ring between $R$ and $S$ which is a VD in $S$. Thus, $S$ is local by [4], Corollary 1.6. Now, by Corollary 2.1, we have that $(R,S)$ is a normal pair, $R$ is a semi local domain with exactly two maximal ideals $N_1$ and $N_2$, and either (a) $S = R_{N_1}$ and $[(0),N_2[\subseteq [(0),N_1[$, or (b) $S = R_{N_2}$ and $[(0),N_1[\subseteq [(0),N_2[$, or (c) there exists a prime ideal $Q$ of $R$ such that $Q \subset N_1 \cap N_2$, $S = R_Q$ and $[(0),N_1[= [(0),N_2[$.

We claim that only (c) can hold. If possible, suppose that (a) holds. Then $|\mathrm{Supp}(S/R)| = 1$ and hence $\dim(R,S) = 1$. Let $T$ be a ring such that $R \subset T \subset S$. Then by Theorem 2.2, $T$ is local with maximal ideal, say $L$. Now, by Theorem 2.1, we get that $(R,T)$ is a residually algebraic pair. Therefore, we have $T = T_L = R_{L\cap R}$ by [5], Lemma 2.9. Thus, $N_1 = L\cap R$, which is a contradiction. Hence, we get $[R,S] = \{R,S\}$, which again contradicts that $|[R,S]| > 2$. This proves that (a) does not hold. Similarly, (b) does not hold. Thus, only (c) can hold. Then $R \subset R_{N_1} \subset S$ and hence $R_{N_1}$ is a VD in $S$. Therefore, by [4], Theorem 1.5, there exists a divided prime ideal $PR_{N_1} \in \mathrm{Spec}(R_{N_1})$ such that $S = (R_{N_1})_{PR_{N_1}} = R_P$. Thus, $Q = P$. Since $[(0),N_1[= [(0),N_2[$, $P$ is a divided prime ideal in $R$. Now, we assert that there is a one to one order preserving correspondence between the elements of $\mathrm{Supp}(S/R)$ and the elements of $\{T\colon\ R \subset T \subset S,\ T$ is a subring of $S\}$. First, we show that $R_{P'} \in [R,S]$ for all $P' \in \mathrm{Supp}(S/R)$. Suppose that $P' \in \mathrm{Supp}(S/R)$. Then either $Q \subseteq P'$ or $P' \subset Q$. If $P' \subset Q$, then $P' \notin \mathrm{Supp}(S/R)$ as for any $(r/s)/(t/1) = r/st \in S_{P'}$, we have $r/st \in R_{P'}$ for $r \in R$, $s \in R\setminus Q$ and $t \in R\setminus P'$. Thus, $Q \subseteq P'$. Hence, $R_{P'} \in [R,S]$ for all $P' \in \mathrm{Supp}(S/R)$. Note that $R$ is a maximal non local subring of $S$, by Theorem 2.2. Thus, for any ring $T$ such that $R \subset T \subset S$, there exists $V \in \mathrm{Spec}(R)$ such that $T = R_V$ by [17], Lemma 2. We claim that $V \in \mathrm{Supp}(S/R)$. If possible, suppose that $R_V = S_V$. Then $Q = V$, which is a contradiction as $T \neq S$. Therefore, our assertion holds. Note that the elements of $\mathrm{Supp}(S/R)\setminus\{N_2\}$ are totally ordered. Suppose, $Q_1, Q_2 \in \mathrm{Supp}(S/R)\setminus\{N_2\}$. We may assume that $Q_i \neq N_1$ for $i = 1,2$. Then $R \subset R_{Q_1}\cap R_{Q_2} \subset S$. Since $R$ is a maximal non local subring of $S$, $R_{Q_1}\cap R_{Q_2}$ is local, which is a contradiction. Thus, we have $\mathrm{Supp}(S/R) = \{Q_1 \subset Q_2 \subset \ldots \subset Q_{n-1} \subset N_1, N_2\}$, where $\dim(R,S) = n$. Therefore, $[R,S] = \{R, R_{Q_1}, R_{Q_2}, \ldots, R_{Q_{n-1}}, R_{N_1}, R_{N_2}, S\}$ and hence we get $|[R,S]| = 3+\dim(R,S)$. □

From Remark 2.2 and Theorem 2.3, it is clear that if $R$ is integrally closed in $S$, $R \subset S$, and $\dim(R,S)$ is finite, then the conditions that the pair $(R,S)$ is normal and $R$ is local are not necessary for $|[R,S]| = 1+\dim(R,S)$. In the next theorem, we present a necessary and sufficient condition for the same.

