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FORMAL DEFORMATIONS AND PRINCIPAL SERIES REPRESENTATIONS OF SL(2, $\mathbb{R}$) AND SL(2, $\mathbb{C}$)

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Abstract. In this note, we study formal deformations of derived representations of the principal series representations of SL(2, $\mathbb{R}$). In particular, we recover all the representations of the derived principal series by deforming one of them. Similar results are also obtained for SL(2, $\mathbb{C}$).

Keywords: deformation of representation; Lie algebra; Chevalley-Eilenberg cohomology; Moyal star product; Weyl correspondence; minimal realization

MSC 2020: 17B10, 17B20, 17B56, 22E46, 53D55

1. Introduction

Deformations appear in a natural way in physics: for instance, relativistic mechanics can be considered as a deformation of Galilean mechanics, the deformation parameter being $1/c$, the inverse of the velocity of light.

In mathematics, deformations of Lie algebra structures have been extensively studied since the fundamental works of Gerstenhaber, see [13], Nijenhuis and Richardson, see [24], [26] and are still objects of current research, see, in particular, [9], [10], [11].

In this note, we focus on the problem of deforming Lie algebra representations which has been investigated by various authors, see [16], [21], [22], [23], [25].

Let $\mathfrak{g}$ be a (real or complex) Lie algebra and let π be a representation of $\mathfrak{g}$ on a vector space V. It is well-known that the existence and classification problems for the formal deformations of π depend on the Chevalley-Eilenberg cohomology spaces $H^1(\mathfrak{g}, W)$ and $H^2(\mathfrak{g}, W)$, where W is some subalgebra of $\operatorname{End}(V)$, which are generally difficult to compute, see, for instance, [4] and [23].

As noticed in [7], if we suppose that π has nontrivial formal deformations then, taking for the deformation parameter a real or complex number, we can expect to obtain a one-parameter family of representations of $\mathfrak{g}$ and, conversely, such a one-parameter family (π_ν) (indexed by $\mathbb{R}$ or $\mathbb{C}$) being given, we can expect to recover it by deforming the representation $\pi := \pi_0$.

Then we see that deforming representations is a way to produce many representations of $\mathfrak{g}$ from a few ones and we can hope for the applications of deformations to the classification of representations of $\mathfrak{g}$ and also to the description of unitary duals of Lie groups.

For instance, in [7], we recovered the discrete series representations of $SU(1,n)$ starting from a single minimal realization of $\mathrm{sl}(n+1,\mathbb{C})$, see [18]. We also refer to [4], [23] for other interesting examples of deformations, especially of representations of the Poincaré group.

Here we continue to study representations of semisimple Lie algebras in the light of deformation theory and consider the family $(\varrho_\lambda)_{\lambda\in\mathbb{R}}$ of representations of $\mathrm{sl}(2,\mathbb{R})$ on $C^\infty(\mathbb{R})$ which is obtained by differentiating the so-called principal series representations of $\mathrm{SL}(2,\mathbb{R})$, see [20]. Then we show that the deformation process when applied to ϱ_0 gives the representations ϱ_λ and only them. This is done by computing the cohomology spaces $H^1(\mathrm{sl}(2,\mathbb{R}),\mathcal{D}(\mathbb{R}))$ and $H^2(\mathrm{sl}(2,\mathbb{R}),\mathcal{D}(\mathbb{R}))$, where $\mathcal{D}(\mathbb{R})$ denotes the algebra of differential operators on $\mathbb{R}$. In fact, the computations are simplified by the use of the Weyl correspondence and the Moyal associative product which allow us to replace operators by functions as in [1], [2], [4] and [7]. Similar results are also obtained for the differentials of the principal series representations of $\mathrm{SL}(2,\mathbb{C})$.

Naturally, we could hope to extend our results to principal series representations of general semisimple Lie groups but even in the case of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$ (for arbitrary n) the computations seem to be difficult.

This note is organized as follows. We start with some generalities about formal deformations of Lie algebra homomorphisms (see Section 2) and about the Weyl correspondence and the Moyal product (see Section 3). Then we introduce the principal series representations of $\mathrm{SL}(2,\mathbb{R})$ and their differentials ϱ_λ, $\lambda\in\mathbb{R}$ (see Section 4). The problem of deforming ϱ_0 is considered in Section 5 and, in Sections 6 and 7 we treat similarly the case of the differentials of the principal series representations of $\mathrm{SL}(2,\mathbb{C})$ which is a little more complicated to calculate.

2. Some generalities on deformations

Here we recall some definitions and results of deformation theory. The material of this section is taken from [14], [16], [23], [25] and the exposition essentially follows [4] and [7].

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let A be an associative algebra over $\mathbb{K}$ with the unit element 1. Then A is also a Lie algebra for the commutator $[a, b] := ab - ba$. Let $\varphi\colon\ \mathfrak{g} \to A$ be a Lie algebra homomorphism.

Definition 2.1.

(1) A formal deformation of φ is a formal series $\Phi = \sum_{k\geqslant 0} t^k \Phi_k$, where $\Phi_0 = \varphi$ and, for each $k \geqslant 1$, Φ_k is a linear map from $\mathfrak{g}$ to A such that

$$\Phi([X, Y]) = [\Phi(X), \Phi(Y)] \tag{2.1}$$

for any X and Y in $\mathfrak{g}$. Here we have extended the bracket of A to the formal series by bilinearity.

(2) Two formal deformations Φ and Ψ of φ are said to be equivalent if there exists a series $a = 1 + ta_1 + t^2 a_2 + \ldots \in A[[t]]$ such that for any $X \in \mathfrak{g}$, we have

$$a^{-1}\Phi(X)a = \Psi(X). \tag{2.2}$$

Now we can introduce the structure of a $\mathfrak{g}$-module on A defined by $X \cdot a = [\varphi(X), a]$ for $X \in \mathfrak{g}$ and $a \in A$ and the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in A. Recall that the differential $\partial\psi$ of the p-cochain ψ is the $(p+1)$-cochain given by

$$\begin{aligned}\partial\psi(X_1, X_2, \ldots, X_{p+1}) = &\sum_{i=1}^{n} (-1)^{i+1} X_i \cdot \psi(X_1, \ldots, \widehat{X}_i, \ldots, X_{p+1}) \\ &+ \sum_{1\leqslant i<j\leqslant p+1} (-1)^{i+j} \psi([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{p+1})\end{aligned}$$

for $X_1, X_2, \ldots, X_{p+1} \in \mathfrak{g}$.

