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OPTIMIZATION PROBLEM UNDER TWO-SIDED $(\max,+)/(\min,+)$ INEQUALITY CONSTRAINTS

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Abstract. $(\max,+)$-linear functions are functions which can be expressed as the maximum of a finite number of linear functions of one variable having the form $f(x_1,\dots,x_h) = \max\limits_j(a_j+x_j)$, where a_j, $j=1,\dots,h$, are real numbers. Similarly $(\min,+)$-linear functions are defined. We will consider optimization problems in which the set of feasible solutions is the solution set of a finite inequality system, where the inequalities have $(\max,+)$-linear functions of variables x on one side and $(\min,+)$-linear functions of variables y on the other side. Such systems can be applied e.g. to operations research problems in which we need to coordinate or synchronize release and completion times of operations or departure and arrival times of passengers. A motivation example is presented and the proposed solution method is demonstrated on a small numerical example.

Keywords: nonconvex optimization; $(\max,+)/(\min,+)$-linear functions; OR - arrival-departure coordination

MSC 2020: 90C26, 90C30

1. Introduction, motivation

The paper studies the properties of one class of optimization problems formulated by means of the so-called tropical algebra, which has appeared in the literature since the 1960s, sometimes under various different names, e.g. max-algebra, extremal algebra (see e.g. [1], [3], [4]). This part of mathematics has found application since its origin in various parts of operations research, fuzzy sets and others. The main idea of this type of algebra consists in the replacement of addition and multiplication by a pair of two other operations. Addition is replaced by one of the extremal operations max or min and multiplication is replaced by addition. In this way, new

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versions of "linearity" arise, e.g. $(\max,+)$- or $(\min,+)$-linearity. The $(\max,+)$-linear and $(\min,+)$-linear functions of the form $f(x) = \max_j(c_j + x_j)$, $g(y) = \min_k(d_k + y_k)$, where c_j, d_k are given constants, are called $(\max,+)$-linear, $(\min,+)$-linear functions, respectively. In this paper we study the properties of inequality systems with $(\max,+)$-linear functions on one side and $(\min,+)$-linear functions on the other side and propose a method for solving one class of optimization problems whose set of feasible solutions is described by such inequality system. The following example shows a motivation for the research presented in this paper.

Example 1. Passengers travel from stations (cities) S_j, $j = 1,\ldots,n$ to stations R_j, $j = 1,\ldots,n$, via transit stations T_i, $1,\ldots,m$. The passangers must change at the transit stations. Travelling times between S_j and T_i are equal to a_{ij}, travelling times between T_i and R_j are b_{ij}. Departure times at S_j will be denoted x_j, arrival times to destinations R_j will be denoted y_j. We have to find x and y such that

(i) every passenger will be able to get a connection at the transit stations (i.e. does not miss a connection to the final destination);

(ii) no unnecessary delay (waiting time) at the transit stations will take place;

(iii) the departure-time vector x is as close as possible to a recommended fixed vector $\widetilde{x}$.

Condition (i) is satisfied if x, y satisfy the inequality system

$$\max_{1\leqslant j\leqslant n}(a_{ij} + x_j) \leqslant \min_{1\leqslant k\leqslant n}(y_k - b_{ik}), \quad i = 1,\ldots,m.$$

Let the set of all solutions satisfying the inequality system above be denoted M.

Condition (ii) will be ensured for any y by the maximum element $x(y)$, i.e. $(x,y) \in M$ implies that $x \leqslant x(y)$.

Condition (iii) will be satisfied by the optimal solution of the problem

$$\max_{1\leqslant j\leqslant n}|\overline{x}_j(y) - \widetilde{x}_j| \to \min \text{ subject to } (x(y), y) \in M.$$

The aim of the paper is to propose a method for solving optimization problems of the type given in Example 1.

2. Notations, preliminary results

Let us introduce the following notations: $\mathbb{R}$ is the set of real numbers, $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$, A, B are matrices with elements $a_{ij}, b_{ij} \in \mathbb{R}$ for all $i \in I$, $j \in J$,

$$(A \circ x)_i = \max_{j \in J}(a_{ij} + x_j), \quad (B \circ' y)_i = \min_{j \in J}(b_{ij} + y_j), \quad i \in I,$$

$x^\top = (x_1, \dots, x_n)$, $A \circ x = ((A \circ x)_1, \dots, (A \circ x)_n)^\top$, $B \circ' y = ((B \circ' y)_1, \dots, (B \circ' y)_n)^\top$ (superscript $\top$ denotes transposition). The norm will be the Chebyshev norm, i.e. $\|x\| = \max_j |x_j|$.