**Theorem 2.4.** *Let $R \subset S$ be an extension of integral domains. If $R$ is integrally closed in $S$ and $\dim(R,S)$ is finite, then $|[R,S]| = 1 + \dim(R,S)$ if and only if $R \subset S$ is a $\lambda$-extension and $\operatorname{Supp}(S/R)$ is finite with a unique maximal element.*

Proof. Let $|[R,S]| = 1 + \dim(R,S)$. Then by [2], Theorem 9, $(R,S)$ is a normal pair and $\operatorname{Supp}(S/R)$ is finite. Now, by [3], Theorem 4, there exists a semi local Prüfer domain $T$ such that $|[T, qf(T)]| = 1 + \dim(T)$. Therefore, $T$ is a valuation domain by [19], Theorem 7 and hence $T$ is a $\lambda$-domain by [14], Corollary 1.5. Thus, $R \subset S$ is a $\lambda$-extension and $\operatorname{Supp}(S/R)$ is finite with a unique maximal element by [3], Theorem 4.

Conversely, assume that $R \subset S$ is a $\lambda$-extension and $\operatorname{Supp}(S/R)$ is finite with a unique maximal element. Then $R_{\mathfrak{m}} \subseteq S_{\mathfrak{m}}$ is a $\lambda$-extension for all $\mathfrak{m} \in \operatorname{Max}(R)$. Thus, by [18], Corollary 2.5, $(R_{\mathfrak{m}}, S_{\mathfrak{m}})$ is a normal pair for all $\mathfrak{m} \in \operatorname{Max}(R)$. Now, by [12], Lemma 6.2, $(R,S)$ is a normal pair. Therefore, by [3], Theorem 4, there exists a semi local Prüfer domain $T$ such that $[R,S] \cong [T, \operatorname{qf}(T)]$ (as partially ordered sets) and $\dim(R,S) = \dim(T)$. Thus, $|[R,S]| = |[T, \operatorname{qf}(T)]|$ and $T$ is a $\lambda$-domain, and hence a valuation domain, by [14], Corollary 1.5. Thus, by [19], Theorem 7, $|[T, \operatorname{qf}(T)]| = 1 + \dim(T)$ and hence $|[R,S]| = |[T, \operatorname{qf}(T)]| = 1 + \dim(T) = 1 + \dim(R,S)$. □

Next we offer the following companion for Theorem 2.4.

**Corollary 2.2.** *Let $R \subset S$ be an extension of integral domains. Assume that $\dim(R,S)$ is finite, $R$ is integrally closed in $S$, and there is a maximal ideal $M$ in $R$ such that $|[R_M, S_M]| = 2$. Then $|[R,S]| = 1 + \dim(R,S)$ if and only if $|[R,S]| = 2$.*

Proof. If $|(R,S)| = 1 + \dim(R,S)$, then the result follows from Theorem 2.4 and [9], Theorem 2.7. The converse follows from [2], Lemma 5. □

**Remark 2.3.** As we have already seen that if $R$ is integrally closed in $S$, $R \subset S$, and $\dim(R,S)$ is finite, then the conditions that the pair $(R,S)$ is normal and $R$ is local are not necessary for $|[R,S]| = 1 + \dim(R,S)$, however these are sufficient. To see this, first note that $R \subset S$ is a $\lambda$-extension by [18], Corollary 2.5. Then for every $T \in [R,S] \setminus \{S\}$, $T = R_Q$ for some $Q \in \operatorname{Supp}(S/R)$ by [18], Proposition 2.4. Now, if $[R,S]$ is infinite, then $\dim(R,S)$ is infinite, which is a contradiction. Therefore, $[R,S]$ is finite and hence by [18], Theorem 2.8, $\operatorname{Supp}(S/R)$ is finite. Thus, $|[R,S]| = 1 + \dim(R,S)$ by Theorem 2.4.