Then we immediately see that equation (2.1) is equivalent to the fact that for each $n \geqslant 0$ and any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}(\partial\Phi_n)[X, Y] &:= [\varphi(X), \Phi_n(Y)] + [\Phi_n(X), \varphi(Y)] - \Phi_n([X, Y]) \\ &= -\sum_{k=1}^{n-1} [\Phi_k(X), \Phi_{n-k}(Y)].\end{aligned}$$

In particular, we see that if such a deformation Φ exists then Φ_1 is a 1-cocycle.

We have the following result, see, for instance, [16], Section III and [23], Section I.

Proposition 2.1.

(1) *If $H^2(\mathfrak{g}, A) = (0)$ then for every 1-cocycle $\alpha\colon\ \mathfrak{g} \to A$, there exists a formal deformation Φ such that $\Phi_1 = \alpha$.*

(2) *If $H^1(\mathfrak{g}, A) = (0)$ then every formal deformation Φ of φ is equivalent to φ.*

In [4], we proved the following result.

Proposition 2.2. *Assume that $H^1(\mathfrak{g}, A)$ is one-dimensional and that there exists a formal deformation Φ of φ such that the cohomology class of Φ_1 generates $H^1(\mathfrak{g}, A)$. For each sequence $c = (c_k)_{k\geqslant 1}$ of $\mathbb{K}$, consider the formal series $S_c(t) := \sum_{k\geqslant 1} c_k t^k$ and the formal deformation Φ^c of φ defined by $\Phi^c(X) = \sum_{r\geqslant 0} S_c(t)^r \Phi_r(X)$ for every $X \in \mathfrak{g}$.*

Then the map $c \to \Phi^c$ is a bijection from the set of all sequences $c = (c_k)_{k\geqslant 1}$ of $\mathbb{K}$ onto the set of all equivalence classes of formal deformations of φ.

We need to adapt Proposition 2.3 to the special cases that will be considered in this note.

Proposition 2.3.

(1) *Assume that there exists a formal deformation Φ of φ of the form $\Phi = \varphi + t\Phi_1$ such that the cohomology class of Φ_1 generates $H^1(\mathfrak{g}, A)$. Then every formal deformation of φ is equivalent to a deformation of the form $\varphi + \Big(\sum_{k\geqslant 1} \lambda_k t^k\Big)\Phi_1$, where $\lambda_k \in \mathbb{K}$ for each $k \geqslant 1$.*

(2) *Assume that $H^1(\mathfrak{g}, A)$ has dimension 2 and that there exist two 1-cocycles φ_1, φ_1' whose cohomology classes generate $H^1(\mathfrak{g}, A)$ and such that for every $\lambda, \lambda' \in \mathbb{K}$, $\varphi + t(\lambda\varphi_1 + \lambda'\varphi_1')$ is a formal deformation of φ. Then every formal deformation of φ is equivalent to a deformation of the form*

$$\varphi + \Big(\sum_{k\geqslant 1} \lambda_k t^k\Big)\varphi_1 + \Big(\sum_{k\geqslant 1} \lambda_k' t^k\Big)\varphi_1',$$

where $\lambda_k, \lambda_k' \in \mathbb{K}$ for each $k \geqslant 1$.

P r o o f. Statement (1) is a particular case of Proposition 2.3. The proof of statement (2) is standard and similar to that of Proposition 2.3, see [4]. Let us sketch it briefly. Let Ψ be a formal deformation of φ. The idea is to show the following property by induction: for each integer $p \geqslant 1$, there exist $a_1, a_2, \ldots, a_p \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_1', \lambda_2', \ldots, \lambda_p' \in \mathbb{K}$ such that the formal deformations Ψ^p and Φ^p of φ given by

$$\Psi^p(X) := \exp(t^p a_p) \ldots \exp(ta_1)\Psi(X)\exp(-ta_1)\ldots\exp(-t^p a_p)$$

and

$$\Phi^p(X) := \varphi(X) + \Big(\sum_{k=1}^{p} \lambda_k t^k\Big)\varphi_1 + \Big(\sum_{k=1}^{p} \lambda_k' t^k\Big)\varphi_1'$$

coincide at order p, that is, we have $\Psi_k^p = \Phi_k^p$ for each $k = 1, 2, \ldots, p$.

Indeed, for $p=1$, since Ψ_1 is a 1-cocycle, there exist $\lambda_1, \lambda_1' \in \mathbb{K}$ and $a_1 \in A$ such that $\Psi_1 = \lambda_1\varphi_1 + \lambda_1'\varphi_1' + \partial a_1$, where $(\partial a_1)(X) = [\varphi(X), a_1]$. Then

$$\Psi^1(X) = \exp(ta_1)\Psi(X)\exp(-ta_1)$$

and

$$\Phi^1 = \varphi + t(\lambda_1\varphi_1 + \lambda_1'\varphi_1')$$

are formal deformations of φ that coincide at order 1, since we have that $\Psi_1^1 = \lambda_1\varphi_1 + \lambda_1'\varphi_1'$.

Now, assume that the property is true for p and prove it for $p+1$. Equation (2.1) for Ψ^p gives

$$\begin{aligned}(\partial\Psi_{p+1}^p)(X,Y) = {} & -\sum_{k=1}^{p}[\Psi_k^p(X), \Psi_{p+1-k}^p(Y)] = -\sum_{k=1}^{p}[\Phi_k^p(X), \Phi_{p+1-k}^p(Y)] \\ = {} & (\partial\Phi_{p+1}^p)(X,Y)\end{aligned}$$

since Ψ^p and Φ^p coincide at order p. Then there exist $\lambda_{p+1}, \lambda_{p+1}' \in \mathbb{K}$ and $a_{p+1} \in A$ such that

$$\Psi_{p+1}^p = \Phi_{p+1}^p + \lambda_{p+1}\varphi_1 + \lambda_{p+1}'\varphi_1' + \partial a_{p+1}$$

and we can easily verify that Ψ^{p+1} and Φ^{p+1} coincide at order $p+1$. $\square$

Note that (2) of Proposition 2.4 can be extended to the case, where $\mathrm{Dim}(H^1(\mathfrak{g}, A))$ is arbitrary without notable modification.