Let

$$\begin{aligned} &M_1(A, b) = \{x \in \mathbb{R}^n;\ A \circ x \leqslant b\}, \quad M_2(B, b) = (y \in \mathbb{R}^n;\ B \circ' y \geqslant b), \\ &x_j(A, b) = \min_{k \in I}(b_k - a_{kj}), \quad \hat{y}_j(B, b) = \max_{k \in I}(b_k - b_{kj}), \quad j \in J, \end{aligned} \tag{1}$$

Using the matrix-vector notation, we obtain

$$x(A, b) = b^\top \circ' (-A),\ \hat{y}(B, b) = b^\top \circ (-B). \tag{2}$$

Let us summarize some known results, which were proved e.g. in [1], [2].

Lemma 1.
(a) *$x(A, b)$ is the maximum element of $M_1(A, b)$, i.e. $x(A, b) \in M_1(A, b)$ and $x \leqslant x(A, b)$ for all $x \in M_1(A, b)$.*
(b) *$\hat{y}(B, b)$ is the minimum element of $M_2(B, b)$, i.e. $\hat{y}(B, b) \in M_2(B, b)$ and $y \geqslant \hat{y}(B, b)$ for all $y \in M_2(B, b)$.*

3. Problem formulation and solution

We will consider the two-sided inequality system

$$\max_{j \in J}(a_{ij} + x_j) \leqslant \min_{k \in J}(b_{ik} + y_k), \quad i \in I, \tag{3}$$

where $I = \{1, 2, \dots, m\}$, $J = \{1, 2, \dots, n\}$, $a_{ij}, b_{ik} \in \mathbb{R}$ or in matrix-vector notation

$$A \circ x \leqslant B \circ' y. \tag{4}$$

The set of all solutions of system (3) will be denoted M. Let us set $b_i(y) = (B \circ' y)_i$, $i \in I$. Then the system can be written in the form

$$\max_{j \in J}(a_{ij} + x_j) \leqslant b_i(y), \quad i \in I. \tag{5}$$

The set M has in accordance with Lemma 1 a maximum element $x(b(y))$, which is defined as follows:

$$x_j(b(y)) = \min_{i \in I}(b_i(y) - a_{ij}) = \min_{i \in I} \min_{k \in J}(b_{ik} - a_{ij} + y_k). \tag{6}$$

Interchanging the minimization operations in (6), we obtain

$$x_j(b(y)) = \min_{k \in J} \min_{i \in I}(b_{ik} - a_{ij} + y_k), \quad j \in J, \tag{7}$$

so that in matrix-vector notation we get

$$x(b(y)) = B^\top \circ (-A) \circ' y = Q \circ' y, \tag{8}$$

where we set $Q = B^\top \circ' (-A)$. We will solve the following optimization problem:

$$\|x(b(y)) - \widetilde{x}\| \to \min \text{ subject to } y \in \mathbb{R}^n, \tag{9}$$

where $\widetilde{x} \in \mathbb{R}^n$ is a given fixed element. Taking (8) into account, the problem is equivalent to the problem

$$t \to \min \text{ subject to } \|Q \circ' y - \widetilde{x}\| \leqslant t. \tag{10}$$

Let us note that since we can set $x_j(b(y)) := x_j(b(y)) - \widetilde{x}_j$, $j \in J$, we can assume w.l.o.g. that $\widetilde{x} = 0$. We will therefore further accept this asumption.

Taking into account, definition of the norm, problem (10) can be transformed (under the assumption that $\widetilde{x} = 0$) as follows:

$$t \to \min \text{ subject to } -t \leqslant \min_{k \in J}(q_{jk} + y_k) \leqslant t, \quad j \in J. \tag{11}$$

Let $M(t)$ denote the set of feasible solutions of problem (11). To analyze the structure of the set $M(t)$, let us set

$$M_1(t) = \{y \in \mathbb{R}^n;\ \min_{k \in J}(q_{jk} + y_k) \geqslant -t,\ j \in J\}, \tag{12}$$

$$M_2(t) = \{y \in \mathbb{R}^n;\ \min_{k \in J}(q_{jk} + y_k) \leqslant t,\ j \in J\}, \tag{13}$$

so that $M(t) = M_1(t) \cap M_2(t)$. Let us note that using this notation, problem (11) can be reformulated as follows:

$$t \to \min \text{ subject to } M(t) \neq \emptyset. \tag{14}$$

We will investigate the conditions under which $M(t) \neq \emptyset$.

Let us derive conditions under which $y \in M_1(t)$, i.e. solve the inequality system $Q \circ' y \geqslant -t$.

$$y \in M_1(t) \Leftrightarrow y_k \geqslant -q_{jk} - t \quad \forall\, k \in J,\ j \in J, \tag{15}$$

so that $y_k \geqslant \max\limits_{j \in J}(-q_{jk}) - t$ for all $k \in J$.

Using the matrix-vector notation we obtain

$$y \in M_1(t) \Leftrightarrow y \geqslant \underline{y} - t, \tag{16}$$

where we set $\underline{y}_k = \max\limits_{j \in J}(-q_{jk})$, $k \in J$.