In Theorem 2.5, we characterize a maximal non VD in an integral domain $S$ that is not integrally closed in $S$, which can be seen as a generalization of [7], Theorem 3.3. First, we prove the following lemma:

**Lemma 2.4.** *Let $R \subset S$ be an extension of integral domains. If $R$ is a maximal non VD in $S$, then either $R = R'^S$ or $R \subset R'^S$ has no intermediate ring.*

P r o o f. Let $R \subset R'^S$. Assume that there is a ring $T$ such that $R \subset T \subseteq R'^S$. Then $T$ is a VD in $S$ and hence $T$ is integrally closed in $S$. Thus, $T = R'^S$. □

**Theorem 2.5.** *Let $R \subset S$ be an extension of integral domains. If $R$ is not integrally closed in $S$, then the following statements are equivalent:*

(i) *$R$ is a maximal non VD in $S$.*

(ii) $|[R, R'^S]| = 2$, *either $R'^S$ is a VD in $S$ or $R'^S = S$, and $S$ is an overring of $R'^S$.*

(iii) $[R, S] = \{R\} \cup [R'^S, S]$, *$R'^S$ is a VD in $S$ or $R'^S = S$, and $S$ is an overring of $R'^S$.*

P r o o f. (i) ⇒ (ii) By Lemma 2.4, $|[R, R'^S]| = 2$. Since $R \subset R'^S \subseteq S$, $R'^S$ is a VD in $S$ or $R'^S = S$. Note that if $R$ is a field, then $R'^S$ is a field and hence $R'^S = S$. We may now assume that $R$ is not a field. Then by Proposition 2.1, $S$ is an overring of $R'^S$.

(ii) ⇒ (iii) If $R'^S = S$, then we are done. Let $T \in [R, S] \setminus \{R, S\}$. Since $R'^S$ is a VD in $S$, $R'^S$ is local by [4], Corollary 1.6. Let $M$ be the maximal ideal of $R'^S$ and $N = (R : R'^S)$. Then $N \in \operatorname{Max}(R)$ by [20], Theorem 1. Thus, by [15], Theorem 2.8, either $N \in \operatorname{Max}(R'^S)$ or $M^2 \subseteq N \subset M$. If the former holds, then $N = M$. Now, we claim that $R'^S \subseteq T$ or $T \subseteq R'^S$. If possible, suppose there exist $x \in R'^S \setminus T$ and $y \in T \setminus R'^S$. Then $y^{-1} \in M = N$. Therefore, we have $xy^{-1} \in R \subset T$. Thus, $x = xy^{-1}y \in T$, which is a contradiction. Hence, $[R, S] = \{R\} \cup [R'^S, S]$. Next, assume that $M^2 \subseteq N \subset M$. Again, if there exist $x \in R'^S \setminus T$ and $y \in T \setminus R'^S$, then $y^{-1} \in M$. Therefore, $y^{-2} \in M^2 \subseteq N$. Thus, we have $xy^{-2} \in R \subset T$. Hence, $x = xy^{-2}y^2 \in T$, which is a contradiction.

(iii) ⇒ (i) Note that $R$ is not a VD in $S$ as $R$ is not integrally closed in $S$. If $|[R, S]| = 2$, then we are done. Now, suppose that $R \subset T \subset S$. Then $R'^S \subseteq T \subset S$. Thus, $T$ is a VD in $S$. Hence, $R$ is a maximal non VD in $S$. □

**Remark 2.4.** If $R \subset S$ is an extension of integral domains such that $R \subset R'^S \subset S$, then, in general, $\dim(R, S)$ may not be equal to $\dim(R'^S, S)$. For example, consider $R = \mathbb{Z}$, $S = \mathbb{Z}[\sqrt{2}, X]$, where $X$ is indeterminate. Then $R'^S = \mathbb{Z}[\sqrt{2}]$. Clearly, $\dim(R, S) \neq \dim(R'^S, S)$. However, $\dim(R, S) = \dim(R'^S, S)$ if $R$ is a maximal non VD in $S$ such that $R \subset R'^S \subset S$ and $\dim(R, S)$ is finite. This is our next proposition.