Note also that the preceding definitions and results can be applied to the particular case of a representation φ of $\mathfrak{g}$ in a real or complex vector space V, since φ is also a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}(V)$, or, more generally, to a subalgebra A of $\mathrm{End}(V)$.

3. Weyl correspondence and Moyal product

Here we first recall the Weyl correspondence on $\mathbb{R}^{2n}$. In fact, we just need in this note the cases $n=1$ and $n=2$.

The Weyl correspondence on $\mathbb{R}^{2n}$ can be defined as follows, see [8], [12], [17]. For every f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we define the operator $W(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$(W(f)\varphi)(y) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \mathrm{e}^{\mathrm{i}xz} f\Big(y+\frac{x}{2}, z\Big)\varphi(y+x)\,\mathrm{d}x\,\mathrm{d}z.$$

It is well-known that the Weyl calculus can be extended to much larger classes of symbols (see, for instance, [17]). In particular, W induces a linear isomorphism from the space $\mathcal{P}(\mathbb{R}^{2n})$ of C^∞-functions $f(y,z)$ on $\mathbb{R}^{2n}$, which are polynomials with respect to the variables $z_1, z_2, \dots, z_n$, onto the space $\mathcal{D}(\mathbb{R}^n)$ of differential operators on $\mathbb{R}^n$ with coefficients in $C^\infty(\mathbb{R}^n)$.

More precisely, if $f(y,z) = v(y)z^\alpha$, where $v \in C^\infty(\mathbb{R}^n)$, then we have

$$(W(f)\varphi)(y) = \Bigl(\mathrm{i}\frac{\partial}{\partial x}\Bigr)^\alpha \Bigl(v\Bigl(y + \frac{x}{2}\Bigr)\varphi(y+x)\Bigr)\Big|_{x=0},$$

see, for instance, [27]. Here we use the multi-index notation $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$.

In particular, if $f(y,z) = v(y)$ then $(W(f)\varphi)(y) = v(y)\varphi(y)$ and if $f(y,z) = v(y)z_k$ then

$$(W(f)\varphi)(y) = \mathrm{i}(\tfrac{1}{2}\partial_k v(y)\,\varphi(y) + v(y)\partial_k\varphi(y)). \tag{3.1}$$

Now, let us introduce the associative product $*$ on $\mathcal{P}(\mathbb{R}^{2n})$, called the *Moyal product*, which corresponds via W to the composition of differential operators, that is, for every $f_1, f_2 \in \mathcal{P}(\mathbb{R}^{2n})$, we have $W(f_1 * f_2) = W(f_1)W(f_2)$.

Then an expansion of $*$ can be obtained as follows, see [12]. Take coordinates (y,z) on $\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n$ and let $x = (y,z)$. Then one has $x_i = y_i$ for $1 \leqslant i \leqslant n$ and $x_i = z_{i-n}$ for $n+1 \leqslant i \leqslant 2n$.

Let $\Lambda = (\Lambda^{ij})$ be the $(2n \times 2n)$-matrix whose only nonzero entries are $\Lambda^{ii+n} = 1$ and $\Lambda^{i+ni} = -1$ for $1 \leqslant i \leqslant n$. For $f_1, f_2 \in \mathcal{P}(\mathbb{R}^{2n})$, let $P^0(f_1,f_2) := f_1 f_2$,

$$P^1(f_1,f_2) := \sum_{1 \leqslant i,j \leqslant n} \Lambda^{ij}\partial_{x_i} f_1 \partial_{x_j} f_2 = \sum_{k=1}^{n}\Bigl(\frac{\partial f_1}{\partial y_k}\frac{\partial f_2}{\partial z_k} - \frac{\partial f_1}{\partial z_k}\frac{\partial f_2}{\partial y_k}\Bigr)$$

(the usual Poisson brackets) and, more generally, for $l \geqslant 2$,

$$P^l(f_1,f_2) := \sum_{1 \leqslant i_1,\dots,i_l,j_1,\dots,j_l \leqslant n} \Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\dots\Lambda^{i_l j_l}\partial^l_{x_{i_1}\dots x_{i_l}} f_1\, \partial^l_{x_{j_1}\dots x_{j_l}} f_2.$$

Then for every $f_1, f_2 \in \mathcal{P}(\mathbb{R}^{2n})$, we have

$$f_1 * f_2 = \sum_{l \geqslant 0} \frac{(-\mathrm{i})^l}{2^l l!} P^l(f_1,f_2).$$

We also need the Moyal brackets given by

$$[f_1,f_2]_* := \mathrm{i}(f_1 * f_2 - f_2 * f_1) = \sum_{l \geqslant 0} \frac{(-\mathrm{i})^{2l}}{2^{2l}(2l+1)!} P^{2l+1}(f_1,f_2).$$

In Sections 5 and 7, the Moyal product will be used to simplify the computations of some cohomology spaces.

4. Principal series representations of $\mathrm{SL}(2,\mathbb{R})$

When G is a general connected semisimple Lie group with finite center the principal series of G can be introduced as follows, see [20], [28]. Let $G = KAN$ be an Iwasawa decomposition of G (see [15], [28]) and let M be the centralizer of A in K. Then, for any unitary irreducible representation σ of M and any unitary character χ of A, we can consider the representation $\pi_{\chi,\sigma}$ which is obtained by unitary induction to G of the representation $\sigma \otimes \chi \otimes 1_N$ of MAN. Hence these representations $\pi_{\chi,\sigma}$ form the principal series of G.