We investigate now the properties of the set $M_2(t)$. We have that

$$y \in M_2(t) \Leftrightarrow\ \forall\, j \in J\ \exists\, k(j) \in J \text{ such that } q_{jk(j)} + y_k(j) \leqslant t. \tag{17}$$

Since $M(t) = M_1(t) \cap M_2(t)$, we obtain

$$y \in M(t) \Leftrightarrow\ \forall\, j \in J\ \exists\, k(j) \in J \text{ such that } -t + \underline{y}_{k(j)} \leqslant y_{k(j)} \leqslant -q_{jk(j)} + t. \tag{18}$$

Let us set for all $j, k \in J$:

$$T_{jk}(t) = \{y_k;\ -t + \underline{y}_k \leqslant y_k \leqslant -q_{jk} + t\}. \tag{19}$$

Then (18) can be reformulated as follows:

$$y \in M(t) \Leftrightarrow\ \forall\, j \in J\ \exists\, k(j) \in J \text{ such that } T_{jk(j)}(t) \neq \emptyset. \tag{20}$$

We will find the minimum t such that $T_{jk}(t) \neq \emptyset$. This minimum is equal to the value τ_{jk} of t at which the left- and right-hand-side of the inequalities defining $T_{jk}(t)$ in (19) are equal, i.e. at which $-t + \underline{y}_k = -q_{jk} + t$ so that

$$\tau_{jk} = \frac{1}{2}(\underline{y}_k - q_{jk}). \tag{21}$$

Taking into account relations (17)–(21), we obtain

$$M(t) \neq \emptyset \Leftrightarrow t \geqslant \tau \equiv \max_{j \in j} \min_{k \in J} \tau_{jk}. \tag{22}$$

It follows that the optimal value t^{opt} of t in optimization problem (14) is equal to τ.

It remains to find the corresponding optimal solution y^{opt} of the original optimization problem (9). As an optimal y solving (9) with $\widetilde{x} = 0$ can be accepted any solution which satisfies the inequality system

$$-t^{\mathrm{opt}} \leqslant \min_{k \in J}(q_{jk} + y_k) \leqslant t^{\mathrm{opt}}, \quad j \in J. \tag{23}$$

Note that it follows from the theoretical results above that the set of solutions of system (23) is always nonempty and if $t < t^{\mathrm{opt}}$, the set of feasible solutions $M(t)$ is empty. If y satisfies system (23), then $\|x(b(y))\| = t^{\mathrm{opt}}$. The vector $x(b(y)) + \widetilde{x}$ is the closest vector to any given $\widetilde{x}$ and at the same time $x(b(y))$ is the maximum element x satisfying the system $A \circ x \leqslant B \circ' y = b(y)$ for the chosen y satisfying system (23). In the context of Example 2, the element $x(y) + \widetilde{x}$ minimizes the distance from the given (recommended) departure times $\widetilde{x}$ and ensures no unnecessary delay at the transit stations.

In the sequel, we will illustrate the theoretical results by a small numerical example.

Example 2. Let $\widetilde{x} = (0, 0)^\top$,

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 1 \\ 7 & 0 \end{pmatrix}, \quad Q = B^\top \circ' -A = \begin{pmatrix} 7 & 8 \\ 1 & 4 \end{pmatrix},$$

$x_1(b(y)) = (Q \circ' y)_1 = \min(7 + y_1, 8 + y_2)$, $x_2(b(y)) = (Q \circ' y)_2 = \min(1 + y_1, 4 + y_2)$. Let us consider the inequality system $t \geqslant (Q \circ' y)_1 - \widetilde{x}_1 = \min(7 + y_1, 8 + y_2) \geqslant -t$, $t \geqslant (Q \circ' y)_2 - \widetilde{x}_2 = \min(1 + y_1, 4 + y_2) \geqslant -t$. We have $\tau_{11} = 3$, $\tau_{12} = 2$, $\tau_{21} = 0$, $\tau_{22} = 0$ so that

$$t^{\mathrm{opt}} = \max_i \min_j \tau_{ij} = \max(2, 0) = 2.$$

The system

$$2 \geqslant (Q \circ' y)_1 - \widetilde{x}_1 = \min(7 + y_1, 8 + y_2) \geqslant -2,$$
$$2 \geqslant (Q \circ' y)_2 - \widetilde{x}_2 = \min(1 + y_1, 4 + y_2) \geqslant -2$$

has solution $\underline{y} = (-3, -6)^\top$, so that $x(b(\underline{y})) = Q \circ' \underline{y} = (2, -2)^\top$ and therefore $\|x(b(\underline{y})) - \widetilde{x}\| = 2 = t^{\mathrm{opt}}$.

4. Conclusion

We proposed an algorithm for finding the optimal solution of problem (9). The algorithm needs to compute mn numbers τ_{jk} and after that to compute t^{opt} according to formula (21). Example 1 shows a possible application to synchronization of activities. Since in real world conditions some input data of the synchronization problems may be uncertain, further research may be oriented to problems with uncertain input coefficients. For instance, we can consider problems with interval, fuzzy or stochastic input matrices whose elements give the processing times of the activities.

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