**Proposition 2.4.** *Let $R \subset S$ be an extension of integral domains such that $R \subset R'^S \subset S$ and $\dim(R, S)$ is finite. If $R$ is a maximal non VD in $S$, then $\dim(R, S) = \dim(R'^S, S)$.*

P r o o f. We claim that there is a one to one correspondence between the elements of $\operatorname{Supp}(S/R)$ and $\operatorname{Supp}(S/R'^S)$. Now, by Theorem 2.5, $|[R, R'^S]| = 2$ and $R'^S$ is a VD in $S$. Thus, $R'^S$ is local by [4], Corollary 1.6 and hence $R$ is local. Let $M$ be the maximal ideal of $R$ and $M'$ be the maximal ideal of $R'^S$. Now, suppose that

$P \in \mathrm{Supp}(S/R)\backslash\{M\}$. We claim that $P' \in \mathrm{Supp}(S/R'^S)$, where $P = P' \cap R$. Suppose that $S_{P'} = (R'^S)_{P'}$. Note that by [20], Corollary 1, $R_P = (R'^S)_P$. Now, by [12], Lemma 2.4, $(R'^S)_P = (R'^S)_{P'}$. Thus, $S_{P'} = (R'^S)_{P'} = (R'^S)_P = R_P$ and hence $R_P = S_P$, which is a contradiction. Now, assume that $P' \in \mathrm{Supp}(S/R'^S) \setminus \{M'\}$. We want to show that $P \in \mathrm{Supp}(S/R)$, where $P = P' \cap R$. If possible, suppose that $R_P = S_P$. Then by [12], Lemma 2.4, $(R'^S)_P = (R'^S)_{P'}$ and $S_P = S_{P'}$. Therefore, $R_P = (R'^S)_P = (R'^S)_{P'} = S_P = S_{P'}$, which is a contradiction. Now, it remains to show that $M \in \mathrm{Supp}(S/R)$ and $M' \in \mathrm{Supp}(S/R'^S)$. If possible, suppose that $S_{M'} = (R'^S)_{M'}$. Then $S_{M'} = R'^S$ and hence $S = R'^S$, a contradiction. Thus, $M' \in \mathrm{Supp}(S/R'^S)$. Now, if $R_M = S_M$, then $R_M = (R'^S)_M$. Therefore, by [12], Lemma 2.4, $(R'^S)_M = (R'^S)_{M'}$ and $S_M = S_{M'}$. Thus, $R_M = (R'^S)_M = (R'^S)_{M'} = S_M = S_{M'}$, which is a contradiction. Hence, $M \in \mathrm{Supp}(S/R)$. Note that this correspondence is an order isomorphism as $R \subset R'^S$ is an integral extension. Thus, the corresponding map of spectra is closed and hence $\dim(R,S) = \dim(R'^S, S)$. □

In Theorem 2.3, we have shown that $|[R,S]| = 1 + \dim(R,S)$ or $|[R,S]| = 3 + \dim(R,S)$ if $R$ is integrally closed, a maximal non VD in $S$ and $\dim(R,S)$ is finite. A somewhat similar statement is true even if $R$ is not integrally closed in $S$ as we show in the next theorem.

**Theorem 2.6.** *Let $R \subset S$ be an extension of integral domains. Assume that $\dim(R,S)$ is finite and $R$ is not integrally closed in $S$. If $R$ is a maximal non VD in $S$, then either $|[R,S]| = 1 + \dim(R,S)$ or $|[R,S]| = 2 + \dim(R,S)$.*

Proof. As $R$ is a maximal non VD in $S$, either $|[R,S]| = 2$ or $R'^S$ is a VD in $S$ by Theorem 2.5. If the former holds, then $|\mathrm{Supp}(S/R)| = 1$ by [2], Lemma 5. Thus, $|[R,S]| = 1 + \dim(R,S)$. Assume now that $R'^S$ is a VD in $S$. Then $(R'^S, S)$ is a normal pair. Also, by [4], Corollary 1.6, $R'^S$ is local. Thus, $|[R'^S, S]| = 1 + \dim(R'^S, S)$ by Proposition 2.4 and Remark 2.3. Now, by Theorem 2.5, we have $|[R,S]| = 1 + |[R'^S, S]|$. Hence, $|[R,S]| = 2 + \dim(R'^S, S)$. Now, the result follows by Proposition 2.4. □

Let $T$ be a domain and $I$ be an ideal of $T$. If $D$ is a subring of $T/I$ and $R = \varphi^{-1}(D)$, where $\varphi\colon T \to T/I$ is the canonical homomorphism, then we write $R := (T, I, D)$. This pullback construction was introduced by Fontana in [13]. The next lemma can be viewed as an extension of [6], Lemma 1.3. For the sake of completeness, we sketch the proof.