Here we consider the case $G = \mathrm{SL}(2,\mathbb{R})$ and we can take $K = SO(2)$,

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Then $M = (\pm\mathrm{Id})$ has two characters σ_ε, $\varepsilon = 0,1$, defined by $\sigma_\varepsilon(-\mathrm{Id}) = (-1)^\varepsilon$. This implies that the principal series of $\mathrm{SL}(2,\mathbb{R})$ is indexed by pairs (ν,ε), where $\varepsilon = 0,1$ and $\nu \in \mathbb{R}$. More precisely, we can easily verify that $\pi_{\nu,\varepsilon}$ can be realized in the Hilbert space $L^2(\mathbb{R})$ as

$$(\pi_{\nu,\varepsilon}(g)u)(y) = \mathrm{sgn}(-by+d)^\varepsilon |-by+d|^{-1-\mathrm{i}\nu} u\Big(\frac{ay-c}{-by+d}\Big),$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$, $u \in L^2(\mathbb{R})$ and $y \in \mathbb{R}$, see [20].

This is the 'noncompact' realization of $\pi_{\nu,\varepsilon}$. Note that all these representations except $\pi_{0,1}$ are irreducible and that $\pi_{\nu,\varepsilon}$ is unitarily equivalent to $\pi_{-\nu,\varepsilon}$, see [20], Chapter II.

By a simple computation, we can verify that for any $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \in \mathrm{sl}(2,\mathbb{R})$, $u \in C_0^\infty(\mathbb{R})$ and $y \in \mathbb{R}$, we have

$$(\mathrm{d}\pi_{\nu,\varepsilon}(X)u)(y) = (1+\mathrm{i}\nu)(\beta y + \alpha)u(y) + (\beta y^2 + 2\alpha y - \gamma)u'(y).$$

For every $\nu \in \mathbb{R}$, let us denote by ϱ_ν the representation of $\mathrm{sl}(2,\mathbb{R})$ in $\mathcal{P}(\mathbb{R})$ defined by the same formula as $\mathrm{d}\pi_{\nu,\varepsilon}$:

$$(\varrho_\nu(X)u)(y) = (1+\mathrm{i}\nu)(\beta y + \alpha)u(y) + (\beta y^2 + 2\alpha y - \gamma)u'(y).$$

Let (e_1, e_2, e_3) be the basis of $\mathfrak{g}$ given by

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that we have the following result, as a particular case of [3], Proposition 6. We define the trace form on $\mathrm{sl}(2,\mathbb{R})$ (which is a multiple of the Killing form) by $\langle X, Y\rangle = \mathrm{Tr}(XY)$.

Proposition 4.1.

(1) *The map*

$$\psi\colon\ (y,z) \mapsto \begin{pmatrix} \frac{1}{2}\nu - yz & z \\ \nu y - y^2 z & yz - \frac{1}{2}\nu \end{pmatrix}$$

is a diffeomorphism from $\mathbb{R}^2$ *onto the set of all matrices of the form* $\mathrm{Ad}(g)(\frac{1}{2}\nu e_3)$, *where* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ *with* $a \neq 0$, *which is a dense open subset of the orbit of* $\frac{1}{2}\nu e_3$ *under the adjoint action of* $\mathrm{SL}(2,\mathbb{R})$.

(2) *Let* W *be the Weyl correspondence on* $\mathbb{R}^2$, *see Section* 3. *Then, for any* $X \in \mathrm{sl}(2,\mathbb{R})$ *and* $(y,z) \in \mathbb{R}^2$, *we have*

$$W^{-1}(\varrho_\nu(X))(y,z) = \mathrm{i}\langle \psi(y,z)\,,\, X\rangle.$$

P r o o f. (1) Simple computation.

(2) Write $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right)$. Then, from equation (3.1), we obtain

$$W^{-1}(\varrho_\nu(X))(y,z) = \mathrm{i}\nu(\beta y + \alpha) - \mathrm{i}(\beta y^2 + 2\alpha y - \gamma)z. \tag{4.1}$$

The result follows. $\square$

In other words, the map $X \to -\mathrm{i}W^{-1}(\varrho_\nu(X))(y,z)$ is a parametrization of a dense open subset of the orbit. In the terminology of [3], we say that W is an *adapted Weyl correspondence*, see also [5] and [6]. Since $\pi_{\nu,\varepsilon}$ is associated with the adjoint orbit of $\frac{1}{2}\nu e_3$ by the Kostant-Kirillov method of orbits, see [3], [19], we can see that W provides another way to connect $\pi_{\nu,\varepsilon}$ to the orbit.

5. Deformations of ϱ_0

Here we aim to study the formal deformations of ϱ_0 in $\mathcal{D}(\mathbb{R})$. We start with the following proposition.

Proposition 5.1.

(1) *The map* $\Phi_0\colon\ X \to -\mathrm{i}W^{-1}(\varrho_0(X))$ *is a Lie algebra homomorphism from* $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})$ *to* $\mathcal{P}(\mathbb{R}^2)$.

(2) *We have that* $\tilde{\varrho} := \varrho_0 + \sum_{k\geqslant 1} t^k \tilde{\varrho}_k$ *is a formal deformation of* ϱ_0 *in* $\mathcal{D}(\mathbb{R})$ *if and only if* $\Phi(X) := \Phi_0(X) - \mathrm{i}\sum_{k\geqslant 1} t^k W^{-1}(\tilde{\varrho}_k(X))$ *is a formal deformation of* Φ_0 *in* $\mathcal{P}(\mathbb{R}^2)$.

P r o o f. (1) For any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}[\Phi_0(X), \Phi_0(Y)]_* &= \mathrm{i}(\Phi_0(X) * \Phi_0(Y) - \Phi_0(Y) * \Phi_0(X)) \\ &= -\mathrm{i}(W^{-1}(\varrho_0(X)) * W^{-1}(\varrho_0(Y)) - W^{-1}(\varrho_0(Y)) * W^{-1}(\varrho_0(Y))) \\ &= -\mathrm{i}W^{-1}(\varrho_0(X)\varrho_0(Y) - \varrho_0(Y)\varrho_0(X)) \\ &= -\mathrm{i}W^{-1}(\varrho_0([X,Y])) = \Phi_0([X,Y]).\end{aligned}$$

(2) Easy to verify. □

In other words, the problem of deforming ϱ_0 is equivalent to that of deforming Φ_0 which is more accessible to calculation. The reason is that it is simpler to compute $f * g$ (for $f, g \in \mathcal{P}(\mathbb{R}^2)$) than $W(f)W(g)$ since the terms that cancel in the expansion of $f * g$ can be easily identified.