**Lemma 2.5.** *Let $V$ be a VD in $S$ with maximal ideal $M$ and $K = V/M$. Let $D$ be a subring of $K$ and $R := (V, M, D)$. If $T$ is a subring of $S$ which contains $R$, then either $V \subset T$ or $T \subseteq V$.*

P r o o f. Let $V \not\subset T$ and $v \in V \setminus T$. To show that $T \subseteq V$, let $t \in T$. If $t \notin V$, then $t^{-1} \in M$ and hence $t^{-1}v \in M$. Thus, $v = tt^{-1}v \in T$, which is a contradiction. □

Let $R \subset S$ be a ring extension of integral domains. Then $R$ is said to be a pseudovaluation subring of $S$ ($R$ is a PV in $S$ for short), see [4] if $x^{-1}a \in R$ for all $x \in S \setminus R$ and for all non-unit $a \in R$. Note that if $S = \mathrm{qf}(R)$, then $R$ is a pseudovaluation domain, see [16]. Now recall from [4], Proposition 3.3 that a local ring $R$, with a maximal ideal $M$, is a PV in $S$ if and only if there is a unique ring between $R$ and $S$ which is a VD in $S$ with a maximal ideal $M$. We call this the associated VD in $S$ of $R$. The next proposition is a generalization of [7], Proposition 5, where we characterize a maximal non VD in $S$ which is a PV in $S$.

**Proposition 2.5.** *Let $R$ be a PV in $S$ such that $R$ is not a VD in $S$ and $V$ be its associated VD in $S$. Assume that $M$ is the maximal ideal of $V$, $F = R/M$ and $K = V/M$. Then the following statements are equivalent:*

(i) *$R$ is a maximal non VD in $V$;*

(ii) *$R$ is a maximal non VD in $S$;*

(iii) *$[R, S] = \{R\} \cup [V, S]$;*

(iv) *$K$ is algebraic over $F$ and $F \subset K$ has no intermediate ring.*

P r o o f. (i) ⇒ (ii) Note that if $R$ is a VD in $S$, then $R$ is a VD in $V$, a contradiction. Thus, $R$ is not a VD in $S$. If $|[R, S]| = 2$, then we are done. Now, suppose that $T$ is a ring such that $R \subset T \subset S$. Then either $T \subset V$ or $V \subseteq T$ by Lemma 2.5. Let $T \subset V$. Then $T$ is a VD in $V$. Since $V$ is a VD in $S$, $T$ is a VD in $S$. Now, if $V \subseteq T$, then clearly $T$ is a VD in $S$.

(ii) ⇒ (i) If $R$ is a VD in $V$, then $R$ is a VD in $S$, a contradiction. Thus, $R$ is not a VD in $V$. If $|[R, V]| = 2$, then we are done. Let $T$ be a ring such that $R \subset T \subset V$. Then $T$ is a VD in $S$. Thus, $T$ is a VD in $V$.

(i) ⇒ (iii) If $|[R, V]| = 2$, then, by Lemma 2.5, we are done. Now, suppose that $T$ is a ring such that $R \subset T \subset V$. Then $T$ is a VD in $V$. Therefore, $V = T$ by [5], Lemma 2.9, which is a contradiction.

(iii) ⇒ (i) If $R$ is a VD in $V$, then $R$ is a VD in $S$, a contradiction. Thus, $R$ is not a VD in $V$ and hence $R$ is a maximal non VD in $V$.

(iii) ⇒ (iv) Since $R \subset V$ has no intermediate ring, $R \subset V$ is an algebraic extension. For if $x \in V \setminus R$, then either $x^2 \in R$ or $R[x^2] = R[x]$ and hence $x$ is algebraic over $R$. Therefore, $K$ is algebraic over $F$ and $F \subset K$ has no intermediate ring.

(iv) ⇒ (iii) Note that $|[R, V]| = 2$. Then, (iii) follows by Lemma 2.5. □

Now, we discuss a few examples of a maximal non VD in an integral domain.

**Example 2.1.** Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2})$. Take $R = F + XK[[X]]$ and $S = K[[X]]$. Then $R$ is a PV in $S$ by [4], Corollary 2.2. Clearly, $R$ is not a VD in $S$. Thus, by Proposition 2.5, $R$ is a maximal non VD in $S$.