Note that by equation (4.1), we have

$$\Phi_0(e_1)(y,z) = -y^2 z; \quad \Phi_0(e_2)(y,z) = z; \quad \Phi_0(e_3)(y,z) = -2yz.$$

As explained in Section 2, we need to compute $H^1(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2))$.

Proposition 5.2. *The space $H^1(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2))$ is one-dimensional, generated by the class of the 1-cocycle $\varphi_0\colon\ \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ defined by $\varphi_0(e_1)(y,z) = y$, $\varphi_0(e_2)(y,z) = 0$ and $\varphi_0(e_3)(y,z) = 1$.*

P r o o f. Let $\varphi\colon\ \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ be a 1-cocycle. Then, for any $X, Y \in \mathfrak{g}$, we have

$$[\Phi_0(X), \varphi(Y)]_* + [\varphi(X), \Phi_0(Y)]_* - \varphi([X,Y]) = 0. \tag{5.1}$$

The idea of the proof is to transform φ gradually to a multiple of φ_0 by adding 1-coboundaries.

First, let us put

$$f(y,z) = \int_0^y \varphi_0(e_2)(y',z)\,\mathrm{d}y' \tag{5.2}$$

and consider the 1-cocycle $\varphi_1\colon\ X \to \varphi(X) + [\Phi_0(X), f]_*$ which is equivalent to φ. Then we have

$$\varphi_1(e_2) = \varphi(e_2) + [z, f]_* = \varphi(e_2) - \partial_y f = 0.$$

Now, applying equation (5.1) to φ_1, $X = e_2$ and $Y = e_3$, we get $[z, \varphi_1(e_3)]_* = 0$. This implies that $\varphi_1(e_3)$ is a polynomial $p(z)$. Moreover, writing equation (5.1) for $X = e_1$ and $Y = e_2$, we find that $[z, \varphi_1(e_1)]_* - p(z) = 0$. Hence there exists a polynomial $q(z)$ such that $\varphi_1(e_1) = p(z)y + q(z)$.

Then we can choose a polynomial $r(z)$ such that $p(z) - p(0) - 2z\partial_z r = 0$ and consider the 1-cocycle $\varphi_2\colon X \to \varphi_1(X) + [\Phi_0(X), r(z)]_*$ which is equivalent to φ_1 and hence to φ. Thus we have not only $\varphi_2(e_2) = 0$ but also

$$\varphi_2(e_3) = p(z) + [-2yz, r(z)]_* = p(z) - 2z\partial_z r = p(0).$$

Hence $\varphi_2(e_3)$ is the constant $\lambda := p(0)$ and we have

$$\varphi_2(e_1) = \varphi_1(e_1) - [y^2 z, r(z)]_* = p(z)y + q(z) - 2yz\partial_z r = \lambda y + q(z).$$

Finally, applying equation (5.1) to φ_2 and $X = e_1$, $Y = e_2$, we get $z(\partial_z q) + q(z) = 0$, hence $q(z) = 0$ and $\varphi(e_1) = \lambda y$, $\varphi(e_2) = 0$ and $\varphi(e_3) = \lambda$. Since we can easily verify that φ_0 is not a 1-coboundary, this ends the proof. $\square$

Although it is not essential for our purposes, we compute the space $H^2(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2))$, too.

Proposition 5.3. *We have* $H^2(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2)) = (0)$.

Proof. Let $\beta\colon \mathfrak{g} \times \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ be a 2-cocycle. Then we have $(\partial\beta)(e_1, e_2, e_3) = 0$, hence

$$[-y^2 z, \beta(e_2, e_3)]_* + [-2yz, \beta(e_1, e_2)]_* + [z, \beta(e_3, e_1)]_* = 0. \tag{5.3}$$

We can choose a linear map $\varphi_1\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ such that $\varphi_1(e_2) = 0$ and $\partial_y\varphi_1(e_3) = \beta(e_2, e_3)$. Then the 2-cocycle $\beta_1 := \beta + \partial\varphi_1$ is equivalent to β and satisfies $\beta_1(e_2, e_3) = 0$.

Similarly we can take $\varphi_2\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ such that $\varphi_2(e_2) = 0$, $\varphi_2(e_3) = 0$ and $\partial_y\varphi_2(e_1) = \beta_1(e_1, e_2)$. Then $\beta_2 := \beta_1 - \partial\varphi_2$ is a 2-cocycle which is also equivalent to β and we have $\beta_2(e_2, e_3) = \beta_2(e_1, e_2) = 0$. Thus equation (5.3) for β_2 implies that $\partial_y\beta_2(e_1, e_3) = 0$, hence there exists a polynomial $P(z)$ such that $\beta_2(e_1, e_3) = P(z)$.

Now, let

$$Q(z) = \frac{1}{2}\int_0^1 P(tz)\,\mathrm{d}t$$

and let $\varphi_3\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ be the linear map defined by $\varphi_3(e_1) = Q(z)$, $\varphi_3(e_2) = \varphi_3(e_3) = 0$. Then we see easily that $Q(z) + z(\partial_z Q) = \frac{1}{2}P(z)$ and, consequently, we have

$$(\partial\varphi_3)(e_1, e_3) = [\varphi_3(e_1), -2yz]_* + 2\varphi_3(e_1) = 2z(\partial_z Q) + 2Q(z) = P(z) = \beta_2(e_1, e_3)$$

and also

$$(\partial\varphi_3)(e_2, e_3) = 0 = \beta_2(e_2, e_3) \quad \text{and} \quad (\partial\varphi_3)(e_1, e_2) = \partial_y\varphi_3(e_1) = 0 = \beta_2(e_1, e_2).$$

Finally we have $\beta_2 = \partial\varphi_3$. The result then follows. $\square$