**Example 2.2.** Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2})$. Take

$$S = K + X_1K[X_1]_{(X_1)} + X_2K(X_1)[X_2]_{(X_2)} + \ldots + X_nK(X_1, X_2, \ldots, X_{n-1})[X_n]_{(X_n)}.$$

Clearly, $S$ is local with the maximal ideal $M$, where

$$M = X_1K[X_1]_{(X_1)} + X_2K(X_1)[X_2]_{(X_2)} + \ldots + X_nK(X_1, X_2, \ldots, X_{n-1})[X_n]_{(X_n)}.$$

Let $R := (S, M, F)$. Then $R$ is a PV in $S$ by [4], Corollary 2.2. Thus, by Proposition 2.5, $R$ is a maximal non VD in $S$.

**Example 2.3.** Let $S = \mathbb{Q}[[X]]$, $T = \{p/q\colon\ p, q \in \mathbb{Z}, q \notin 2\mathbb{Z}, 3\mathbb{Z}\}$, $T_1 = \{p/q\colon\ p, q \in \mathbb{Z}, q \notin 2\mathbb{Z}\}$, and $T_2 = \{p/q\colon\ p, q \in \mathbb{Z}, q \notin 3\mathbb{Z}\}$. Let $R$ be the subring of $S$ consisting of the power series whose constant term is in $T$. Then $[R, S] = \{R, V_1, V_2, S\}$, where the constant term of each power series in $V_1$ and $V_2$ is in $T_1$ and $T_2$, respectively. Then clearly $V_1$ and $V_2$ are VD in $S$ but $R$ is not a VD in $S$. Thus, $R$ is a maximal non VD in $S$. Note that there is nothing special in 2, 3 as we can take any distinct prime numbers in this example.

Recall that in the beginning, we have defined that a proper subring $R$ of an integral domain $S$ is a maximal non VD if $R$ is not a VD in $S$ and every proper subring of $S$ properly containing $R$ is a VD in $S$. Now, the natural question arises if we can define a minimal non VD extension, that is, an extension $R \subset S$ where $R$ is not a VD in $S$ and $R$ is a VD in each proper subring of $S$ properly containing $R$. In Theorem 2.7, we show that with this definition, no new class of ring extension is obtained. In the next lemma, first we show that such an extension is a residually algebraic pair.

**Lemma 2.6.** *Let $R \subset S$ be an extension of integral domains. If $R$ is a VD in each proper subring of $S$ properly containing $R$, then $(R, S)$ is a residually algebraic pair.*

P r o o f. Let $T$ be a ring such that $R \subset T \subseteq S$. It is enough to show that $R \subset T$ is a residually algebraic extension. Let $Q \in \operatorname{Spec}(T)$ and set $P = Q \cap R$. Suppose that $T/Q$ is not algebraic over $R/P$. Then there exists $t \in T$ such that $\bar{t} = t + Q \in T/Q$ is transcendental over $R/P$. Consider $T' = (R/P)[\bar{t}^2]$. Then $R \subset U \subset T$, where $T' = U/(Q \cap U)$. Therefore, $R$ is a VD in $U$. Thus, $U$ is local by [4], Corollary 1.6, which is a contradiction. $\square$

**Theorem 2.7.** *Let $R \subset S$ be an extension of integral domains. If $R$ is a VD in each proper subring of $S$ properly containing $R$, then either $R$ is a VD in $S$ or $R \subset S$ has no intermediate ring.*

Proof. *Case 1*: Let $R = R'^S$. Assume that $T$ is a ring such that $R \subset T \subset S$. Then $R$ is a VD in $T$. Therefore, $R$ is local by [4], Corollary 1.6. Now, by Lemma 2.6, $(R, S)$ is a residually algebraic pair. Thus, $R$ is a VD in $S$ by [5], Theorem 2.5.

*Case 2*: Let $R'^S = S$. Assume that $T$ is a ring such that $R \subset T \subset S$. Then $R$ is a VD in $T$ and hence is integrally closed in $T$, which is a contradiction.

*Case 3*: Let $R \subset R'^S \subset S$. Then $R$ is a VD in $R'^S$ and hence is integrally closed in $R'^S$, which is a contradiction. □

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