Proposition 5.4. *Let φ_0 be as in Proposition* 5.2. *Then the map $\Phi\colon X \to \Phi_0(X) + t\varphi_0(X)$ is a formal deformation of Φ_0.*

P r o o f. Recall that $\Phi_0\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ is a Lie algebra homomorphism and that φ_0 is a 1-cocycle. Moreover, note that for any $X, Y \in \mathfrak{g}$, we have $[\varphi_0(X), \varphi_0(Y)]_* = 0$. Then, for any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}
[\Phi(X), \Phi(Y)]_* &= [\Phi_0(X), \Phi_0(Y)]_* + t([\Phi_0(X), \varphi_0(Y)]_* + [\varphi_0(X), \Phi_0(Y)]_*)\\
&= \Phi_0([X,Y]) + t\varphi_0([X,Y]) = \Phi([X,Y]).
\end{aligned}$$

□

Then we can apply Proposition 2.4 and obtain a description of all formal deformations of Φ_0 (hence of ϱ_0). Moreover, we can also recover the representations ϱ_ν as shown by the following proposition.

Proposition 5.5. *For any $\nu \in \mathbb{R}$ and any $X \in \mathfrak{g}$, we have $\varrho_\nu(X) = \varrho_0(X) + \mathrm{i}\nu W(\varphi_0(X))$.*

P r o o f. This is immediate by equation (4.1). □

Note that we do not need to continue the deformation process further. Indeed, let us fix $\nu \in \mathbb{R}$ and consider the problem of deforming ϱ_ν or, equivalently, the Lie algebra homomorphism $\Phi_\nu\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^2)$ defined by $\Phi_\nu(X) = -\mathrm{i}W^{-1}(\varrho_\nu(X))$. Then, denoting by $H^1_\nu(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2))$ the first cohomology space corresponding to the $\mathfrak{g}$-module structure defined on $\mathcal{P}(\mathbb{R}^2)$ by $X \cdot f := [\Phi_\nu(X), f]_*$, we have the following result that implies that deforming ϱ_ν (for a given ν) produces nothing but the series $(\varrho_{\nu'})$ again.

Proposition 5.6. *The space $H^1_\nu(\mathfrak{g}, \mathcal{P}(\mathbb{R}^2))$ is one-dimensional and generated by the cohomology class of φ_0 defined at Proposition* 5.2.

P r o o f. Similar to that of Proposition 5.2. □

6. Principal series representations of $\mathrm{SL}(2,\mathbb{C})$

In this and the next section we take $G = \mathrm{SL}(2,\mathbb{C})$, $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$. Let $K = SU(2)$,

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Then we have

$$M = \left\{ \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} : u \in \mathbb{C},\ |u| = 1 \right\}.$$

The principal series of G is then indexed by pairs (ν, n), where $\nu \in \mathbb{R}$ determines the character $\chi_\nu\colon \left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \to a^\nu$ of A and the integer n determines the character $\chi'_n\colon \left(\begin{smallmatrix} u & 0 \\ 0 & \bar{u} \end{smallmatrix}\right) \to u^n$ of M. We can then verify that $\pi_{\nu,n} := \mathrm{Ind}_{MAN}^G(\chi'_n \otimes \chi_\nu \otimes 1_N)$ can be realized in $L^2(\mathbb{C})$ as

$$(\pi_{\nu,n}(g)u)(y) = (-by+d)^n|-by+d|^{-2-\mathrm{i}\nu-n}u\Big(\frac{ay-c}{-by+d}\Big),$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$, $u \in L^2(\mathbb{C})$ and $y \in \mathbb{C}$, see [20], [28].

By differentiating the preceding equation, we can also verify that for any $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \in \mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$, $u \in C_0^\infty(\mathbb{C})$ and $y \in \mathbb{C}$, we have

$$\begin{aligned}(\mathrm{d}\pi_{\nu,n}(X)u)(y) &= \Big(1+\mathrm{i}\frac{\nu}{2}\Big)(\beta y + \overline{\beta}\overline{y} + \alpha + \overline{\alpha})u(y)\\ &\quad + \frac{n}{2}(\beta y - \overline{\beta}\overline{y} + \alpha - \overline{\alpha})u(y) + (\beta y^2 + 2\alpha y - \gamma)\partial_y u\\ &\quad + (\overline{\beta}\overline{y}^2 + 2\overline{\alpha y} - \overline{\gamma})\partial_{\overline{y}} u.\end{aligned}$$

In order to use the Weyl correspondence, it is convenient to consider the equivalent representation $\tilde{\varrho}_{\nu,n}$ which is obtained by transferring $\mathrm{d}\pi_{\nu,n}$ to functions on $\mathbb{R}^2$ taking into account the identification $\mathbb{C} \cong \mathbb{R}^2$ given by $(y_1, y_2) \to y_1 + \mathrm{i}y_2$, $(y_1, y_2) \in \mathbb{R}^2$. More precisely, for any $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \in \mathrm{sl}(2,\mathbb{C})$, $u \in C_0^\infty(\mathbb{R}^2)$ and $(y_1, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}(\tilde{\varrho}_{\nu,n}(X)u)(y_1,y_2) &= \Big(1+\mathrm{i}\frac{\nu}{2}\Big)(\beta(y_1+\mathrm{i}y_2) + \overline{\beta}(y_1-\mathrm{i}y_2) + \alpha + \overline{\alpha})u(y_1,y_2)\\ &\quad + \frac{n}{2}(\beta(y_1+\mathrm{i}y_2) - \overline{\beta}(y_1-\mathrm{i}y_2) + \alpha - \overline{\alpha})u(y_1,y_2)\\ &\quad + \frac{1}{2}(\beta(y_1+\mathrm{i}y_2)^2 + 2\alpha(y_1+\mathrm{i}y_2) - \gamma)(\partial_{y_1}u - \mathrm{i}\partial_{y_2}u)\\ &\quad + \frac{1}{2}(\overline{\beta}(y_1-\mathrm{i}y_2)^2 + 2\overline{\alpha}(y_1-\mathrm{i}y_2) - \overline{\gamma})(\partial_{y_1}u + \mathrm{i}\partial_{y_2}u).\end{aligned}$$

We denote by $\varrho_{\nu,n}$ the representation of $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$ on $\mathcal{P}(\mathbb{R}^2)$ defined by the same formula as $\tilde{\varrho}_{\nu,n}$.

Let us introduce the scalar product on $\mathfrak{g}$ by $\langle X, Y\rangle = \Re\mathrm{Tr}(XY)$. We give now a result that is analogous to Proposition 4.1.

Proposition 6.1.

(1) *The map*

$$\psi'\colon\ (y,z) \to \begin{pmatrix} \frac{1}{2}(\nu - \mathrm{i}n) - y\overline{z} & \overline{z} \\ (\nu - \mathrm{i}n)y - y^2\overline{z} & y\overline{z} - \frac{1}{2}(\nu - \mathrm{i}n) \end{pmatrix}$$

is a diffeomorphism from $\mathbb{C}^2$ *to the set of all matrices of the form*

$$\frac{1}{2}(\nu - \mathrm{i}n)\mathrm{Ad}(g)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

where $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ *with* $a \neq 0$, *which is a dense open subset of the orbit of* $\frac{1}{2}(\nu - \mathrm{i}n)\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ *under the adjoint action of* G.

(2) *Let* W *be the Weyl correspondence on* $\mathbb{R}^4$, *see Section* 3. *Then, for any* $X \in \mathfrak{g}$ *and* $(y_1, y_2, z_1, z_2) \in \mathbb{R}^4$, *we have*

$$W^{-1}(\varrho_{\nu,n}(X))(y_1, y_2, z_1, z_2) = \mathrm{i}\langle \psi'(y,z), X\rangle,$$

where $y = y_1 + \mathrm{i}y_2$ *and* $z = z_1 + \mathrm{i}z_2$.

Proof. Statement (1) can be verified easily. To prove statement (2), let $X = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & -\alpha\end{smallmatrix}\right) \in \mathfrak{g}$. Write $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\beta = \beta_1 + \mathrm{i}\beta_2$ and $\gamma = \gamma_1 + \mathrm{i}\gamma_2$ with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$. Then, using equation (3.1), we find

$$\begin{aligned}-\mathrm{i}W^{-1}(\varrho_{\nu,n}(X))(y_1, y_2, z_1, z_2) &= \nu(\beta_1 y_1 - \beta_2 y_2 + \alpha_1) + n(\beta_2 y_1 + \beta_1 y_2 + \alpha_2)\\ &\quad - (\beta_1 y_1^2 - 2\beta_2 y_1 y_2 - \beta_1 y_2^2 + 2\alpha_1 y_1 - 2\alpha_2 y_2 - \gamma_1)z_1\\ &\quad - (\beta_2 y_1^2 + 2\beta_1 y_1 y_2 - \beta_2 y_2^2 + 2\alpha_2 y_1 + 2\alpha_1 y_2 - \gamma_2)z_2.\end{aligned}$$

The result follows. □

7. Deformations of $\varrho_{0,0}$

In this section, we study the formal deformations of $\varrho_{0,0}$ in $\mathcal{D}(\mathbb{R}^2)$ by following the same lines as in Section 5. As already explained, it is equivalent to studying the formal deformations of the Lie algebra homomorphism $\Psi_0\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^4)$ defined by

$$\Psi_0(X) = -\mathrm{i}W^{-1}(\varrho_{0,0}(X)).$$

More precisely, let us introduce the following basis of $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$ (considered as a real Lie algebra)

$$e_1 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};\quad e_2 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix};\quad e_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};$$
$$f_1 = \begin{pmatrix}0 & \mathrm{i}\\ 0 & 0\end{pmatrix};\quad f_2 = \begin{pmatrix}0 & 0\\ \mathrm{i} & 0\end{pmatrix};\quad f_3 = \begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}.$$

Then, by using the expression for $W^{-1}(\varrho_{0,0}(X))$ given in the proof of Proposition 6.1, we get

$$\Psi_0(e_1) = (-y_1^2 + y_2^2)z_1 - 2y_1y_2z_2; \quad \Psi_0(f_1) = 2y_1y_2z_1 - (y_1^2 - y_2^2)z_2;$$
$$\Psi_0(e_2) = z_1; \quad \Psi_0(f_2) = z_2; \quad \Psi_0(e_3) = -2(y_1z_1 + y_2z_2); \quad \Psi_0(f_3) = 2(y_2z_1 - y_1z_2).$$

Proposition 7.1. *The space $H^1(\mathfrak{g}, \mathcal{P}(\mathbb{R}^4))$ is of dimension 2 and consists of the classes of 1-cocycles $\psi_{\lambda,\mu}\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^4)$ defined by*

$$\psi_{\lambda,\mu}(e_1) = \lambda y_1 + \mu y_2; \quad \psi_{\lambda,\mu}(f_1) = \mu y_1 - \lambda y_2;$$
$$\psi_{\lambda,\mu}(e_2) = \psi_{\lambda,\mu}(f_2) = 0; \quad \psi_{\lambda,\mu}(e_3) = \lambda; \quad \psi_{\lambda,\mu}(f_3) = \mu$$

for $\lambda, \mu \in \mathbb{R}$.

Proof. The proof follows the same lines as that of Proposition 5.2 and thus we only sketch it.

Let $\psi\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^4)$ be a 1-cocycle. Then, for any $X, Y \in \mathfrak{g}$, we have

$$[\Psi_0(X), \psi(Y)]_* + [\psi(X), \Psi_0(Y)]_* - \psi([X, Y]) = 0. \tag{7.1}$$

Step 1. We first apply equation (7.1) to $X = e_2$ and $Y = f_2$. Then we get $\partial_{y_2}\psi(e_2) = \partial_{y_1}\psi(f_2)$. Thus, by Poincaré's Lemma, there exists $u \in \mathcal{P}(\mathbb{R}^4)$ such that $\partial_{y_1}u = \psi(e_2)$ and $\partial_{y_2}u = \psi(f_2)$. Hence, replacing ψ by the equivalent 1-cocycle $\psi + [\Psi_0(\cdot), u]_*$ we can assume that $\psi(e_2) = \psi(f_2) = 0$.

Step 2. By successively applying equation (7.1) to (e_2, e_3), (f_2, e_3), (f_2, f_3) and (e_2, f_3) we verify that $\psi(e_3)$ and $\psi(f_3)$ do not depend on y_1, y_2. Then there exist two polynomials $v(z_1, z_2)$ and $w(z_1, z_2)$ such that $\psi(e_3) = v$ and $\psi(f_3) = w$.

Step 3. Now we apply equation (7.1) successively to the cases $(X, Y) = (e_1, e_2), (e_1, f_2), (e_2, f_1)$ and (f_1, f_2). Then we get

$$\partial_{y_1}\psi(e_1) = v; \quad \partial_{y_2}\psi(e_1) = w; \quad \partial_{y_1}\psi(f_1) = w; \quad \partial_{y_2}\psi(f_1) = -v.$$

This implies that there exist two polynomials $s(z_1, z_2)$ and $r(z_1, z_2)$ such that

$$\psi(e_1) = vy_1 + wy_2 + s; \quad \psi(f_1) = wy_1 - vy_2 + r.$$

Step 4. We choose a polynomial $h(z_1, z_2)$ such that

$$2(z_1\partial_{z_1}h + z_2\partial_{z_2}h) = v(z_1, z_2) - v(0, 0)$$

and replace ψ by the equivalent 1-cocycle $\psi + [\Psi_0(\cdot), h]_*$. This does not modify the equalities $\psi(e_2) = \psi(f_2) = 0$ and, moreover, we have

$$\psi(e_3) + [\Psi_0(e_3), h]_* = v(0,0).$$

We see that one can assume that $\Psi_0(e_3) = v$ is a constant λ.

Step 5. We write equation (7.1) for $(X, Y) = (e_1, e_3)$. Then, on the one hand, we get

$$z_1 \partial_{z_1} w + z_2 \partial_{z_2} w = 0$$

and, since w is a polynomial, we see that w is a constant μ.

On the other hand, we also obtain

$$z_1 \partial_{z_1} s + z_2 \partial_{z_2} s + s = 0.$$

Then, since s is a polynomial, we find that $s = 0$.

Finally, by writing equation (7.1) for $(X, Y) = (e_1, f_3)$, we also obtain that $r = 0$.

Step 6. It remains to show that, for any $\lambda, \mu \in \mathbb{R}$ such that $(\lambda, \mu) \neq (0,0)$, $\psi_{\lambda,\mu}$ is not a 1-coboundary.

Assume that $(\lambda, \mu) \neq (0,0)$ and that there exists $F \in \mathcal{P}(\mathbb{R}^4)$ such that for any $X \in \mathfrak{g}$, we have $\psi_{\lambda,\mu}(X) = [\Psi_0(X), F]_*$. Then by taking $X = e_2$ and $X = f_2$, we get $[z_1, F]_* = [z_2, F]_* = 0$ and we see that F does not depend on y_1, y_2. Moreover, by taking $X = e_3$, we obtain

$$-2z_1 \partial_{z_1} F - 2z_2 \partial_{z_2} F = \lambda.$$

Since $F(z_1, z_2)$ is a polynomial this gives $\lambda = 0$ and F constant. But then we get $\psi_{\lambda,\mu}(f_3) = [\Psi_0(f_3), F]_* = 0$ and hence $\mu = 0$. This contradicts $(\lambda, \mu) \neq (0,0)$ and then the proof is finished. $\square$

As in Section 5, we can verify that for any $(\lambda, \mu) \in \mathbb{R}^2$, the map $\Psi\colon X \to \Psi_0(X) + t\psi_{\lambda,\mu(X)}$ is a formal deformation of Ψ_0 on $\mathcal{P}(\mathbb{R}^4)$ and we obtain a description of all formal deformations of Ψ_0 on $\mathcal{P}(\mathbb{R}^4)$ – hence of all formal deformations of $\varrho_{0,0}$ – by using Proposition 2.4. Moreover, we can also recover the representations $\tilde{\varrho}_{\nu,n}$ of Section 6 by considering the maps $X \to \tilde{\varrho}_{0,0}(X) + \mathrm{i}W(\psi_{\nu,n}(X))$.

Note that the direct computation of $H^2(\mathfrak{g}, \mathcal{P}(\mathbb{R}^4))$ is rather complicated and here we have not succeeded in performing it.

As in the case of $\mathrm{sl}(2,\mathbb{R})$, we can verify that the deformation process when applied to a given representation $\varrho_{\nu,\mu}$ does not produce 'more' representations. Indeed, the problem of deforming $\varrho_{\nu,\mu}$ is equivalent to that of deforming the homomorphism $\Psi_{\nu,\mu}\colon \mathfrak{g} \to \mathcal{P}(\mathbb{R}^4)$ defined by $\Psi_{\nu,\mu}(X) = -\mathrm{i}W^{-1}(\varrho_{\nu,\mu}(X))$. Then we can endow $\mathcal{P}(\mathbb{R}^4)$

with the $\mathfrak{g}$-module structure defined by $X \cdot F := [\Psi_{\nu,\mu}(X), F]_*$ and we denote by $H^1_{\nu,\mu}(\mathfrak{g}, \mathcal{P}(\mathbb{R}^4))$ the corresponding first cohomology space. Thus we have the following result.

Proposition 7.2. *The space $H^1_{\nu,\mu}(\mathfrak{g}, \mathcal{P}(\mathbb{R}^4))$ is generated by classes of the 1-cocycles ψ^1 and ψ^2 defined by*

$$\begin{aligned}
&\psi^1(e_1) = y_1; \quad \psi^1(f_1) = -y_2; \quad \psi^1(e_2) = \psi^1(f_2) = \psi^1(f_3) = 0; \quad \psi^1(e_3) = 1;\\
&\psi^2(e_1) = y_2; \quad \psi^2(f_1) = y_1; \quad \psi^2(e_2) = \psi^2(f_2) = \psi^2(e_3) = 0; \quad \psi^2(f_3) = 1.
\end{aligned}$$

P r o o f. The proof is analogous to that of Proposition 5.2. □

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